Online Supplement for
Optimal Sampling Laws for Stochastically Constrained Simulation Optimization on Finite Sets

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A. Useful Results

Proposition A.1 (Principle of the slowest term (see, e.g., Ganesh et al., 2004, Lemma 2.1))**.** *Let $a_i(n), i = 1, 2, \ldots, k$, be a finite number of sequences in $\mathbb{R}^+$, the set of positive reals. If $\lim_{n\to\infty} \frac{1}{n} \log a_i(n)$ exists for all i, then $\lim_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{k} a_i(n) = \max_i \left(\lim_{n\to\infty} \frac{1}{n} \log a_i(n) \right)$.*

A consequence of the principle of the slowest term, Proposition A.2 states that the slowest among a set of rate functions is equivalent to the rate function of the slowest sequence.

Proposition A.2. *Let $a_i(n)$ be defined as in Proposition A.1. If $\lim_{n\to\infty} \frac{1}{n} \log a_i(n)$ exists for all i, then $\max_i \lim_{n\to\infty} \frac{1}{n} \log a_i(n) = \lim_{n\to\infty} \frac{1}{n} \log\left(\max_i a_i(n)\right)$.*

Proof. The lower bound is $\max_i \lim_{n\to\infty} \frac{1}{n} \log a_i(n) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{k} a_i(n) \geq \lim_{n\to\infty} \frac{1}{n} \log \max_i a_i(n)$. The upper bound is $\lim_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{k} a_i(n) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(k \max_i a_i(n)\right) = \lim_{n\to\infty} \frac{1}{n} \log \max_i a_i(n)$. □

B. Proof of Proposition 2

Proof. Suppose $\boldsymbol{\alpha}^*$ is not unique. Then there exists another optimal point $\check{\boldsymbol{\alpha}}^*$ such that $\boldsymbol{\alpha}^* \neq \check{\boldsymbol{\alpha}}^*$, but $z^* = \check{z}^*$. We contradict this statement in each of the following cases.

Case 1: $\alpha_1^* = \check{\alpha}_1^*$. Suppose system 1 receives the same allocation in each of the optimal solutions $\boldsymbol{\alpha}^*$ and $\check{\boldsymbol{\alpha}}^*$. Since $\sum_{i=1}^{r} \alpha_i^* = \sum_{i=1}^{r} \check{\alpha}_i^* = 1$, and $\alpha_1^* = \check{\alpha}_1^*$, then there exist systems k and ℓ for which $\alpha_k^* > \check{\alpha}_k^*$ and $\alpha_\ell^* < \check{\alpha}_\ell^*$. Let the rate function for $P\{FS_i\}, i \neq 1$, be denoted $K_i(\alpha_1, \alpha_i)$, where this expression will vary depending on whether $i \in \Gamma, \mathcal{S}_b$, or $\mathcal{S}_w$. Then since the rate functions have strictly positive partial derivatives $\partial K_i(\alpha_1, \alpha_i)/\partial \alpha_i$ for all $i = 2, \ldots, r$ (see equation (6) for the partial derivatives for $i \in \Gamma$; the others follow trivially), it must be the case that $K_k(\alpha_1^*, \alpha_k^*) > K_k(\alpha_1^*, \check{\alpha}_k^*)$ and $K_\ell(\alpha_1^*, \alpha_\ell^*) < K_\ell(\alpha_1^*, \check{\alpha}_\ell^*)$. However this provides a contradiction, since feasibility in Problem Q^* requires $K_k(\alpha_1^*, \alpha_k^*) = K_\ell(\alpha_1^*, \alpha_\ell^*)$ and $K_k(\alpha_1^*, \check{\alpha}_k^*) = K_\ell(\alpha_1^*, \check{\alpha}_\ell^*)$.

Case 2: $\alpha_1^* \neq \check{\alpha}_1^*$ and $\alpha_i^* \neq \check{\alpha}_i^*$ for at least one $i \in \mathcal{S}_b$. Suppose $\alpha_k^* < \check{\alpha}_k^*$ for $k \in \mathcal{S}_b$. Then feasibility in Problem Q^* implies $\alpha_k^* \sum_{j \in \mathcal{C}_I^k} J_{kj}(\gamma_j) = z^* < \check{\alpha}_k^* \sum_{j \in \mathcal{C}_I^k} J_{kj}(\gamma_j) = \check{z}^*$, which is a contradiction.

Case 3: $\alpha_1^* \neq \check{\alpha}_1^*$ and $\alpha_i^* = \check{\alpha}_i^*$ for all $i \in \mathcal{S}_b$. Since $\alpha_1^* \neq \check{\alpha}_1^*$, the feasibility constraint for system 1 cannot be binding at optimality in Problem Q^*. Suppose we pre-allocate a total, fixed, and optimal portion of the sample $\alpha_{\mathcal{S}_b}^* = \check{\alpha}_{\mathcal{S}_b}^*$ to the systems in $\mathcal{S}_b$. Then we may consider a modified version of the problem in (5),

$$\begin{aligned} \max \quad & \min\left(\min_{i \in \Gamma} \left(\inf_x (\alpha_1 I_1(x) + \alpha_i I_i(x)) \right), \min_{i \in \mathcal{S}_w} \left(\inf_x (\alpha_1 I_1(x) + \alpha_i I_i(x)) + \alpha_i \sum_{j \in \mathcal{C}_I^i} J_{ij}(\gamma_j) \right) \right) \quad \text{(B.1)} \\ s.t. \quad & \sum_{i \in \{1\} \cup \Gamma \cup \mathcal{S}_w} \alpha_i = 1 - \alpha_{\mathcal{S}_b}^*, \quad \alpha_i \geq 0. \end{aligned}$$

We now establish the strict concavity of the rate functions of systems in Γ. Denote the rate function as $K_k(\alpha_1, \alpha_k)$ for $k \in \Gamma$, and let the exponent T denote the matrix transpose. Since $(\alpha_1^*, \alpha_k^*) \neq (\check{\alpha}_1^*, \check{\alpha}_k^*)$, consider

$$\begin{aligned} &(\nabla K_k(\alpha_1, \alpha_i)|_{(\alpha_1^*, \alpha_k^*)} - \nabla K_k(\alpha_1, \alpha_i)|_{(\check{\alpha}_1^*, \check{\alpha}_k^*)})^T ((\alpha_1^*, \alpha_k^*)^T - (\check{\alpha}_1^*, \check{\alpha}_k^*)^T) \\ &= (I_1(x(\alpha_1^*, \alpha_k^*)) - I_1(x(\check{\alpha}_1^*, \check{\alpha}_k^*)))(\alpha_1^* - \check{\alpha}_1^*) + (I_k(x(\alpha_1^*, \alpha_k^*)) - I_k(x(\check{\alpha}_1^*, \check{\alpha}_k^*)))(\alpha_k^* - \check{\alpha}_k^*) \\ &= (\alpha_1^* I_1(x(\alpha_1^*, \alpha_k^*)) + \alpha_k^* I_k(x(\alpha_1^*, \alpha_k^*))) - (\alpha_1^* I_1(x(\check{\alpha}_1^*, \check{\alpha}_k^*)) + \alpha_k^* I_k(x(\check{\alpha}_1^*, \check{\alpha}_k^*))) \\ &\quad + (\check{\alpha}_1^* I_1(x(\check{\alpha}_1^*, \check{\alpha}_k^*)) + \check{\alpha}_k^* I_k(x(\check{\alpha}_1^*, \check{\alpha}_k^*))) - (\check{\alpha}_1^* I_1(x(\alpha_1^*, \alpha_k^*)) + \check{\alpha}_k^* I_k(x(\alpha_1^*, \alpha_k^*))). \quad \text{(B.2)} \end{aligned}$$

By the strict convexity of $I_1(x)$ and $I_i(x)$, $x(\alpha_1, \alpha_i)$ uniquely minimizes $\inf_x(\alpha_1 I_1(x) + \alpha_i I_i(x))$. Therefore (B.2) is strictly less than zero, and hence the rate functions in Γ are strictly concave in (α_1, α_i). By a similar proof, the rate functions for $i \in \mathcal{S}_w$ are also strictly concave in (α_1, α_i).

Since the minimum of strictly concave functions is strictly concave, the objective in (B.1) is strictly concave. Since the objective is strictly concave and the feasible set is convex, the problem in (B.1) has a unique solution. Let the rate achieved at this solution be z_b^*. If $z_b^* \neq z^* = \check{z}^*$, then $\alpha_{\mathcal{S}_b}^*$ cannot be optimal to the original Problem Q^*. Therefore $z_b^* = z^* = \check{z}^*$, but z_b^* is the optimal value at the unique optimal solution to (B.1). Therefore $\boldsymbol{\alpha}^* = \check{\boldsymbol{\alpha}}^*$, and we have a contradiction. □

C. Proofs for Section 6

Proof of Proposition 3. We only prove that $\widehat{\mathcal{C}}_F^i \to \mathcal{C}_F^i$ wp1 as $m \to \infty$. The proofs for the other parts of the proposition follow in a similar fashion.

By Assumption 2,$\hat{G}_{ij}(m) \to g_{ij}$ wp1 for all $i \leq r$ and $j \leq s$. We know that $g_{ij} < \gamma_j$ for each $j \in \mathcal{C}_F^i$. Since $|\mathcal{C}_F^i| < \infty$, we conclude that for large enough m, $\hat{G}_{ij}(m) < \gamma_j$ uniformly in $j \in \mathcal{C}_F^i$ wp1, and hence the assertion holds. $\square$

We omit the proof of Lemma 1 since it follows closely along the lines of the proofs presented in Glynn and Juneja (2004).

Proof of Lemma 2. We prove that the theorem holds in two steps. We first show $\alpha_1 \hat{I}_1^m(\hat{x}_m(\alpha_1, \alpha_i) + \alpha_i \hat{I}_i^m(\hat{x}_m(\alpha_1, \alpha_i))$ converges uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1 for all $i \in \Gamma \cup \mathcal{S}_w$, where $\hat{x}_m(\alpha_1, \alpha_i) = \arg\inf_x(\alpha_1 \hat{I}_1^m(x) + \alpha_i \hat{I}_i^m(x))$. Next we show $\alpha_i \sum_{j \in \mathcal{C}_I^i} \hat{J}_{ij}^m(\gamma_j), i \in \mathcal{S}_b \cup \mathcal{S}_w$ and $\alpha_1 \hat{J}_{1j}^m(\gamma_j), j \in \mathcal{C}_F^1$ converge uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1. These assertions, together with the observation that we search only in the set $\{\boldsymbol{\alpha} : \sum_{i=1}^r \alpha_i = 1, \alpha_i > 0\}$, and hence $I_i(x(\alpha_1, \alpha_i)) > \delta > 0$, which implies for large enough m, $\hat{I}_i^m(\hat{x}_m(\alpha_1, \alpha_i)) > \delta$, proves the theorem.

By Lemma 1, $\hat{I}_i^m(x) \to I_i(x)$ uniformly in x on $[h_\ell - \epsilon, h_u + \epsilon]$ wp1 for some $\epsilon > 0$. By Glynn and Juneja (2004), $\hat{x}_m(\alpha_1, \alpha_i) \to x(\alpha_1, \alpha_i)$ wp1, where $x(\alpha_1, \alpha_i) = \arg\inf_x(\alpha_1 I_1(x) + \alpha_i I_i(x)) \in [h_\ell, h_u]$. Therefore for m large enough and for all feasible α_1, α_i, we have $\hat{x}_m(\alpha_1, \alpha_i) \in [h_\ell - \epsilon/2, h_u + \epsilon/2]$ wp1 for all $i \in \{1\} \cup \Gamma \cup \mathcal{S}_w$. It then follows that $\alpha_1 \hat{I}_1^m(\hat{x}_m(\alpha_1, \alpha_i)) + \alpha_i \hat{I}_i^m(\hat{x}_m(\alpha_1, \alpha_i))$ converges uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1, for all $i \in \Gamma \cup \mathcal{S}_w$.

Under Assumption 3, it follows from analogous arguments to those in Glynn and Juneja (2004) that $\hat{J}_{ij}^m(\gamma_j) \to J_{ij}(\gamma_j)$ as $m \to \infty$ wp1, for all $i \in \mathcal{S}_b \cup \mathcal{S}_w$ and $j \leq s$. Therefore the terms $\alpha_i \sum_{j \in \mathcal{C}_I^i} \hat{J}_{ij}^m(\gamma_j)$ converge uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1. Likewise, for all $j \in \mathcal{C}_F^1$, $\alpha_1 \hat{J}_{1j}^m(\gamma_j)$ converges uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1. $\square$

Proof of Theorem 5. As argued previously, $f_1(\boldsymbol{\alpha})$ and $f_2(\boldsymbol{\alpha})$ are continuous functions of $\boldsymbol{\alpha}$ on a compact set. Further, the solutions $f_1(\boldsymbol{\alpha}) = 0$ and $f_2(\boldsymbol{\alpha}) = 0$ exist. If we replace each rate function in Problem Q with estimated rate functions, these new problems remain continuous, concave maximization problems on a compact set, which attain their maxima. Therefore the systems $\hat{F}_1^m(\boldsymbol{\alpha}) = 0$ and $\hat{F}_2^m(\boldsymbol{\alpha}) = 0$ have a solution for large enough m wp1. By Lemma 2 we also have that $\hat{F}_1^m(\boldsymbol{\alpha}) \to f_1(\boldsymbol{\alpha})$ and $\hat{F}_2^m(\boldsymbol{\alpha}) \to f_2(\boldsymbol{\alpha})$ uniformly in $\boldsymbol{\alpha}$ as $m \to \infty$ wp1. We have thus satisfied all the requirements for convergence of the sample-path solution $\hat{\boldsymbol{\alpha}}^*(m)$ to its true counterpart $\boldsymbol{\alpha}^*$ as $m \to \infty$ wp1 (see Pasupathy and Kim, 2011, Theorem 5.7). $\square$

References

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