Online Supplement for "'Survivability in Hierarchical Telecommunications Networks under Dual Homing"'

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In this online supplement, we provide the proofs for the theorems of Section 3.

We define the following vectors. For an edge $e \in E$, let χ_e be a vector of size $|E|$ where the entry corresponding to edge e is equal to 1 and the remaining entries are 0. Similarly, for $i \in V \setminus \{0\}$ and distinct nodes $j, k \in V \setminus \{i\}$, we define γ_{ijk} to be a vector of size $|A|$ where the entries corresponding to arcs (i, j) and (i, k) are equal to 1 and the other entries are equal to 0.)

Proof of Theorem 5 Suppose that all solutions $(x, y) \in \mathcal{P}$ satisfy $ax + by = \beta$. Let $e' \in E$ and consider the solutions $(\sum_{e \in E} \chi_e, 0)$ and $(\sum_{e \in E \setminus \{e'\}} \chi_e, 0)$. Both solutions are in $\mathcal{P}$. So we have $a_{e'} = 0$.

Let $(i, j) \in A$ with $j \neq 0$. Consider the solutions $(\sum_{e \in E} \chi_e, 0)$ and $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{ij0})$. As both solutions are in $\mathcal{P}$ and $a = 0$, we have $b_{ij} + b_{i0} = 0$.

Let $i \in V \setminus \{0\}$ and $j, k \in V \setminus \{i, 0\}$ be distinct nodes, and consider solutions $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{ik0})$ and $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{ikj})$. These solutions are in $\mathcal{P}$ implying that $b_{ij} = b_{i0}$. As we also have $b_{ij} + b_{i0} = 0$, we can conclude that $b_{ij} = b_{i0} = 0$.

Now as both a and b are zero, β is also zero. Hence no inequality is satisfied at equality by all points of $\mathcal{P}$. $\square$

Proof of Theorem 6

i. Let $e \in E$ and $\mathcal{F} = \{(x,y) \in \mathcal{P} : x_e = 0\}$. Suppose that all solutions $(x,y) \in \mathcal{F}$ satisfy $ax + by = \beta$. Let $e^{'} \in E \setminus \{e\}$ and consider the solutions $(\sum_{\hat{e} \in E \setminus \{e\}} \chi_{\hat{e}}, 0)$ and $(\sum_{\hat{e} \in E \setminus \{e,e'\}} \chi_{\hat{e}}, 0)$. Both solutions are in $\mathcal{F}$, so $a_{e^{'}} = 0$.

   Let $i \in V \setminus \{0\}$ and $j,k,l \in V \setminus \{i\}$ be distinct nodes. As the solutions $(\sum_{\hat{e} \in E \setminus (\delta(i) \cup \{e\})} \chi_{\hat{e}}, \gamma_{ijk})$ and $(\sum_{\hat{e} \in E \setminus (\delta(i) \cup \{e\})} \chi_{\hat{e}}, \gamma_{ijl})$ are both in $\mathcal{F}$, we have $b_{ik} = b_{il}$. Hence $b_{ij} = \theta_i$ for some $\theta_i \in \mathbb{R}$ for all $(i,j) \in A$. Now consider the solutions $(\sum_{\hat{e} \in E \setminus \{e\}} \chi_{\hat{e}}, 0)$ and $(\sum_{\hat{e} \in E \setminus (\delta(i) \cup \{e\})} \chi_{\hat{e}}, \gamma_{ijk})$. These solutions are both in $\mathcal{F}$ and $a_{\hat{e}} = 0$ for all $\hat{e} \in E \setminus \{e\}$. Hence, $b_{ij} + b_{ik} = 0$, and thus $2\theta_i = 0$ implying $\theta_i = 0$. Finally, since the solution $(0,0)$ is in $\mathcal{F}$, $\beta = 0$.

   Since all the coefficients except a_e are zero, $ax + by = \beta$ is a multiple of $x_e = 0$. Hence $x_e \geq 0$ defines a facet of $\mathcal{P}$.

ii. Let $(i,j) \in A$ and $\mathcal{F} = \{(x,y) \in \mathcal{P} : y_{ij} = 0\}$. Suppose that all solutions $(x,y) \in \mathcal{F}$ satisfy $ax + by = \beta$. Let $e^{'} \in E$ and consider the solutions $(\sum_{e \in E} \chi_e, 0)$ and $(\sum_{e \in E \setminus \{e'\}} \chi_e, 0)$. Since both solutions are in $\mathcal{F}$, we have $a_{e^{'}} = 0$.

   Let $u \in V \setminus \{0\}$. If $u \neq i$, then let $v,k,l \in V \setminus \{u\}$ be distinct nodes. If $u = i$, then let $v,k,l \in V \setminus \{i,j\}$ be distinct nodes. The solutions $(\sum_{e \in E \setminus \delta(u)} \chi_e, \gamma_{uvk})$ and $(\sum_{e \in E \setminus \delta(u)} \chi_e, \gamma_{uvl})$ are in $\mathcal{F}$. So we can conclude that $b_{uv} = \theta_u$ for some $\theta_u \in \mathbb{R}$ for all $(u,v) \in A \setminus \{(i,j)\}$. Considering the solutions $(\sum_{e \in E} \chi_e, 0)$ and $(\sum_{e \in E \setminus \delta(u)} \chi_e, \gamma_{uvk})$, it can be seen that $2\theta_u = 0$. Moreover, $\beta = 0$ since $(0,0)$ is in $\mathcal{F}$.

   Since all the coefficients except b_{ij} are zero, $ax + by = \beta$ is a multiple of $y_{ij} = 0$ and $y_{ij} \geq 0$ is facet defining for $\mathcal{P}$.

iii. Let $(i,j) \in A$ such that $j \neq 0$, $e = \{i,j\}$, and $\mathcal{F} = \{(x,y) \in \mathcal{P} : 2x_e + 2y_{ij} + \sum_{k \in V \setminus \{j\}} y_{jk} = 2\}$. Suppose that every solution (x,y) in $\mathcal{F}$ also satisfies $ax + by = \beta$.

   Let $e^{'} \in E \setminus \{e\}$. As $(\sum_{\hat{e} \in E} \chi_{\hat{e}}, 0)$ and the solution $(\sum_{\hat{e} \in E \setminus \{e'\}} \chi_{\hat{e}}, 0)$ are both in $\mathcal{F}$, we have $a_{e^{'}} = 0$.

   Let $k \in V \setminus \{0,i,j\}$ and $u,v,w \in V \setminus \{k\}$ be distinct nodes. The solutions $(\sum_{\hat{e} \in E \setminus \delta(k)} \chi_{\hat{e}}, \gamma_{kuv})$ and $(\sum_{\hat{e} \in E \setminus \delta(k)} \chi_{\hat{e}}, \gamma_{kuw})$ are in $\mathcal{F}$. This implies that $b_{kl} = \theta_k$ for some $\theta_k \in \mathbb{R}$ for all $l \in V \setminus \{k\}$. As both $(\sum_{\hat{e} \in E \setminus \delta(k)} \chi_{\hat{e}}, \gamma_{kuv})$ and $(\sum_{\hat{e} \in E} \chi_{\hat{e}}, 0)$ are in $\mathcal{F}$ and $a_{\hat{e}} = 0$ for all $\hat{e} \in E \setminus \{e\}$, we have $2\theta_k = 0$. Therefore $b_{kl} = 0$ for every $(k,l) \in A$ such that $k \in V \setminus \{0,i,j\}$.

Let $u,v,w \in V \setminus \{i,j\}$ be distinct nodes. We define $\overline{x} = \sum_{\hat{e} \in E \setminus (\delta(i) \cup \delta(j))} \chi_{\hat{e}}$. Consider the solutions $(\overline{x}, \gamma_{iuv} + \gamma_{juv})$ and $(\overline{x}, \gamma_{iuw} + \gamma_{juv})$. As these solutions are in $\mathcal{F}$, we conclude that $b_{il} = \theta_i$ for some $\theta_i \in \mathbb{R}$ for every $l \in V \setminus \{i,j\}$. In addition, $(\overline{x}, \gamma_{iuv} + \gamma_{juv})$ and $(\sum_{\hat{e} \in E \setminus \delta(j)} \chi_{\hat{e}}, \gamma_{juv})$ are in $\mathcal{F}$. As $a_{\hat{e}} = 0$ for all $\hat{e} \in E \setminus \{e\}$, we have $b_{iu} + b_{iv} = 2\theta_i = 0$. Therefore $b_{il} = 0$ for every $(i,l) \in A$ for all $l \in V \setminus \{i,j\}$.

Next consider the solutions $(\sum_{\hat{e} \in E \setminus \delta(j)} \chi_{\hat{e}}, \gamma_{juv})$ and $(\sum_{\hat{e} \in E \setminus \delta(j)} \chi_{\hat{e}}, \gamma_{juw})$ where $u,v,w \in V \setminus \{j\}$ are distinct nodes. Both solutions are in $\mathcal{F}$ implying that $b_{jl} = \sigma$ for some $\sigma \in \mathbb{R}$ for every $l \in V \setminus \{j\}$. Note that $(\sum_{\hat{e} \in E \setminus \delta(i)} \chi_{\hat{e}}, \gamma_{ij0})$ is also in $\mathcal{F}$. So we can say that $b_{ij} = 2\sigma$ since $a_{\hat{e}} = 0$ for all $\hat{e} \in E \setminus \{e\}$, and $b_{i0} = 0$. Finally, consider the solution $(\sum_{\hat{e} \in E} \chi_{\hat{e}}, 0) \in \mathcal{F}$. As $a_{\hat{e}} = 0$ for all $\hat{e} \in E \setminus \{e\}$ and $b_{i0} = 0$, it follows that $a_e = 2\sigma$. Also $\beta = 2\sigma$.

Hence $ax + by = \beta$ is a σ multiple of $2x_e + 2y_{ij} + \sum_{k \in V \setminus \{j\}} y_{jk} = 2$ and $\mathcal{F}$ is a facet of $\mathcal{P}$.

iv. Similar to the proof of part iii. □

Proof of Theorem 7 Let $S \subseteq V \setminus \{0\}$ such that $S \neq \emptyset$, $i \in S$, and $\mathcal{F} = \{(x,y) \in \mathcal{P} : x(\delta(S)) + \sum_{j \in V \setminus S} y_{ij} = 2\}$. Suppose that every solution (x,y) in $\mathcal{F}$ also satisfies $ax + by = \beta$.

Suppose that $|V \setminus S| \leq 2$. Since a terminal node is assigned to two hubs, the graph does not contain multiple edges, and the backbone network is to be 2-edge connected, there must be at least three hubs on the backbone network. So there is at least one hub in set S. Then inequality (36) is dominated by the valid inequality $x(\delta(S)) \geq 2$.

If $|S| = 2$, then inequality (36) is $x(\delta(S)) \geq 2t_i + y_{ij}$ with the t_i variable and is dominated by the in-cut inequality (20).

Finally, suppose that $|S| = 3$ and $S = \{i,k,l\}$. Let $(x,y) \in X$ with $x(\delta(S)) + \sum_{j \in V \setminus S} y_{ij} = 2$. If i is a hub node, then $y_{ik} = y_{il} = 0$. Now suppose that node i is not a hub, and note that either $x(\delta(S)) = 2$ or $x(\delta(S)) = 0$. If $x(\delta(S)) = 2$, then $\sum_{j \in V \setminus S} y_{ij} = 0$ and i must be assigned to nodes k and l. So $y_{ik} = y_{il} = 1$. Finally, if $x(\delta(S)) = 0$, then $\sum_{j \in V \setminus S} y_{ij} = 2$, and hence $\sum_{j \in S} y_{ij} = 0$ implying $y_{il} = y_{ik} = 0$. Hence all solutions $(x,y) \in X$ with $x(\delta(S)) + \sum_{j \in V \setminus S} y_{ij} = 2$ also satisfy $y_{ik} = y_{il}$ and therefore the cut inequality is not facet defining.

Now suppose that $|S| \geq 4$ and $|V \setminus S| \geq 3$. Notice that as G is complete, $G(S)$ and $G(V \setminus S)$ are both 2-edge connected.

Let e_1, e_2 be any two edges in $\delta(S)$ and $x = \sum_{e \in E(S) \cup E(V \setminus S)} \chi_e$. Then $(x + \chi_{e_1} + \chi_{e_2}, 0)$ is a solution in $\mathcal{F}$. Let $e' \in \delta(S) \setminus \{e_1, e_2\}$. As the solutions $(x + \chi_{e_1} + \chi_{e'}, 0)$ and $(x + \chi_{e_2} + \chi_{e'}, 0)$ are also in $\mathcal{F}$, we have $a_{e_1} = a_{e_2} = a_{e'}$. Therefore $a_e = \sigma$ for all $e \in \delta(S)$ for some $\sigma \in \mathbb{R}$.

Let $e' \in E(S)$ and e_1 and e_2 be two edges in $\delta(S)$ incident to the two endpoints of e', *i.e.*, $e_1 \cap e_2 \cap e' = \emptyset$. Consider the solutions $(x + \chi_{e_1} + \chi_{e_2}, 0)$ and $(x + \chi_{e_1} + \chi_{e_2} - \chi_{e'}, 0)$. As they are both in $\mathcal{F}$, we have $a_{e'} = 0$. We can show similarly that $a_{e'} = 0$ for all $e' \in E(V \setminus S)$.

Let $j \in V \setminus \{i, 0\}$ and e_1, e_2 be two edges in $\delta(S) \setminus \delta(j)$ with different endpoints in S if $j \in S$ and with different endpoints in $V \setminus S$ if $j \in V \setminus S$. We define $\overline{x} = x - \sum_{e \in \delta(j)} x_e \chi_e$. Let $u, v, w \in V \setminus \{j\}$. Both solutions $(\overline{x} + \chi_{e_1} + \chi_{e_2}, \gamma_{juv})$ and $(\overline{x} + \chi_{e_1} + \chi_{e_2}, \gamma_{juw})$ are in $\mathcal{F}$. So we have $b_{jk} = \theta_j$ for some $\theta_j \in \mathbb{R}$ for every $k \in V \setminus \{j\}$. Moreover, $(x + \chi_{e_1} + \chi_{e_2}, 0)$ is also in $\mathcal{F}$. As $a_e = 0$ for all $e \in E(S) \cup E(V \setminus S)$, we have $2\theta_j = 0$. Therefore $b_{jk} = 0$ for every $(j, k) \in A$ such that $j \neq i$.

Let e_1, e_2 be two edges in $\delta(S) \setminus \delta(i)$ with different endpoints in S. Let $\hat{x} = x - \sum_{e \in \delta(i)} x_e \chi_e$. Let $u, v, w \in S \setminus \{i\}$. Consider the solution $(\hat{x} + \chi_{e_1} + \chi_{e_2}, \gamma_{iuv})$, which is in $\mathcal{F}$. Observe that, $(\hat{x} + \chi_{e_1} + \chi_{e_2}, \gamma_{iuw})$ is also in $\mathcal{F}$. So we have $b_{ik} = \theta$ for some $\theta \in \mathbb{R}$ for every $k \in S \setminus \{i\}$. Besides, we have $(x + \chi_{e_1} + \chi_{e_2}, 0)$ in $\mathcal{F}$. As $a_e = 0$ for all $e \in E(S) \cup E(V \setminus S)$, we have $2\theta = 0$. So we can conclude that $b_{ik} = 0$ for all $k \in S \setminus \{i\}$.

Let $e_1, e_2 \in \delta(S)$ and $u, v, w \in V \setminus S$. We define $x' = \sum_{e \in E(V \setminus S)} \chi_e$ and $y' = \sum_{j \in S} \gamma_{juv}$. Now we can see that the solutions (x', y') and $(x', y' - \gamma_{iuv} + \gamma_{iuw})$ are both in $\mathcal{F}$. So we have $b_{ik} = \xi$ for some $\xi \in \mathbb{R}$ for all $k \in V \setminus S$. Moreover, $(x + \chi_{e_1} + \chi_{e_2}, 0)$ is also in $\mathcal{F}$. As $a_e = 0$ for all $e \in E(S)$, $b_{jk} = 0$ for all $(j, k) \in A$ with $j \neq i$, and $a_e = \sigma$ for all $e \in \delta(S)$ we have $2\xi = 2\sigma$. Therefore $b_{ik} = \sigma$ for all $k \in V \setminus S$.

Finally, as the solution $(x + \chi_{e_1} + \chi_{e_2}, 0)$ is in $\mathcal{F}$, $\beta = 2\sigma$. Hence $ax + by = \beta$ is a σ multiple of $x(\delta(S)) + \sum_{j \in V \setminus S} y_{ij} = 2$ and $\mathcal{F}$ is a facet of $\mathcal{P}$.

The proof for the case with $|S| = 1$ can be done in a similar way. □

Proof of Theorem 8

i. Let $S \subseteq V \setminus \{0\}$ such that $S \neq \emptyset$, $i \in S$, $j \in S \setminus \{i\}$, and $\mathcal{F} = \{(x, y) \in \mathcal{P} : x(\delta(S)) + \sum_{k \in V \setminus \{i,j\}} y_{ik} - y_{ij} = 2\}$. Suppose that $|S| \geq 3$, $|V \setminus S| \geq 3$, and every solution (x, y) in $\mathcal{F}$ also satisfies $ax + by = \beta$.

We can show that $a_e = \sigma$ for some $\sigma \in \mathbb{R}$ for every $e \in \delta(S)$, $a_e = 0$ for every $e \in E \setminus \delta(S)$, $b_{uv} = 0$ for every $(u, v) \in A$ such that $u \neq i$, and $b_{ik} = \sigma$ for every $(i, k) \in A$ such that $k \in V \setminus S$ using the arguments of the proof of Theorem 7.

Let $u \in V\setminus S$, $k \in S\setminus\{i, j\}$, and $e_1, e_2 \in \delta(S)\setminus\delta(i)$. We define $x' = \sum_{e\in E(V\setminus S)\cup E(S)\setminus\delta(i)} \chi_e + \chi_{e_1} + \chi_{e_2}$. Then (x', γ_{iju}) and (x', γ_{ijk}) are both solutions in $\mathcal{F}$, showing that $b_{ik} = \sigma$ for all $k \in V \setminus \{i, j\}$.

Let $k \in S \setminus \{i, j\}$ and u and v be distinct nodes in $V \setminus S$. Consider the solutions (x', γ_{ijk}) and $(\sum_{e\in E(V\setminus S)} \chi_e, \sum_{l\in S} \gamma_{luv})$. Clearly both solutions are in $\mathcal{F}$. Since $a_e = \sigma$ for $e \in \delta(S)$, $b_{ik} = \sigma$ for $k \in V \setminus \{i, j\}$ and the other coefficients are all zero, we have $b_{ij} = -\sigma$.

Hence $ax + by = \beta$ is a multiple of $x(\delta(S)) + \sum_{k\in V\setminus\{i,j\}} y_{ik} - y_{ij} = 2$ and $\mathcal{F}$ is a facet of $\mathcal{P}$.

ii. Let $S \subseteq V \setminus \{0\}$ such that $S \neq \emptyset$, $i \in S$, and $j \in V \setminus S$ with $|S| \geq 4$ and $|V \setminus S| \geq 3$ and $\mathcal{F} = \{(x, y) \in \mathcal{P} : x(\delta(S)) + 2\sum_{l\in V\setminus(S\cup\{j\}} y_{il} = 2\}$. Suppose that every solution (x, y) in $\mathcal{F}$ also satisfies $ax + by = \beta$.

As in the proof of Theorem 7, we can show that $a_e = \sigma$ for some $\sigma \in \mathbb{R}$ for all $e \in \delta(S)$, $a_e = 0$ for all $e \in E \setminus \delta(S)$, $b_{uv} = 0$ for all $(u, v) \in A$ such that $u \neq i$, and $b_{il} = 0$ for all $l \in S \setminus \{i\}$.

Let e_1, e_2 be two edges in $\delta(S) \setminus \delta(i)$ and $k_1, k_2 \in S \setminus \{i\}$. Consider the solution $(x, \gamma_{ik_1k_2})$ where $x = \sum_{e\in E(S)\cup E(V\setminus S)\setminus\delta(i)} \chi_e + \chi_{e_1} + \chi_{e_2}$. This solution and (x, γ_{ijk_1}) are both in $\mathcal{F}$. So we have $b_{ij} = b_{ik_2} = 0$.

Finally, let $k \in V\setminus(S\cup\{j\})$ and $k' \in S\setminus\{i\}$. The solutions $(x, \gamma_{ijk'})$ and $(\sum_{e\in E(V\setminus S)} \chi_e, \sum_{l\in S} \gamma_{ljk})$ are in $\mathcal{F}$. As $a_e = 0$ for $e \in E(S)$, $a_e = \sigma$ for $e \in \delta(S)$, $b_{uv} = 0$ for $(u, v) \in A$ with $u \neq i$ and $b_{ik'} = 0$, we can see that $b_{ik} = 2\sigma$ for every $k \in V \setminus (S \cup \{j\})$.

Therefore $ax + by = \beta$ is a multiple of $x(\delta(S)) + 2\sum_{l\in V\setminus(S\cup\{j\})} y_{il} = 2$ and $\mathcal{F}$ is a facet of $\mathcal{P}$. □

Proof of Theorem 9 Let $(i, j) \in A$ and $\mathcal{F} = \{(x, y) \in \mathcal{P} : y_{ij} - \sum_{k\in V\setminus\{i,j\}} y_{ik} = 0\}$. Suppose that every solution (x, y) in $\mathcal{F}$ also satisfies $ax + by = \beta$.

Let $e \in E$. Consider the solutions $(\sum_{\hat{e}\in E} \chi_{\hat{e}}, 0)$ and $(\sum_{\hat{e}\in E\setminus\{e\}} \chi_{\hat{e}}, 0)$. As they are both in $\mathcal{F}$, we have $a_e = 0$.

Let $k \in V \setminus \{0, i\}$ and $u, v, w \in V \setminus \{k\}$ be distinct nodes. Considering the solutions $(\sum_{e \in E \setminus \delta(k)} \chi_e, \gamma_{kuv})$ and $(\sum_{e \in E \setminus \delta(k)} \chi_e, \gamma_{kuw})$, which are both in $\mathcal{F}$, we obtain $b_{kl} = \theta_k$ for some $\theta_k \in \mathbb{R}$ for every $(k, l) \in A$. In addition, $(\sum_{e \in E \setminus \delta(k)} \chi_e, \gamma_{kuv})$ and $(\sum_{e \in E} \chi_e, 0)$ are both in $\mathcal{F}$. Since $a_e = 0$ for all $e \in E$, we have $2\theta_k = 0$. Hence $b_{kl} = 0$ for every $(k, l) \in A$ such that $k \in V \setminus \{0, i\}$.

Let $u, v \in V \setminus \{i, j\}$ be distinct nodes. The solutions $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{iju})$ and $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{ijv})$ are both in $\mathcal{F}$. This shows that $b_{il} = \sigma$ for some $\sigma \in \mathbb{R}$ for every $l \in V \setminus \{i, j\}$. Finally, we consider the solutions $(\sum_{e \in E \setminus \delta(i)} \chi_e, \gamma_{iju})$ and $(\sum_{e \in E} \chi_e, 0)$. Both solutions are in $\mathcal{F}$ and note that we know $a_e = 0$ for all $e \in E$ and $b_{iu} = \sigma$. Together these imply that $b_{ij} = -\sigma$. It is easy to see that $\beta = 0$.

Now we can conclude that $ax + by = \beta$ is a multiple of $y_{ij} - \sum_{k \in V \setminus \{i,j\}} y_{ik} = 0$ and $\mathcal{F}$ is a facet of $\mathcal{P}$. $\square$

Proof of Theorem 10 Let $\mathcal{F} = \{(x, y) \in \mathcal{P} : x(\delta(V_0, \ldots, V_p) \setminus F) + \frac{\sum_{l=1}^{p} \sum_{j \in V \setminus V_l} y_{i_l j}}{2} = p - k\}$. Assume that every solution $(x, y) \in \mathcal{F}$ also satisfies $ax + by = \beta$. Without loss of generality, assume that $|F \cap \delta(i_l)| = 1$ for $l = 1, \ldots, 2k + 1$. Clearly, $p \geq 2k + 1$.

Let $E_1 = \cup_{l=0}^{p} E(V_l) \cup \{i_1, i_2\} \cup \{i_2, i_{2k+2}\} \cup_{l=2k+2}^{p-1} \{i_l, i_{l+1}\} \cup \{i_p, i_3\} \cup_{l=2}^{k} \{i_{2l}, i_{2l+1}\} \cup F$ and $x = \sum_{e \in E_1} \chi_e$. The edges of the set $E_1 \setminus \cup_{l=0}^{p} E(V_l)$ are given in Figure 1. Here, the empty circles represent the sets of the partition, the big black circles represent the fixed nodes, and the small black circles represent the remaining nodes. The edges incident to V_0 are all in F. Now, it is easy to verify that the solution $(x, 0)$ is in $\mathcal{F}$. The solution $(x - \chi_e, 0)$ is also in $\mathcal{F}$ for every $e \in E(V_l)$ for $l = 0, \ldots, p$ since $G(V_l)$ is 3-edge connected. Therefore $a_e = 0$ for all $e \in E(V_l)$ for $l = 0, \ldots, p$. Similarly, $(x - \chi_e, 0)$ is in $\mathcal{F}$ for $e \in \delta(i_2) \cap F$. By symmetry, we can show that $a_e = 0$ for every edge $e \in F$.

Let $l \in \{1, \ldots, p\}$, $j \in V_l \setminus \{i_l\}$, and $u, v, w \in V \setminus \{j\}$ be distinct nodes. Since the solutions $(x - \sum_{e \in \delta(j)} x_e \chi_e, \gamma_{juv})$ and $(x - \sum_{e \in \delta(j)} x_e \chi_e, \gamma_{juw})$ are both in $\mathcal{F}$ we have $b_{ju} = \xi_j$ for some $\xi_j \in \mathbb{R}$ for $j \in V_l \setminus \{i_l\}$ and $u \in V \setminus \{j\}$. Comparing the solution $(x - \sum_{e \in \delta(j)} x_e \chi_e, \gamma_{juv})$ with $(x, 0)$ yields $\xi_j = 0$ for $j \in V_l \setminus \{i_l\}$ as $a_e = 0$ for all $e \in E(V_l)$. Hence, $b_{ju} = 0$ for all $j \in V_l \setminus \{i_l\}$, $u \in V \setminus \{j\}$, and $l \in \{1, \ldots, p\}$. Now, let $j \in V_0 \setminus \{0\}$. If $F \cap \delta(j) = \emptyset$, then we can use the same argument to prove that $b_{ju} = 0$ for all $u \in V \setminus \{j\}$. Otherwise, let $\{j, s\}$ be the edge in $F \cap \delta(j)$. We let node s play the role of node i_2 and use the same solutions and the fact that $a_e =$ for all $e \in F$ to show that $b_{ju} = 0$ for all $u \in V \setminus \{j\}$.

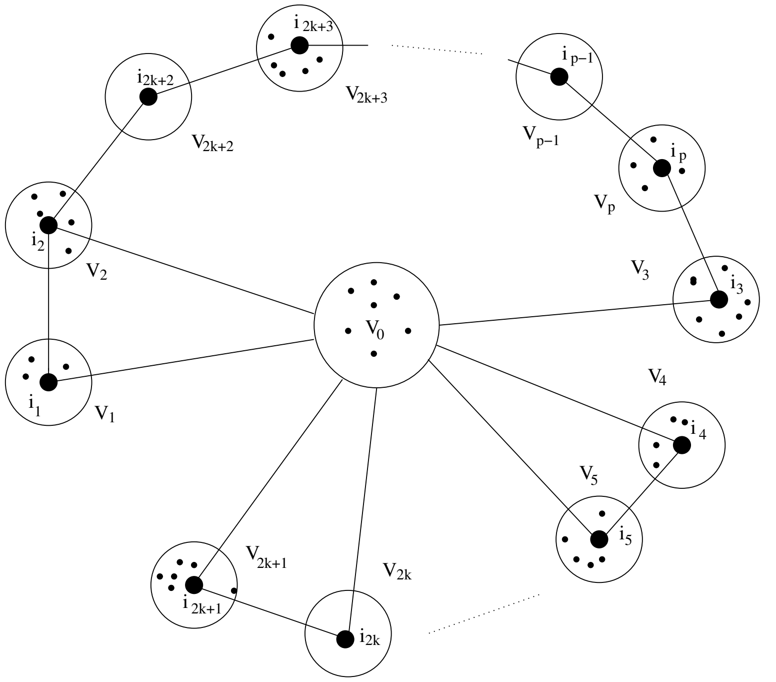

Figure 1: Backbone edges of the solution $(x, 0)$ other than those in $\cup_{l=0}^{p} E(V_l)$

Let $e \in \delta(V_1) \setminus (F \cup \{\{i_1, i_2\}\})$. Observe that the solution $(x - \chi_{i_1 i_2} + \chi_e, 0)$ is in $\mathcal{F}$. So $a_e = \alpha_1$ for all $e \in \delta(V_1) \setminus (F \cup \{\{i_1, i_2\}\})$ and for some $\alpha_1 \in \mathbb{R}$. By symmetry, we can show that $a_{\{i_1,i_2\}} = \alpha_1$ and that $a_e = \alpha_l$ for $e \in \delta(V_l) \setminus F$ for $l = 1, \ldots, 2k+1$.

Now as $(\delta(V_j) \setminus F) \cap (\delta(V_l) \setminus F) \neq \emptyset$ for any j and l such that $1 \leq j < l \leq 2k+1$, we have $\alpha_j = \alpha_l$. Hence we can conclude that $a_e = \alpha$ for some $\alpha \in \mathbb{R}$ for $e \in \delta(V_0, \ldots, V_{2k+1}) \setminus F$.

Let $l \in \{0, \ldots, 2k+1\} \setminus \{3\}$ and $e \in (\delta(V_{2k+2}) \setminus E_1) \cap \delta(V_l)$. The solution $(x - \chi_{i_2 i_{2k+2}} + \chi_e, 0)$ is in $\mathcal{F}$. Hence $a_e = \alpha$. Now by changing the roles of V_3 and V_1, we can also show that $a_e = \alpha$ for all $e \in [V_{2k+2}, V_3]$. By symmetry, we can conclude that $a_e = \alpha$ for all $e \in [V_i, V_j]$ for $i = 0, \ldots, 2k+1$ and $j = 2k+2, \ldots, p$.

If $|V_2| \geq 2$, then since $G(V_2)$ is 3-edge connected and G has no multiple edges, $|V_2| \geq 4$. Let u and v be distinct nodes in $V_2 \setminus \{i_2\}$. Let e_1 be the edge between nodes i_1 and u and e_2 be the edge between nodes v and i_{2k+2}. Consider the solutions $(x, 0)$ and $(x - \sum_{e \in \delta(i_2)} x_e \chi_e + \chi_{e_1} + \chi_{e_2}, \gamma_{i_2 uv})$. These solutions are in $\mathcal{F}$. Now as $a_e = 0$ for $e \in E(V_2) \cup F$ and $a_e = \alpha$ for all $e \in [V_1, V_2]$ and $e \in [V_2, V_2 k + 2]$, we have $b_{i_2 u} + b_{i_2 v} = 0$. Let $w \in V_2 \setminus \{i_2, u, v\}$. As both solutions $(x - \sum_{e \in \delta(i_2)} x_e \chi_e + \chi_{e_1} + \chi_{e_2}, \gamma_{i_2 uw})$ and $(x - \sum_{e \in \delta(i_2)} x_e \chi_e + \chi_{e_1} + \chi_{e_2}, \gamma_{i_2 vw})$ are also in $\mathcal{F}$, we have that $b_{i_2 u} = b_{i_2 v} = 0$. By symmetry, we can show that $b_{i_l u} = 0$ for all $u \in V_l \setminus \{i_l\}$ and

$l \in \{1, \ldots, p\}$.

Let $u, v, w \in V \setminus V_1$ be distinct nodes and define $\overline{x} = x - \sum_{e \in \delta(V_1)} x_e \chi_e - \sum_{e \in E(V_1)} \chi_e$ and $\overline{y} = \sum_{k \in V_1 \setminus \{i_1\}} \gamma_{kuv}$. Clearly, $(\overline{x}, \overline{y} + \gamma_{i_1 uv})$ and $(\overline{x}, \overline{y} + \gamma_{i_1 uw})$ are both solutions in $\mathcal{F}$. Therefore, we have $b_{i_1 u} = \theta_1$ for $u \in V \setminus V_1$. Now let $e' \in F \cap \delta(i_1)$ and note that $(\overline{x} + \chi_{i_1 i_2} + \chi_{e'}, \overline{y})$ is also in $\mathcal{F}$. As $a_{e'} = 0$, $a_{i_1 i_2} = \alpha$, and $b_{i_1 u} = b_{i_1 v} = \theta_1$ we obtain $\theta_1 = \alpha/2$. By symmetry, we can extend this results to $b_{i_l u} = \alpha/2$ for $l = 1, \ldots, 2k+1$ and $u \in V \setminus V_l$.

Let $u, v, w \in V \setminus V_{2k+2}$ be distinct nodes. Let $x' = x - \sum_{e \in E(V_{2k+2})} \chi_e - \chi_{i_2 i_{2k+2}} - \chi_{i_{2k+2} i_{2k+3}} + \chi_{i_2 i_{2k+3}}$ and $y' = \sum_{k \in V_{2k+2} \setminus \{i_{2k+2}\}} \gamma_{kuv}$. It can be seen that the solutions $(x', y' + \gamma_{i_{2k+2} uv})$ and $(x', y' + \gamma_{i_{2k+2} uw})$ are both in $\mathcal{F}$. This implies that $b_{i_{2k+2} u} = \theta_{2k+2}$ for some $\theta_{2k+2} \in \mathbb{R}$ for all $u \in V \setminus \{i_{2k+2}\}$ and by symmetry, we can conclude that $b_{i_l m} = \theta_l$ for all $l = 2k+2, \ldots, p$ and $u \in V \setminus V_l$. Let $E_2 = (\cup_{l=0}^{2k+1} E(V_l)) \cup \{i_1, i_2\} \cup (\cup_{l=1}^{k} \{i_{2l}, i_{2l+1}\}) \cup F$ and $u, v \in V_0$. We can define $x'' = \sum_{e \in E_2} \chi_e$ and $y'' = \sum_{l=2k+2}^{p} \sum_{j \in V_l} \gamma_{juv}$. Observe that (x'', y'') is a solution in $\mathcal{F}$. We can also construct a solution in $\mathcal{F}$ as $(x'' - \chi_{i_2 i_3} + \chi_{e_1} + \chi_{e_2} + \sum_{e \in E(V_{2k+2})} \chi_e, y^{''} - \sum_{j \in V_{2k+2}} \gamma_{juv})$, where $e_1 \in [V_2, V_{2k+2}]$ and $e_2 \in [V_3, V_{2k+2}]$. Since $a_{i_2 i_3} = a_{e_1} = a_{e_2} = \alpha$, we can see that $\theta_{2k+2} = \alpha/2$. By symmetry, we have $\theta_l = \alpha/2$ for $l = 2k+2, \ldots, p$. Hence $b_{i_l u} = \alpha/2$ for $l = 1, \ldots, p$ and $u \in V \setminus V_l$.

Now considering $(x, 0)$ and $(x', y' + \gamma_{2k+2uv})$ together and using symmetry, we obtain $a_e = \alpha$ for $e \in [V_j, V_l]$ with j and l in $\{2k+2, \ldots, p\}$. Thus, $a_e = \alpha$ for all $e \in \delta(V_0, \ldots, V_p) \setminus F$.

It is easy to see that $\beta = \alpha(p-k)$ and $ax + by = \beta$ is an α multiple of $x(\delta(V_0, \ldots, V_p) \setminus F) + \frac{\sum_{l=1}^{p} \sum_{j \in V \setminus V_l} y_{i_l j}}{2} = p - k$. $\square$