Online Supplement to "Rapid Screening Procedures for Zero-One Optimization via Simulation"

Shing Chih Tsai
Department of Industrial and Information Management, National Cheng Kung University, Tainan City, Taiwan,
sctsai@mail.ncku.edu.tw

1. Proof of Approach A

Without loss of generality we assume that there are k solutions evaluated by the proposed procedure. That is, $\mathbf{X}_{[k]}$ is the best solution among the union of solution sets $\bigcup_{r=0}^{R-1} \tilde{I}(r)$. Let us consider the case when $\mathbf{X}_{[k]}$ is generated from the 0-th iteration (i.e., one of the initial solutions), that is, $[k] \in I(0)$. In this case, we should obtain the smallest PCS because $\mathbf{X}_{[k]}$ has to survive all R screening iterations to be selected at the end of the procedure. We want to show that our procedure can satisfy the PCS requirement even under the most difficult scenario. Therefore,

$$\begin{aligned} \text{PCS}\left[\bigcup_{r=0}^{R-1} \tilde{I}(r)\right] &\geq \Pr\{\text{select } \mathbf{X}_{[k]} | [k] \in I(0)\} \\ &= 1 - \Pr\{[k] \notin I'(R) | [k] \in I(0)\} - \Pr\{\text{select } \mathbf{X}_i, i \neq [k] | [k] \in I'(R)\}. \end{aligned}$$

The above equality follows because there are only two cases when $\mathbf{X}_{[k]}$ will not be chosen at the end: (i) $\mathbf{X}_{[k]}$ does not survive through R iterations, and (ii) $\mathbf{X}_{[k]}$ is contained in $I'(R)$ but is not selected in the selection iteration. Note that cases (i) and (ii) are mutually exclusive events.

Let PICS(r) denote the probability of the event that $\mathbf{X}_{[k]}$ is eliminated at iteration r (i.e., an incorrect selection is made). The Bonferroni inequality (e.g., Chapter 7 of Tong (1980)) is then applied to yield

$$\Pr\left\{[k] \notin I'(R) | [k] \in I(0)\right\} \leq \sum_{r=0}^{R-1} \text{PICS}(r), \qquad (1)$$

where

$$
\begin{aligned}
\text{PICS}(r) &= \Pr\left\{\exists \ell \in I(r), \bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \le -W_{[k]\ell}(r), \ell \ne [k]\right\} \\
&= 1 - \Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \ge -W_{[k]\ell}(r), \forall \ell \in I(r), \ell \ne [k]\right\} \\
&\le 1 - \Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \ge -W_{[k]\ell}(r), \forall \ell \in \{1, 2, \ldots, B(r)\}, \ell \ne [k]\right\} \qquad (2) \\
&\le 1 - \prod_{\substack{\ell=1 \\ \ell \ne [k]}}^{B(r)} \Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \ge -W_{[k]\ell}(r)\right\} \qquad (3) \\
&\le \frac{\alpha_1}{R}. \qquad (4)
\end{aligned}
$$

Inequality (2) follows because $\{\forall \ell \in \{1, 2, \ldots, B(r)\}\}$ is a bigger set than $\{\forall \ell \in I(r)\}$, which makes the condition more restrictive. Inequality (3) holds by application of Slepian's inequality (e.g., Tong, 1980) since $\text{Cov}[\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r), \bar{Y}_{[k]}(r) - \bar{Y}_{m}(r)] = \text{Var}[\bar{Y}_{[k]}(r)]$, which is nonnegative for any system $\ell \ne m$ (note that CRN is not used). Following similar developments in Appendix 7.1 of Boesel et al. (2003) we can show that

$$
\Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \ge -W_{[k]\ell}(r)\right\} \ge \left(1 - \alpha'\right)^{1/(B(r)-1)},
$$

regardless of the configuration of the true means. Then we obtain Inequality (4) from the way α' is chosen. Combining Inequality (1) and Inequality (4) we can conclude that $\Pr\{[k] \notin I'(R) | [k] \in I(0)\} \le \alpha_1$. This also implies that $\Pr\{[k] \in I'(R)\} \ge 1 - \alpha_1$ for all configurations of the true means. Furthermore, we know $\Pr\{\text{select } \mathbf{X}_i, i \ne [k] | [k] \in I'(R)\} \le \alpha_2$ whenever $\mu_{[k]} - \mu_{[k-1]} \ge \delta$, by the similar derivation of the Extended Rinott's Procedure from Boesel et al. (2003). Even if $\mu_{[k]} - \mu_{[k-1]} < \delta$ we can guarantee that the solution we select is within δ of the truly best solution in $I'(R)$ with probability $\ge 1 - \alpha_2$ (see Theorem 1 of Nelson and Matejcik (1995)). Notice that the Rinott's constant h used to determine N_i depends on the total number of solutions sampled in the procedure (i.e., $B(R-1)$), rather than just those solutions remaining in the subset $I'(R)$. This is because the conditional probability of selecting the best solution, given it survives through the R screening iterations, depends on whether or not the previous samples are maintained. Finally, assuming $\mu_{[k]} - \mu_{[k-1]} \ge \delta$ and combining the above results yield $\text{PCS}\left[\bigcup_{r=0}^{R-1} \tilde{I}(r)\right] \ge 1 - \alpha$.

2. Proof of Approach B

Similar to the derivation of the validity of approach A, we assume $[k] \in I(0)$ to argue that PCS can hold even in the worst case. First, we want to show that $\Pr\{[k] \in I'(R)|[k] \in I(0)\} \geq 1 - \alpha_1$ for all configurations of the true means. We consider

$$\begin{aligned}
\Pr\left\{[k] \notin I'(r+1)|[k] \in I'(r)\right\} &= \Pr\left\{\bigcup_{\substack{\ell \in I(r)\\ \ell \neq [k]}} \left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \leq -W_{[k]\ell}(r)\right\}\right\} \\
&\leq \sum_{\substack{\ell \in I(r)\\ \ell \neq [k]}} \Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \leq -W_{[k]\ell}(r)\right\} \\
&\leq \sum_{\substack{\ell \in I(r)\\ \ell \neq [k]}} \frac{\alpha'}{|I(r)|-1} \qquad (5) \\
&= 1 - (1-\alpha_1)^{1/R}
\end{aligned}$$

where Inequality (5) follows because the Bonferroni inequality is applied to yield

$$\Pr\left\{\bar{Y}_{[k]}(r) - \bar{Y}_{\ell}(r) \leq -W_{[k]\ell}(r)\right\} \leq \frac{\alpha'}{|I(r)|-1}$$

which is analogous to the case employing CRN in Nelson et al. (2001). Therefore, we can conclude $\Pr\left\{[k] \in I'(r+1)|[k] \in I'(r)\right\} \geq (1-\alpha_1)^{1/R}$ for all configurations of the true means. Because the screening statistic we use in each iteration is based on only new samples, these R screening iterations can be deemed independent of each other. Therefore, under any configuration of the true means we can obtain

$$\begin{aligned}
\Pr\{[k] \in I'(R)|[k] \in I(0)\} &= \Pr\{[k] \in I'(1)|[k] \in I(0)\} \times \prod_{r=1}^{R-1} \Pr\left\{[k] \in I'(r+1)|[k] \in I'(r)\right\} \\
&\geq 1 - \alpha_1.
\end{aligned}$$

The proof is thus attained.

References

Boesel, J., B.L. Nelson, S.-H. Kim. 2003. Using ranking and selection to "clean up" after simulation optimization. *Oper. Res.* **54** 814–825.

Nelson, B.L., F.J. Matejcik. 1995. Using common random numbers for indifference-zone selection and multiple comparisons in simulation. *Management Sci.* **41** 1935–1945.

Nelson, B.L., J. Swann, D. Goldsman, W. Song. 2001. Simple procedures for selecting the best simulated system when the number of alternatives is large. *Oper. Res.* **49** 950–963.

Tong, Y.L. 1980. *Probability Inequalities in Multivariate Distributions.* Academic Press, New York.