

# Online Supplement To: Penalty-Based Algorithms for the Stochastic Obstacle Scene Problem

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## 1 Deciding Where to Disambiguate: Single Disk Case

Consider an instance of continuous-space SOSp with only one disk, as shown in Figure 1. In this instance, the starting point is  $S = (0, 0)$  and the termination point is  $T = (4, 0)$ . The disk is centered at  $(2, 0)$  with a radius of 1. The cost of disambiguation is taken as zero. If the mark associated with this disk is  $\rho = 0$ , then the optimal disambiguation point is  $C = (1, 0)$ . Consequently, a disambiguation algorithm that dictates disambiguating at  $C$  would traverse  $S, C, D, T$  with a total length of 4 units.

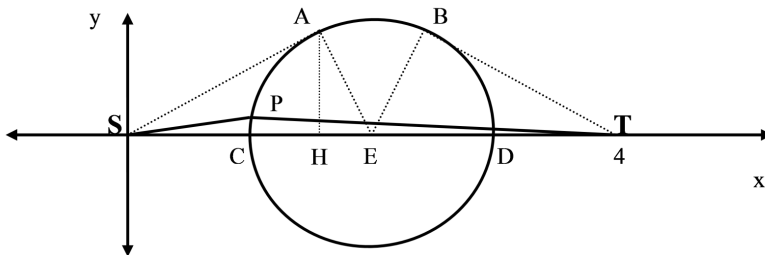


Figure 1: A continuous SOSp instance with a single disk.

On the other hand, if  $\rho = 1$ , then, the optimal disambiguation point is intersection point of the tangent line from  $S$  to the disk, which is denoted by  $A$ . The coordinates of  $A$  can be computed using geometry as follows: the secant line  $SCE$  is related to the tangent line  $SA$  by  $|SA|^2 = |SC||CE|$ . Thus, we get  $|SA| = \sqrt{3}$ . Since  $\widehat{SAE}$  is a right angle, we have  $|SA||AE| = |SE||AH|$ , which yields  $|AH| = \frac{\sqrt{3}}{2}$ . Furthermore, since  $SHA$  is a  $30-60-90^\circ$  triangle, we have  $|SH| = \frac{1}{2}$ . Thus,  $A = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \approx (1.5, 0.866)$ .

The optimal disambiguation point for an arbitrary  $\rho \in (0, 1)$  can be computed using geometry and some algebra as follows: Let  $E(P)$  be the expected length of the traversal when the disk is disambiguated at point

$P = (u, v)$ . Thus, the task is to determine the optimal disambiguation point  $P^*$  where

$$P^* = (u^*, v^*) = \arg \min_{\substack{u \in (1, 1.5) \\ v = \sqrt{1 - (u-2)^2}}} E(P = (u, v))$$

First, note that  $\widehat{CEA} = \widehat{BED} = 60^\circ$ , so  $\widehat{AEB} = 60^\circ$  and therefore  $|\text{arc}AB| = \frac{\pi}{3}$ . Moreover,  $|BT| = |SA| = \sqrt{3}$ , which yields  $|\text{arc}AB| + |BT| = \frac{\pi}{3} + \sqrt{3} \approx 2.7792$ . Thus, for  $\rho \in (0, 1)$ , for any point  $P$  on the arc segment  $\text{arc}CA$ , we have

$$\begin{aligned} E(P) &= |SP| + (1 - \rho)|PT| + (\rho)(|\text{arc}PA| + |\text{arc}AB| + |BT|) \\ &= |SP| + (1 - \rho)|PT| + (\rho)(|\text{arc}PA| + 2.7792) \\ &= \sqrt{u^2 + v^2} + (1 - \rho)(\sqrt{(4 - u)^2 + v^2}) + (\rho)\left(2.7792 + \left(\frac{\pi}{3} - \arctan\left(\frac{v}{2 - u}\right)\right)\right) \end{aligned} \quad (1)$$

Substituting  $v = \sqrt{1 - (u - 2)^2}$ , we get

$$\begin{aligned} &= \sqrt{u^2 + 1 - (u - 2)^2} + (1 - \rho)(\sqrt{(4 - u)^2 + 1 - (u - 2)^2}) \\ &\quad + (\rho)\left(2.7792 + \left(\frac{\pi}{3} - \arctan\left(\frac{\sqrt{1 - (u - 2)^2}}{2 - u}\right)\right)\right) \\ &= \sqrt{4u - 3} + (1 - \rho)(\sqrt{13 - 4u}) + (\rho)\left(2.7792 + \left(\frac{\pi}{3} - \arctan\left(\frac{\sqrt{-u^2 + 4u - 3}}{2 - u}\right)\right)\right) \end{aligned}$$

Thus, given a specific  $\rho \in (0, 1)$ ,  $E(P)$  is a function of a single variable,  $u$ , which we denote by  $E(u)$ . Observe that  $E(u)$  is a convex function, which indicates its unique minimum can be found by setting the following derivative to zero:

$$\frac{d}{du}(E(u)) = \frac{2}{\sqrt{4u - 3}} + (1 - \rho)\left(\frac{-2}{\sqrt{13 - 4u}}\right) - (\rho)\left(\frac{\frac{(-u+2)}{\sqrt{-u^2+4u-3}}(2-u) + \sqrt{-u^2+4u-3}}{\left(1 + \frac{-u^2+4u-3}{(2-u)^2}\right)(2-u)^2}\right)$$

The last term can be simplified as  $(\rho)\left(\frac{1}{\sqrt{(u-1)(3-u)}}\right)$ , yielding  $\frac{d}{du}(E(p(u)))$  as

$$\begin{aligned}
&= \frac{2}{\sqrt{4u-3}} + (1-\rho)\left(\frac{-2}{\sqrt{13-4u}}\right) - (\rho)\left(\frac{1}{\sqrt{(u-1)(3-u)}}\right) \\
&= \left[\sqrt{(4u-3)(13-4u)(u-1)(3-u)}\right]^{-1} \left[2\sqrt{4u-3}(13-4u)(u-1)(3-u) \right. \\
&\quad \left. - 2(1-\rho)\sqrt{13-4u}(4u-3)(u-1)(3-u) - \rho\sqrt{(u-1)(3-u)}(4u-3)(13-4u)\right]
\end{aligned}$$

Thus we have the following result: For  $\rho \in (0, 1)$ , the optimal disambiguation point is  $P^* = (u^*, \sqrt{1 - (u^* - 2)^2})$  where  $u^*$  is the unique solution of the following equation in the interval  $(1, 1.5)$ :

$$\begin{aligned}
0 &= 2\sqrt{4u-3}(13-4u)(u-1)(3-u) - 2(1-\rho)\sqrt{13-4u}(4u-3)(u-1)(3-u) \\
&\quad - \rho\sqrt{(u-1)(3-u)}(4u-3)(13-4u) \tag{2}
\end{aligned}$$

Using MATLAB, we tabulated  $\rho$  versus  $(u^*, v^*)$  for several different values of  $\rho$  in Table 1. It can be seen that the closer  $\rho$  is to 1, the closer the optimal disambiguation point is to  $A = (1.5, .87)$ .

$\rho$	$(u^*, v^*)$	$E(u^*, v^*)$
0	(1, 0)	4
.5	(1.06, .34)	4.33
.75	(1.15, .53)	4.44
.9	(1.27, .68)	4.49
1	(1.5, .87)	4.51

Table 1: Optimal disambiguation points and corresponding expected lengths for different  $\rho$ 's

## 2 Deciding Where to Disambiguate: Two Disks Case

In this section, we illustrate the fact that the optimal disambiguation point of a particular disk does not just depend on this disk's mark, but also on the location and mark of the other disks present in the obstacle field.

Consider the SOSF instance with two disks in Figure 2. In this instance, the starting point is  $S = (0, 0)$  and the termination point is  $T = (8, 0)$ .

The first disk is centered at  $(2,0)$  and the second at  $(6,0)$ , both with a radius of 1. The cost of disambiguation is taken as zero. Let  $x_1, x_2$  be the first and second disks, respectively, and  $\rho_1, \rho_2$  be the marks of these disks. Furthermore, let  $W_{1,2}$  denote the walk associated with the algorithm that calls for first disambiguating  $x_1$  and then  $x_2$  regardless of the outcome of the disambiguation. Now, let  $P^* = (u_1^*, v_1^*)$  and  $Q^* = (u_2^*, v_2^*)$  be the optimal disambiguation points associated with  $W_{1,2}$ . If  $\rho_1 = 0, \rho_2 = 0$ , then  $P^* = C = (1, 0)$  and  $Q^* = G = (5, 0)$ . However, if  $\rho_1 = 0, \rho_2 = 1$ , then  $P^* = F$  and  $Q^* = H$ . Thus, the optimal disambiguation point for  $x_1$  associated with the policy  $W_{1,2}$  depends on the location and mark of  $x_2$ .

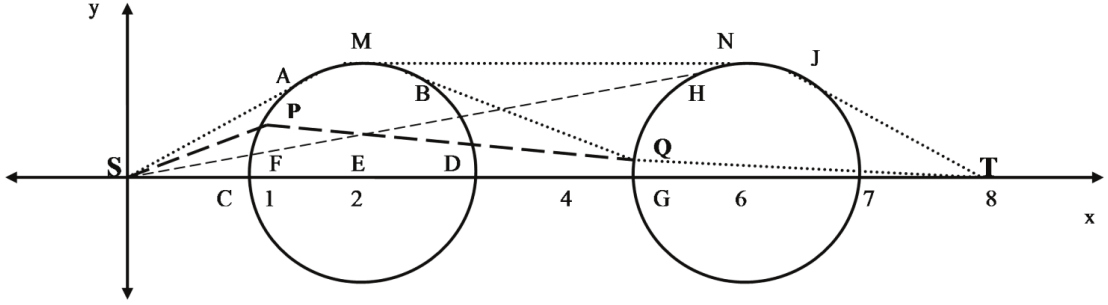


Figure 2: A continuous SOSp instance with two disks.

In fact, we can compute the optimal disambiguation point  $P^*$  associated with  $W_{1,2}$  as follows:

$$E(W_{1,2}(P, Q)) = |SP| + (1 - \rho_1)(|PQ| + (1 - \rho_2)|QT| + (\rho_2)(|\text{arc}QHJ| + |JT|)) \\ + (\rho_1)(|\text{arc}PAB| + |BQ| + (1 - \rho_2)|QT| + (\rho_2)(|\text{arc}QHJ| + |JT|))$$

Due to the fact that  $P$  and  $Q$  are points on  $x_1$  and  $x_2$  respectively, it holds that  $v_1 = \sqrt{1 - (u_1 - 2)^2}$  and  $v_2 = \sqrt{1 - (u_2 - 6)^2}$ . For this reason,  $E(p_{1,2}(P, Q))$  can be expressed as a function of  $u_1$  and  $u_2$  similar to equation (1). One can then compute partial derivatives of  $E(W_{1,2}(u_1, u_2))$  with respect to  $u_1$  and  $u_2$  and determine the roots in the interval  $(1, 1.5)$  for  $u_1$  and  $(5, 6)$  for  $u_2$  to obtain  $P^*$  and  $Q^*$ . That is,

$$(u_1^*, u_2^*) = \arg \min_{\substack{u_1 \in (1, 1.5) \\ u_2 \in (5, 6)}} E(W_{1,2}(u_1, u_2)) \quad (3)$$

Note that  $Q^*$  is computed in (3) as it is needed to determine  $P^*$  in an expected sense. Once  $x_1$  is disambiguated, a new  $Q^*$  needs to be re-calculated as in the single disk case based upon the actual outcome of the disambiguation. It should also be noted that the solution of (3) involves nontrivial geometric calculations and solving a highly nonlinear system with two equations in two unknowns, namely,  $u_1^*$  and  $u_2^*$ .