e-companion

EC.1. Derivation Details

Unbiasedness and Variances of DiGAR Least-Squares Estimators

The unweighted OLS case is considered here, as the extension to weights α is similar (basically insert $\frac{1-\alpha}{\alpha}$ in last terms of both numerator and denominator at beginning of derivations). The parameter estimators given by (7) are unbiased, since

$$E[\hat{\beta}_1] = \frac{\sum_{i=1}^{n} x_i E[y_i] - n\bar{x}E[\bar{y}] + nE[\bar{g}]}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n} = \beta_1,$$

$$E[\hat{\beta}_0] = E[\bar{y}] - E[\hat{\beta}_1]\bar{x} = \beta_0,$$

and the variances of these two estimators are calculated as follows:

$$\begin{aligned}
\mathrm{Var}(\hat{\beta}_1) &= \frac{\sum_{i=1}^{n} x_i^2 \mathrm{Var}(y_i) - n^2\bar{x}^2\mathrm{Var}(\bar{y}) + n^2\mathrm{Var}(\bar{g})}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2} \\
&= \frac{\sum_{i=1}^{n} x_i^2\sigma^2 - n^2\bar{x}^2\sigma^2/n + n^2\sigma_g^2/n}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2} \\
&= \frac{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)\sigma^2 + n\sigma_g^2}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2}, \\
\mathrm{Var}(\hat{\beta}_0) &= \mathrm{Var}(\bar{y}) + \bar{x}^2\mathrm{Var}\hat{\beta}_1 \\
&= \frac{\sigma^2}{n} + \bar{x}^2 \frac{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)\sigma^2 + n\sigma_g^2}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2} \\
&= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2}\right) + \sigma_g^2\left(\frac{n\bar{x}^2}{\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + n\right)^2}\right).
\end{aligned}$$

Proof of Proposition 4

Comparing the variances given by (12) and (14), the inequality holds if and only if

$$\frac{\sigma^2}{n}\frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2+(\frac{1-\alpha}{\alpha})^2\frac{\sigma_g^2}{\sigma^2}}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1-\alpha}{\alpha}\right)^2}\leq\frac{\sigma^2}{n}\frac{1}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$\iff\left(\frac{\sigma_g}{\sigma}\right)^2\leq\left(\frac{\alpha}{1-\alpha}\right)^2\left(\frac{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1-\alpha}{\alpha}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}-\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)$$

$$=2\left(\frac{\alpha}{1-\alpha}\right)+\frac{1}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Unbiasedness and Variances of DiGAR MLEs

Under the normality assumption, the respective probability density functions for y_i and g_i are given by

$$f(y_i;\beta_0,\beta_1,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(y_i-\beta_0-\beta_1x_i)^2}{2\sigma^2}\right\},$$
$$f(g_i;\beta_1,\sigma_g^2)=\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left\{-\frac{(g_i-\beta_1)^2}{2\sigma_g^2}\right\}.$$

Since they are mutually independent, the likelihood function is simply the product given by

$$L(\beta_0,\beta_1,\sigma^2,\sigma_g^2)=(2\pi)^{-n}\sigma^{-n}\sigma_g^{-n}\exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i^2)-\frac{1}{2\sigma_g^2}\sum_{i=1}^{n}(g_i-\beta_1)^2\right\},$$

and the log-likelihood function is

$$\log(L)=-n\ln(2\pi\sigma\sigma_g)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i^2)-\frac{1}{2\sigma_g^2}\sum_{i=1}^{n}(g_i-\beta_1)^2.$$

Differentiating the log-likelihood function with respect to the parameters,

$$\frac{\partial\log(L)}{\partial\beta_1}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)x_i+\frac{1}{\sigma_g^2}\sum_{i=1}^{n}(g_i-\beta_1),$$
$$\frac{\partial\log(L)}{\partial\beta_0}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i),$$
$$\frac{\partial\log(L)}{\partial\sigma^2}=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)^2,$$
$$\frac{\partial\log(L)}{\partial\sigma_g^2}=-\frac{n}{2\sigma_g^2}+\frac{1}{2\sigma_g^4}\sum_{i=1}^{n}(g_i-\beta_1)^2.$$

Setting each derivative equal to 0 and solving for the estimators yields

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^{n} x_i y_i/\sigma^2 + n\bar{g}/\sigma_g^2 - n\bar{x}\bar{y}/\sigma^2}{\sum_{i=1}^{n} x_i^2/\sigma^2 + n/\sigma_g^2 - n\bar{x}^2/\sigma^2}, \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}.\end{aligned} \tag{EC.1}$$

Since σ^2 and σ_g^2 are unknown, they must be estimated from the sample variances,

$$\hat{\sigma}_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2, \quad \hat{\sigma}_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(g_i - \bar{g})^2,$$

which must be substituted for σ^2 and σ_g^2 in (EC.1).

Proof of BLUE in Correlated DiGAR Model

The proof uses the Gauss-Markov Theorem.

THEOREM EC.1 **(Gauss-Markov Theorem)**. *For the regression model (24) with $E[\boldsymbol{\epsilon}] = 0$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$, the OLS estimators have minimum variance among all linear unbiased estimators.*

Since $\mathbf{V}$ is positive definite, there exists an $2n \times 2n$ nonsingular matrix $\mathbf{P}$ such that $\mathbf{V} = \mathbf{P}\mathbf{P}'$. Multiplying $\mathbf{y} = \boldsymbol{x}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{y} = \mathbf{P}^{-1}\mathbf{X}\boldsymbol{\beta} + \mathbf{P}^{-1}\boldsymbol{\epsilon},$$

where $E[\mathbf{P}^{-1}\boldsymbol{\epsilon}] = \mathbf{P}^{-\mathbf{1}}E[\boldsymbol{\epsilon}] = \mathbf{0}$ and

$$\text{Cov}(\mathbf{P}^{-1}\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\text{Cov}(\boldsymbol{\epsilon}(\mathbf{P}^{-1})' = \mathbf{P}^{-1}\sigma^2\mathbf{V}(\mathbf{P}^{-1})' = \sigma^2\mathbf{P}^{-1}\mathbf{P}\mathbf{P}'(\mathbf{P}')^{-1} = \sigma^2\mathbf{I}.$$

Therefore, the assumptions in the Gauss-Markov Theorem are satisfied, and the least-squares estimator

$$\hat{\boldsymbol{\beta}} = \left[(\mathbf{P}^{-\mathbf{1}}\mathbf{X})'(\mathbf{P}^{-\mathbf{1}}\mathbf{X})\right]^{-1}(\mathbf{P}^{-\mathbf{1}}\mathbf{X})'\mathbf{P}^{-\mathbf{1}}\mathbf{y},$$

is the best linear unbiased estimator (BLUE). The estimator $\hat{\boldsymbol{\beta}}$ can be written as

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= \left[(\mathbf{P}^{-\mathbf{1}}\mathbf{X})'(\mathbf{P}^{-\mathbf{1}}\mathbf{X})\right]^{-1}(\mathbf{P}^{-\mathbf{1}}\mathbf{X})'\mathbf{P}^{-\mathbf{1}}\mathbf{y} \\ &= [\mathbf{X}'(\mathbf{P}')^{-\mathbf{1}}\mathbf{P}^{-\mathbf{1}}\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{P}')^{-\mathbf{1}}\mathbf{P}^{-\mathbf{1}}\mathbf{y} \\ &= [\mathbf{X}'(\mathbf{P}\mathbf{P}')^{-\mathbf{1}}\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{P}\mathbf{P}')^{-\mathbf{1}}\mathbf{y} \\ &= (\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{y}.\end{aligned}$$

Correlated DiGAR MLE

If the residuals are normally distributed, i.e., $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{V})$, then $\mathbf{y} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{V})$, and the likelihood function is

$$L(\boldsymbol{\beta}) = \frac{1}{(2\pi)^n |\mathbf{V}|^{1/2}} \exp\left\{-(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{V})^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})/2\right\},$$

so the log-likelihood function is

$$\ln L(\boldsymbol{\beta}) = -n\ln(2\pi) - \frac{1}{2}\ln(|\mathbf{V}|) - \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Differentiating with respect to $\boldsymbol{\beta}$,

$$\frac{\partial \ln L}{\partial \boldsymbol{\beta}} = -\left(\mathbf{X}'\mathbf{V}\mathbf{X}\boldsymbol{\beta} - \mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{y}\right),$$

setting equal to zero and solving for $\boldsymbol{\beta}$ gives the estimator

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{y})^{-1}\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{y},$$

which is the same as the best linear unbiased estimator (BLUE).

OLS Estimators in Quadratic DiGAR

Consider the loss function

$$L = \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_2 x_i^2 - \beta_1 x_i - \beta_0)^2 + \frac{1}{2}\sum_{i=1}^{n}(g_i - 2\beta_2 x_i - \beta_1)^2$$

Differentiating with respect to $\beta_0, \beta_1, \beta_2$,

$$\begin{aligned}
\frac{\partial L}{\partial \beta_0} &= -\sum_{i=1}^{n}(y_i - \beta_2 x_i^2 - \beta_1 x_i - \beta_0) \\
\frac{\partial L}{\partial \beta_1} &= -\sum_{i=1}^{n}(y_i - \beta_2 x_i^2 - \beta_1 x_i - \beta_0)x_i - \sum_{i=1}^{n}(g_i - 2\beta_2 x_i - \beta_1) \\
\frac{\partial L}{\partial \beta_2} &= -\sum_{i=1}^{n}(y_i - \beta_2 x_i^2 - \beta_1 x_i - \beta_0)x_i^2 - \sum_{i=1}^{n}(g_i - 2\beta_2 x_i - \beta_1)2x_i
\end{aligned} \tag{EC.2}$$

Setting them equal to 0 and solving yields the following estimators, $\boldsymbol{\beta} = \mathbf{A}\mathbf{y}$, where

$$\boldsymbol{\beta} \equiv [\beta_0, \beta_1, \beta_2]^T, \ \mathbf{y} = \left[\sum_{i=1}^{n} x_i^2 y_i + 2\sum_{i=1}^{n} x_i g_i, \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} g_i, \sum_{i=1}^{n} y_i\right]^T,$$

and

$$\mathbf{A} = \frac{1}{abc - af^2 - be^2 - cd^2 + 2def} \begin{bmatrix} bc - f^2 & ef - cd & df - be \\ ef - cd & ac - e^2 & de - af \\ df - be & de - af & ab - d^2 \end{bmatrix},$$

with $a = \sum_{i=1}^{n} x_i^4 + 4\sum_{i=1}^{n} x_i^2,\ b = \sum_{i=1}^{n} x_i^2,\ c = n,\ d = \sum_{i=1}^{n} x_i^3 + 2\sum_{i=1}^{n} x_i,\ e = \sum_{i=1}^{n} x_i^2,\ f = \sum_{i=1}^{n} x_i.$

EC.2. Gradient Estimation for $G/G/1$ Queue

Let A_k be the interarrival time between the $(k-1)$st and kth customer (by convention, taking A_1 to be the time of the 1st arrival), and let X_k be the service time of the kth customer. The system time of the kth customer, denoted by T_k, satisfies the well-known Lindley equation:

$$T_{k+1} = X_{k+1} + (T_k - A_{k+1})^+, \tag{EC.3}$$

where $a^+ = \max(a, 0)$. The infinitesimal perturbation analysis (IPA) estimator is then obtained by simple differentiation, which for a general parameter θ is given by ((Suri and Zazanis 1988)):

$$\frac{dT_{k+1}}{d\theta} = \frac{dX_{k+1}}{d\theta} + \left(\frac{dT_k}{d\theta} - \frac{dA_{k+1}}{d\theta}\right) 1\{T_k \geq A_{k+1}\},\ k > 1, \quad \text{with } \frac{dT_1}{d\theta} = \frac{dX_1}{d\theta}. \tag{EC.4}$$

For x a parameter of the (common) customer service time distribution, the unbiased IPA estimator is

$$\frac{dT_{k+1}}{dx} = \frac{dX_{k+1}}{dx} + \frac{dT_k}{dx} 1\{T_k \geq A_{k+1}\},$$

where dX/dx can be calculated based on the distribution for the random variable X. For example, if X is exponentially distributed (with mean x), then dX/dx is simply given by X/x, and (EC.2) becomes

$$\frac{dT_{k+1}}{dx} = \frac{X_{k+1}}{x} + \frac{dT_k}{dx} 1\{T_k \geq A_{k+1}\},\ k > 1, \quad \text{with } \frac{dT_1}{dx} = \frac{X_1}{x},$$

the latter assuming that the system starts empty. This is what is used for the $M/M/1$ queue example.

Similarly, for the $U/U/1$ example, where the interarrival time and service time distributions are $U(\theta_1-\delta_1,\theta_1+\delta_1)$ $U(\theta_2-\delta_2,\theta_2+\delta_2)$, respectively, the four unbiased IPA estimators are

$$
\begin{aligned}
\frac{\partial T_{k+1}}{\partial\theta_1} &= \left(\frac{\partial T_k}{\partial\theta_1}-1\right)1\{T_k\ge A_{k+1}\},\ k>1,\quad \text{with } \frac{\partial T_1}{\partial\theta_1}=0,\\
\frac{\partial T_{k+1}}{\partial\theta_2} &= 1+\frac{\partial T_k}{\partial\theta_2}1\{T_k\ge A_{k+1}\},\ k>1,\quad \text{with } \frac{\partial T_1}{\partial\theta_2}=1,\\
\frac{\partial T_{k+1}}{\partial\delta_1} &= \left(\frac{\partial T_k}{\partial\delta_1}-\frac{A_{k+1}-\theta_1}{\delta_1}\right)1\{T_k\ge A_{k+1}\},\ k>1,\quad \text{with } \frac{\partial T_1}{\partial\delta_1}=0,\\
\frac{\partial T_{k+1}}{\partial\delta_2} &= \frac{X_{k+1}-\theta_2}{\delta_2}+\frac{\partial T_k}{\partial\delta_2}1\{T_k\ge A_{k+1}\},\ k>1,\quad \text{with } \frac{\partial T_1}{\partial\delta_2}=\frac{X_1-\theta_2}{\delta_2},
\end{aligned}
$$

EC.3. Analytical Results for $M/M/1$ and $U/U/1$ Queues

For the $M/M/1$ queue, the true models for an interarrival mean of 0.2 are given by

$$
\begin{aligned}
y^{(2)}(x) &= x+\frac{x^2}{5+x},\\
y^{(3)}(x) &= x+\frac{5x^2}{(5+x)^2}+\frac{x^3(15+2x)}{(5+x)^3},\\
y^{(4)}(x) &= x+\frac{25x^2}{(5+x)^3}+\frac{25x^3}{(5+x)^4}+\frac{5x^3(15+2x)}{(5+x)^4}+\frac{x^4(225+50x+3x^2)}{(5+x)^5},\\
y^{(5)}(x) &= x+\frac{125x^2}{(5+x)^4}+\frac{250x^3}{(5+x)^5}+\frac{25x^3(15+2x)}{(5+x)^5}+\frac{5x^4(225+50x+3x^2)}{(5+x)^6}\\
&\quad+\frac{25x^4(15+2x)}{(5+x)^6}+\frac{250x^4}{(5+x)^6}+\frac{x^5(10+x)(350+65x+4x^2)}{(5+x)^7}.
\end{aligned}
$$

For the $U/U/1$ queue, the true models are given by

$$
\begin{aligned}
y^{(2)} &= \frac{\delta_1}{4}-\frac{\theta_1}{2}+\frac{3\theta_2}{2}+\frac{1}{\delta_1}\left(\frac{\delta_2^2}{12}+\frac{\theta_1^2}{4}-\frac{\theta_1\theta_2}{2}+\frac{\theta_2^2}{4}\right)\\
y^{(3)} &= \frac{5\delta_1}{12}-\theta_1+2\theta_2+\frac{1}{12\delta_1}\left(2\delta_2^2+9\theta_1^2-18\theta_1\theta_2+9\theta_2^2\right)\\
&\quad-\frac{1}{12\delta_1^2}(\theta_1-\theta_2)(\delta_2^2+2\theta_1^2-4\theta_1\theta_2+2\theta_2^2)\\
y^{(4)} &= \frac{107\delta_1}{192}-\frac{25\theta_1}{16}+\frac{41\theta_2}{16}+\frac{1}{2880\delta_1}(750\delta_2^2+4590\theta_1^2-9180\theta_1\theta_2+4590\theta_2^2)\\
&\quad-\frac{1}{48\delta_1^2}(\theta_1-\theta_2)(13\delta_2^2+35\theta_1^2-70\theta_1\theta_2+35\theta_2^2)\\
&\qquad+\frac{1}{2880\delta_1^3}(13\delta_2^4+270\delta_2^2\theta_1^2-540\delta_2^2\theta_1\theta_2+270\delta_2^2\theta_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+405\theta_1^4-1620\theta_1^3\theta_2+2430\theta_1^2\theta_2^2-1620\theta_1\theta_2^3+405\theta_2^4)\\
y^{(5)}=&\frac{221\delta_1}{320}-\frac{107\theta_1}{48}+\frac{155\theta_2}{48}+\frac{1}{2880\delta_1}(1070\delta_2^2+8430\theta_1^2-16860\theta_1\theta_2+8430\theta_2^2)\\
&-\frac{1}{48\delta_1^2}(\theta_1-\theta_2)(29\delta_2^2+99\theta_1^2-198\theta_1\theta_2+99\theta_2^2)\\
&+\frac{1}{2880\delta_1^3}(49\delta_2^4+1230\delta_2^2\theta_1^2-2460\delta_2^2\theta_1\theta_2+1230\delta_2^2\theta_2^2\\
&+2325\theta_1^4-9300\theta_1^3\theta_2+13950\theta_1^2\theta_2^2-9300\theta_1\theta_2^3+2325\theta_2^4)\\
&-\frac{1}{720\delta_1^4}(\theta_1-\theta_2)(9\delta_2^4+80\delta_2^2\theta_1^2-160\delta_2^2\theta_1\theta_2+80\delta_2^2\theta_2^2+96\theta_1^4\\
&-384\theta_1^3\theta_2+576\theta_1^2\theta_2^2-384\theta_1\theta_2^3+96\theta_2^4)
\end{aligned}
$$

EC.4. Raw Simulation Output

Data Set 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Run	Output	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
1	T_2	6.532	1.092	1.492	11.613	0.440	8.782	9.688	5.339	4.418	7.956
	T_3	4.763	2.279	2.487	22.521	1.628	16.681	5.038	11.936	6.522	3.686
	T_4	4.687	1.305	6.765	26.267	2.783	10.602	13.819	8.835	3.694	13.921
	T_5	5.264	1.015	2.824	30.624	3.413	5.710	15.001	5.591	8.627	15.447
	dT_2/dx	1.814	0.295	1.370	3.052	0.652	2.142	2.306	1.241	1.345	1.768
	dT_3/dx	3.078	0.616	0.654	6.547	0.407	4.908	2.664	3.728	1.482	0.819
	dT_4/dx	3.132	0.917	1.780	7.645	0.695	5.341	4.891	2.054	0.839	3.093
	dT_5/dx	1.462	0.274	0.743	9.284	1.274	5.662	6.937	2.657	1.960	4.260
2	T_2	7.608	1.440	7.261	3.873	1.899	3.469	9.984	1.431	12.533	1.701
	T_3	3.995	1.329	5.989	2.709	4.237	7.929	10.364	1.903	15.366	7.894
	T_4	5.239	2.726	6.737	7.645	12.301	17.273	5.696	3.508	17.969	7.872
	T_5	6.579	8.904	5.497	1.152	12.214	17.156	5.301	10.069	18.289	7.744
	dT_2/dx	2.372	0.389	3.180	0.993	0.474	1.018	2.383	0.332	3.063	0.378
	dT_3/dx	2.672	0.359	3.925	1.178	1.059	1.934	3.204	0.442	5.138	2.033
	dT_4/dx	3.678	0.737	4.794	1.960	3.075	5.097	3.715	0.819	6.291	2.862
	dT_5/dx	1.827	2.406	5.046	0.295	3.564	6.265	4.154	2.341	8.002	2.899
3	T_2	7.802	1.613	1.702	4.706	5.489	1.770	2.316	1.640	5.873	25.133
	T_3	9.429	3.031	2.096	11.011	6.641	10.285	0.081	2.125	4.131	30.479
	T_4	7.334	0.803	5.660	12.446	10.994	11.840	4.598	10.431	6.275	50.209
	T_5	6.200	0.019	9.499	13.973	11.455	10.078	6.703	5.172	10.970	43.387
	dT_2/dx	2.758	0.489	0.447	1.905	1.448	0.431	0.809	0.381	1.334	5.585
	dT_3/dx	3.739	0.819	0.551	4.010	1.942	2.508	0.019	0.494	1.374	7.381
	dT_4/dx	3.844	0.217	1.903	4.439	3.384	4.621	1.094	2.797	2.234	11.819
	dT_5/dx	1.722	0.005	3.556	3.582	4.274	5.357	1.596	3.063	3.324	12.319
4	T_2	0.125	1.411	4.335	3.491	6.873	7.390	5.743	4.038	1.589	0.406
	T_3	2.516	0.174	4.563	7.181	8.019	9.975	7.803	9.174	2.573	1.876
	T_4	2.286	13.853	1.195	10.043	9.003	13.860	5.366	15.783	1.691	1.795
	T_5	2.824	18.055	1.507	12.680	3.282	8.599	5.180	12.544	0.200	3.343
	dT_2/dx	0.034	0.381	1.140	0.895	2.377	1.802	1.367	1.058	0.361	0.090
	dT_3/dx	0.698	0.047	1.200	2.444	3.165	3.066	2.285	2.133	0.584	0.417
	dT_4/dx	0.635	3.744	1.261	2.575	4.148	4.822	3.281	4.022	0.384	0.399
	dT_5/dx	1.408	5.016	1.426	3.813	0.820	4.889	3.601	4.405	0.045	0.946
5	T_2	3.936	9.800	3.093	23.809	3.366	8.562	8.976	9.939	17.323	3.326
	T_3	3.543	2.547	2.107	17.240	6.698	9.761	11.276	6.056	23.652	5.795
	T_4	10.669	1.988	10.270	23.081	7.790	2.950	10.300	6.554	21.505	4.160
	T_5	6.482	3.146	7.708	36.288	10.822	2.287	17.472	2.267	18.081	3.569
	dT_2/dx	1.093	2.648	1.139	6.104	1.113	2.569	3.098	2.435	3.937	1.182
	dT_3/dx	1.142	0.688	0.554	6.806	1.986	3.715	3.762	3.762	5.628	2.289
	dT_4/dx	3.353	1.073	3.173	8.460	2.259	0.719	4.233	4.725	5.636	0.924
	dT_5/dx	3.832	1.664	3.622	12.265	3.086	0.782	6.268	0.527	6.192	1.047
6	T_2	13.319	1.065	9.269	13.481	12.798	0.012	15.706	0.920	3.756	6.120
	T_3	28.584	2.677	11.883	18.382	20.917	5.545	17.425	3.142	4.524	18.093
	T_4	28.491	2.071	24.240	21.678	16.693	4.303	18.921	2.669	0.246	25.719
	T_5	19.985	1.752	20.643	18.641	6.876	0.681	34.535	3.477	2.591	22.339
	dT_2/dx	3.699	0.287	3.175	3.566	3.199	0.003	3.747	0.214	0.853	1.405
	dT_3/dx	8.101	0.723	4.153	5.069	5.337	1.352	5.389	0.730	1.045	4.491
	dT_4/dx	8.460	0.917	7.455	6.949	5.890	1.466	6.065	0.620	0.055	6.352
	dT_5/dx	9.784	0.473	7.709	7.863	6.074	0.166	9.902	0.868	0.589	6.474

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		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Run	Output	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
7	T_2	3.035	9.008	5.290	13.933	2.362	0.435	2.850	13.454	1.793	1.246
	T_3	7.335	1.760	23.620	20.391	2.863	7.863	2.401	11.741	0.692	11.946
	T_4	4.561	0.805	22.006	24.583	0.815	6.628	0.933	9.752	0.924	10.891
	T_5	6.735	0.118	22.318	12.336	2.517	11.064	1.835	11.257	11.697	11.729
	dT_2/dx	0.843	2.561	1.392	4.047	0.590	0.106	0.678	4.419	0.407	0.277
	dT_3/dx	2.867	2.607	6.397	6.267	0.865	1.917	0.571	4.756	0.157	2.654
	dT_4/dx	3.690	0.217	7.207	8.641	0.203	2.646	0.222	5.155	0.210	2.814
	dT_5/dx	4.451	0.031	7.601	8.757	0.629	2.698	0.437	5.657	2.853	3.069
8	T_2	7.582	12.953	2.126	3.127	0.766	2.809	4.806	3.495	14.836	1.206
	T_3	0.726	0.979	0.836	3.614	3.303	0.738	4.483	1.985	15.128	0.453
	T_4	1.468	3.932	0.887	8.160	5.279	1.944	8.373	6.136	25.327	4.167
	T_5	2.798	1.834	3.465	5.771	1.635	4.504	8.862	12.133	24.908	4.403
	dT_2/dx	2.223	3.500	0.559	0.802	0.191	0.685	1.426	0.812	3.389	0.268
	dT_3/dx	2.288	0.264	0.220	0.926	0.999	0.180	1.513	0.461	4.097	0.100
	dT_4/dx	0.408	1.062	0.258	2.092	1.319	0.474	2.800	1.460	6.607	0.926
	dT_5/dx	0.777	0.495	0.911	2.722	0.408	1.256	2.110	2.895	8.711	1.128
9	T_2	7.739	0.584	11.405	2.574	3.534	4.415	0.393	14.925	5.133	3.374
	T_3	6.594	0.678	0.570	2.786	6.642	5.930	6.233	5.058	1.054	6.209
	T_4	1.912	6.790	3.886	0.143	11.310	4.598	8.197	0.591	9.717	13.830
	T_5	2.572	8.763	0.058	0.158	11.003	2.924	12.638	4.036	11.999	7.224
	dT_2/dx	2.324	0.157	3.815	0.660	0.935	1.660	0.093	3.958	1.618	0.810
	dT_3/dx	2.609	0.183	0.150	0.714	2.036	2.807	1.484	1.176	1.743	1.823
	dT_4/dx	0.531	1.835	1.022	0.036	2.827	2.920	2.599	0.137	2.208	4.363
	dT_5/dx	0.714	2.967	0.015	0.040	3.621	0.713	3.848	0.938	3.246	1.605
10	T_2	1.777	2.543	8.342	12.069	6.095	3.224	4.560	3.038	1.565	6.383
	T_3	5.056	4.279	0.047	14.939	17.054	3.124	0.719	11.227	1.707	7.470
	T_4	6.177	1.753	4.492	18.968	20.820	6.116	0.801	8.332	7.549	8.350
	T_5	5.778	14.492	2.674	19.473	15.372	3.280	0.274	19.648	27.667	9.712
	dT_2/dx	0.559	0.687	2.301	3.094	3.366	0.786	3.138	0.706	0.355	1.418
	dT_3/dx	1.404	1.156	0.012	3.944	7.103	1.484	0.171	2.611	0.608	1.660
	dT_4/dx	1.808	0.473	1.182	5.427	8.212	2.410	0.190	2.859	2.254	1.967
	dT_5/dx	1.928	3.916	0.703	6.152	8.845	2.531	0.065	6.026	7.227	3.772

Data Set 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Run	Output	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
1	T_2	8.190	0.116	6.911	3.007	7.284	1.120	3.397	0.992	6.040	5.704
	T_3	2.569	6.958	7.710	4.886	28.551	0.306	3.295	3.733	6.028	7.130
	T_4	4.341	10.905	23.154	9.136	36.469	0.328	4.130	3.031	8.830	11.229
	T_5	14.747	11.557	19.359	10.070	38.236	5.095	3.570	3.322	6.056	6.081
	dT_2/dx	2.602	0.031	1.843	0.983	1.990	0.273	0.808	0.230	1.572	1.675
	dT_3/dx	0.714	1.881	2.321	1.787	7.361	0.074	0.784	0.868	1.652	2.268
	dT_4/dx	1.684	3.294	6.721	2.342	9.406	0.080	1.213	0.935	2.613	3.271
	dT_5/dx	5.384	4.224	8.480	2.663	10.081	1.242	1.230	0.772	3.214	3.537
2	T_2	2.379	8.656	3.826	2.121	6.974	2.602	1.816	4.200	0.052	23.445
	T_3	3.258	17.171	14.486	3.828	9.808	5.518	0.284	4.455	1.992	21.362
	T_4	10.032	14.982	9.165	8.560	15.341	9.140	7.160	0.734	3.621	7.824
	T_5	8.865	4.340	11.719	12.588	6.816	10.828	4.641	1.364	11.227	6.484
	dT_2/dx	0.660	2.437	1.356	0.543	2.520	0.634	0.432	0.976	0.011	5.925
	dT_3/dx	1.294	4.964	5.134	1.097	4.177	1.346	0.067	1.098	0.452	6.317
	dT_4/dx	2.786	5.238	5.623	3.125	7.120	2.647	1.704	1.269	0.823	6.343
	dT_5/dx	3.337	5.586	7.995	5.141	7.144	3.137	2.745	0.317	2.780	6.420

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		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Run	Output	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
3	T_2	12.310	6.305	9.926	7.122	6.269	14.391	5.086	7.641	10.910	5.447
	T_3	18.996	6.785	18.615	7.401	3.858	11.364	0.590	2.073	13.296	9.771
	T_4	26.102	2.414	23.592	11.288	8.998	9.585	4.372	2.393	13.475	0.232
	T_5	19.221	3.287	26.823	13.496	7.350	11.470	1.459	3.374	18.527	0.724
	dT_2/dx	3.419	2.201	2.612	1.826	2.009	3.973	1.210	2.006	2.995	4.539
	dT_3/dx	6.208	1.833	5.696	1.897	0.964	4.626	0.140	2.267	4.299	5.585
	dT_4/dx	8.302	0.652	7.185	3.232	2.249	5.234	1.152	2.644	4.429	0.051
	dT_5/dx	10.176	0.888	8.863	4.555	2.297	6.724	0.347	0.784	5.693	0.161
4	T_2	2.346	8.677	1.878	1.166	6.889	2.162	0.229	8.789	2.043	2.480
	T_3	1.886	10.868	11.501	1.255	2.236	1.014	0.919	2.877	0.427	16.610
	T_4	6.077	12.848	10.048	3.623	7.376	0.830	2.546	2.733	5.052	14.196
	T_5	4.640	14.350	13.881	6.277	6.384	0.725	11.834	8.626	3.779	10.624
	dT_2/dx	1.943	2.661	0.494	0.299	1.827	0.527	0.054	2.044	0.586	3.728
	dT_3/dx	0.523	4.090	3.172	0.321	0.559	0.247	0.218	2.466	0.097	7.333
	dT_4/dx	1.688	5.359	3.349	0.929	1.844	0.202	0.606	0.635	1.148	7.412
	dT_5/dx	2.002	7.505	4.621	1.846	1.917	0.176	2.817	2.326	0.858	7.664
5	T_2	10.405	2.731	7.292	7.293	3.397	8.688	2.174	5.617	9.628	13.767
	T_3	7.545	4.220	13.171	4.698	1.085	19.540	1.277	5.435	2.686	11.653
	T_4	11.683	11.793	13.102	4.087	6.226	28.663	1.338	8.498	1.229	11.391
	T_5	5.773	7.515	4.852	2.455	4.454	31.173	3.714	6.671	2.513	5.004
	dT_2/dx	3.782	0.738	1.919	3.435	0.849	2.119	0.517	1.711	2.326	3.153
	dT_3/dx	2.096	1.140	4.310	1.204	0.271	5.311	0.304	2.326	0.610	3.505
	dT_4/dx	3.496	3.187	5.105	1.846	1.556	7.733	0.318	3.313	0.279	3.612
	dT_5/dx	4.677	3.382	6.090	0.629	1.655	8.531	0.884	3.404	0.571	3.965
6	T_2	2.853	1.906	4.025	0.399	5.026	1.765	5.911	0.537	1.068	10.459
	T_3	3.128	3.395	8.237	0.521	1.070	4.918	6.497	3.671	7.572	4.370
	T_4	4.238	1.940	10.432	1.631	5.830	7.237	4.589	6.589	13.366	0.757
	T_5	2.634	2.361	8.417	1.585	5.014	11.334	5.702	7.553	5.849	2.092
	dT_2/dx	1.374	0.515	1.300	0.102	1.256	0.430	1.407	0.124	0.526	2.707
	dT_3/dx	1.799	0.917	2.167	0.133	1.304	1.199	1.804	0.853	1.721	0.971
	dT_4/dx	1.177	1.138	2.745	0.418	2.638	2.218	1.900	1.532	3.354	1.105
	dT_5/dx	1.300	0.638	3.507	0.406	2.763	2.764	2.166	1.977	3.702	0.464
7	T_2	9.513	2.017	5.752	2.837	7.821	7.026	0.471	7.703	1.097	11.008
	T_3	12.459	2.720	10.734	3.116	7.766	3.172	2.099	10.088	0.902	20.644
	T_4	14.616	2.132	11.499	0.920	8.622	3.820	10.078	21.770	2.002	13.215
	T_5	9.273	5.375	9.442	0.874	0.608	2.402	18.413	19.897	0.729	10.482
	dT_2/dx	2.971	1.275	1.513	0.727	1.955	1.713	0.112	1.975	1.946	4.249
	dT_3/dx	4.662	0.735	3.499	0.799	1.941	2.290	0.499	3.143	0.205	7.161
	dT_4/dx	5.657	0.576	3.889	0.236	3.092	2.747	2.399	5.970	0.455	7.325
	dT_5/dx	6.432	1.452	4.468	0.224	0.152	3.139	4.524	6.967	0.165	7.916
8	T_2	3.127	5.131	1.584	13.773	1.278	1.402	2.886	5.073	0.381	4.472
	T_3	1.206	6.881	6.959	23.595	5.051	0.804	2.579	12.005	3.509	1.415
	T_4	3.027	9.205	4.704	22.190	8.459	0.757	6.168	12.417	11.938	0.872
	T_5	13.277	7.234	5.844	15.554	8.569	5.100	8.749	20.168	4.010	3.307
	dT_2/dx	0.983	2.288	1.305	3.611	0.421	0.342	0.787	2.535	0.086	0.993
	dT_3/dx	0.335	2.869	1.831	7.315	1.517	0.196	0.614	4.164	0.797	0.314
	dT_4/dx	0.840	4.002	1.238	7.397	2.514	0.295	1.914	4.267	3.409	0.193
	dT_5/dx	4.126	4.064	1.663	7.467	2.694	1.244	2.663	6.602	3.421	0.735
9	T_2	7.319	0.167	4.658	1.715	0.625	5.846	6.939	19.009	6.107	9.265
	T_3	5.050	1.656	6.451	1.994	2.580	10.248	9.344	19.833	10.785	12.439
	T_4	5.588	9.350	0.388	11.459	4.474	11.550	6.113	18.371	12.631	13.216
	T_5	1.754	3.652	1.109	16.238	0.833	4.293	0.802	12.879	18.544	15.676

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		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Run	Output	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
	dT_2/dx	2.033	0.045	1.226	0.439	0.156	1.426	1.652	4.420	1.684	2.784
	dT_3/dx	1.402	0.447	1.697	0.511	0.645	2.499	2.395	6.930	2.451	3.686
	dT_4/dx	2.110	2.941	0.102	2.938	1.508	4.228	3.184	7.224	3.119	3.911
	dT_5/dx	0.487	3.380	0.291	5.663	0.208	4.332	0.191	7.417	5.163	4.815
10	T_2	1.565	9.568	0.707	5.825	1.357	1.070	0.729	4.065	0.496	6.337
	T_3	1.363	9.299	6.004	8.644	1.340	4.394	3.744	0.146	1.433	13.040
	T_4	2.122	8.986	5.487	6.589	2.982	3.578	1.899	2.277	0.218	1.156
	T_5	6.935	12.252	0.661	6.301	0.950	2.733	7.220	2.547	5.747	4.979
	dT_2/dx	0.434	2.586	0.186	1.493	0.339	0.260	0.173	0.945	0.112	2.370
	dT_3/dx	0.687	2.915	1.580	3.239	0.335	1.071	0.891	0.034	0.325	4.876
	dT_4/dx	0.589	3.086	2.242	1.689	0.745	1.871	0.452	0.529	0.049	0.256
	dT_5/dx	2.245	4.134	0.174	1.615	0.237	0.666	1.719	0.592	1.306	1.106

EC.5. Additional Tables and Graphs

Table EC.1 Parameter estimates and performance metrics for M/M/1 queue data set 2

i	$y^{(i)}$ model	$\hat{\beta}_1$	$\hat{\beta}_0$	L_2	$\widehat{\mathrm{Var}}(\hat{\beta}_1)$	$s^2(\hat{\beta}_1)$
2	standard	0.62	3.32	0.073	3.22	5.68
	DiGAR	1.55	-0.47	0.004	0.038	0.047
	DiGAR ($\alpha=0.25$)	1.43	0.04	0.007	0.019	0.029
	DiGAR ($\alpha=0.75$)	1.60	-0.67	0.003	0.216	0.217
	DiGAR ($\alpha\propto 1/\sigma^2$)	1.62	-0.71	0.001	0.018	0.030
	DiGARn	1.62	-0.74	0.003	0.033	0.029
	DiGAR*	1.25	-0.15	0.82	0.633	0.036
	DiGAR**	1.63	-0.77	0.001	0.018	0.030
	"true" linear	1.69	-1.00	4E-6		
3	standard	2.83	-4.24	0.026	8.22	7.17
	DiGAR	2.32	-2.16	0.009	0.095	0.066
	DiGAR ($\alpha=0.25$)	2.39	-2.44	0.009	0.063	0.032
	DiGAR ($\alpha=0.75$)	2.29	-2.05	0.009	0.361	0.300
	DiGAR ($\alpha\propto 1/\sigma^2$)	2.28	-2.04	0.002	0.059	0.029
	DiGARn	2.28	-2.00	0.009	0.094	0.029
	DiGAR*	1.33	-1.44	9.18	0.346	0.098
	DiGAR**	2.28	-2.02	0.006	0.060	0.028
	"true" linear	2.30	-2.18	2E-5		
4	standard	4.31	-8.35	1.02	13.5	5.35
	DiGAR	3.08	-3.35	0.90	0.15	0.074
	DiGAR ($\alpha=0.25$)	3.24	-2.97	0.90	0.090	0.041
	DiGAR ($\alpha=0.75$)	3.01	-4.01	0.90	0.60	0.265
	DiGAR ($\alpha\propto 1/\sigma^2$)	2.99	-3.06	0.78	0.084	0.035
	DiGARn	2.98	-2.97	0.90	0.15	0.038
	DiGAR*	1.88	-1.15	2.50	1.63	0.110
	DiGAR**	2.98	-2.97	0.89	0.085	0.035
	"true" linear	2.85	-3.43	7E-5		
5	standard	6.18	-15.5	0.80	10.1	9.19
	DiGAR	3.63	-5.18	0.32	0.178	0.114
	DiGAR ($\alpha=0.25$)	3.97	-6.54	0.34	0.104	0.068
	DiGAR ($\alpha=0.75$)	3.50	-4.46	0.32	0.585	0.42
	DiGAR ($\alpha\propto 1/\sigma^2$)	3.45	-4.58	0.18	0.099	0.064
	DiGARn	3.44	-4.41	0.32	0.141	0.072
	DiGAR*	2.08	-1.74	4.73	2.070	0.096
	DiGAR**	3.43	-4.40	0.26	0.100	0.064
	"true" linear	3.37	-4.70	2E-4		

Table EC.2 Parameter estimates and performance metrics for linear fit (data set 1)

		standard model				DiGAR model				$\widehat{\text{Var}}$
Run		$\hat{\beta}_1$	$\hat{\beta}_0$	$\widehat{\text{Var}}(\hat{\beta}_1)$	L_2	$\hat{\beta}_1$	$\hat{\beta}_0$	$\widehat{\text{Var}}(\hat{\beta}_1)$	L_2	Ratio
1	$y^{(2)}$	-0.94	8.08	11.540	2.692	1.04	0.07	0.150	2.297	76
	$y^{(3)}$	-1.11	11.62	84.557	0.707	1.74	0.09	0.862	0.019	98
	$y^{(4)}$	-5.98	35.30	158.298	12.617	2.46	1.11	2.009	7.887	78
	$y^{(5)}$	-15.11	73.01	125.698	28.162	2.62	1.20	3.529	7.454	35
2	$y^{(2)}$	7.63	-25.31	55.041	2.202	2.01	-2.55	0.761	0.066	72
	$y^{(3)}$	-0.51	10.28	69.931	1.547	2.36	-1.34	0.862	1.069	81
	$y^{(4)}$	-9.21	45.95	16.390	9.098	2.69	-2.23	1.405	0.264	11
	$y^{(5)}$	-2.72	18.89	18.219	3.243	3.91	-7.96	0.830	1.012	21
3	$y^{(2)}$	-2.36	18.10	13.009	7.446	2.30	-0.76	0.301	6.469	43
	$y^{(3)}$	-8.07	41.94	47.758	10.679	2.48	-0.78	1.369	4.155	34
	$y^{(4)}$	-17.06	79.36	69.784	28.173	1.95	2.37	3.350	4.127	20
	$y^{(5)}$	-12.67	61.88	87.034	18.029	2.78	-0.67	3.117	2.428	27
4	$y^{(2)}$	-1.10	8.14	13.643	4.826	1.22	-1.29	0.227	4.363	60
	$y^{(3)}$	0.85	1.50	45.485	4.355	1.82	-2.42	0.708	4.242	64
	$y^{(4)}$	-1.69	13.37	27.735	3.509	2.01	-1.62	0.681	2.300	40
	$y^{(5)}$	-2.15	16.80	27.360	2.461	2.77	-3.10	0.837	0.636	32
5	$y^{(2)}$	3.64	-7.65	17.072	1.604	2.18	-1.71	0.214	1.387	79
	$y^{(3)}$	-0.26	8.18	48.315	0.397	1.93	-0.68	0.549	0.008	88
	$y^{(4)}$	-5.18	30.76	82.857	6.468	2.42	0.01	1.247	2.565	66
	$y^{(5)}$	-0.14	7.99	98.658	2.843	3.11	-5.19	1.159	2.099	85
6	$y^{(2)}$	3.54	-10.95	11.617	5.695	1.17	-1.34	0.153	5.505	75
	$y^{(3)}$	2.39	-5.32	8.185	7.000	1.37	-1.21	0.082	7.052	99
	$y^{(4)}$	2.41	-4.09	20.915	5.439	1.87	-1.90	0.187	5.486	111
	$y^{(5)}$	2.05	-3.06	13.345	12.320	1.98	-2.75	0.194	12.333	68
7	$y^{(2)}$	0.54	3.35	18.596	0.185	1.74	-1.54	0.218	0.104	85
	$y^{(3)}$	3.03	-4.92	51.567	0.086	2.53	-2.90	0.672	0.057	76
	$y^{(4)}$	3.67	-5.99	60.135	0.548	3.27	-4.37	0.822	0.518	73
	$y^{(5)}$	5.15	-13.12	65.729	1.465	3.67	-7.10	1.074	1.277	61
8	$y^{(2)}$	-2.20	12.80	19.575	4.354	1.07	-0.41	0.281	3.459	69
	$y^{(3)}$	-3.87	22.06	64.124	2.784	1.55	0.13	0.958	0.509	66
	$y^{(4)}$	-1.06	12.26	57.473	0.947	2.33	-1.45	0.792	0.035	72
	$y^{(5)}$	-3.91	25.02	39.222	3.273	2.91	-2.59	0.903	0.066	43
9	$y^{(2)}$	9.20	-31.08	29.635	3.501	2.17	-2.61	0.628	0.095	47
	$y^{(3)}$	13.76	-47.69	20.445	8.728	3.14	-4.69	1.145	0.791	17
	$y^{(4)}$	10.46	-33.05	23.459	4.807	3.69	-5.61	0.705	1.333	33
	$y^{(5)}$	14.88	-52.69	45.395	9.718	4.09	-8.97	1.584	1.696	28
10	$y^{(2)}$	-1.17	7.92	13.201	7.023	0.73	0.20	0.168	6.580	78
	$y^{(3)}$	0.55	2.71	24.543	4.486	1.52	-1.20	0.340	4.338	72
	$y^{(4)}$	-6.04	27.98	5.457	23.736	0.60	1.09	0.398	19.245	13
	$y^{(5)}$	-2.98	17.10	15.539	16.176	1.05	0.79	0.303	14.055	51
Mean	$y^{(2)}$	1.68	-1.66	20.293	3.953	1.56	-1.19	0.310	3.033	69
	$y^{(3)}$	0.68	4.036	46.491	4.077	2.04	-1.50	0.755	2.224	70
	$y^{(4)}$	-2.97	20.19	52.250	9.534	2.33	-1.26	1.160	4.376	51
	$y^{(5)}$	-1.76	15.18	53.620	9.769	2.89	-3.63	1.353	4.306	45

Table EC.3 Parameter estimates and performance metrics for linear fit (data set 2)

		standard model				DiGAR model				$\widehat{\text{Var}}$
Run		$\hat{\beta}_1$	$\hat{\beta}_0$	$\widehat{\text{Var}}(\hat{\beta}_1)$	L_2	$\hat{\beta}_1$	$\hat{\beta}_0$	$\widehat{\text{Var}}(\hat{\beta}_1)$	L_2	Ratio
1	$y^{(2)}$	3.51	-8.48	18.843	0.215	1.74	-1.33	0.179	0.015	105
	$y^{(3)}$	1.81	0.43	66.998	0.369	2.44	-2.12	0.727	0.355	92
	$y^{(4)}$	4.89	-10.53	72.328	1.415	3.18	-3.61	0.843	1.200	86
	$y^{(5)}$	6.92	-18.68	102.134	0.921	3.72	-5.70	1.349	0.162	76
2	$y^{(2)}$	0.92	1.38	21.543	0.535	1.42	-0.62	0.231	0.503	93
	$y^{(3)}$	8.46	-28.09	16.866	3.127	2.67	-4.65	0.503	0.830	34
	$y^{(4)}$	6.87	-19.13	33.746	1.285	3.58	-5.78	0.519	0.335	65
	$y^{(5)}$	7.06	-19.29	32.933	0.938	3.94	-6.66	0.651	0.132	51
3	$y^{(2)}$	10.58	-37.05	55.559	4.802	2.25	-3.29	0.977	0.022	57
	$y^{(3)}$	10.19	-33.34	92.343	4.366	2.89	-3.76	1.274	0.603	72
	$y^{(4)}$	25.78	-92.34	180.341	45.919	5.32	-9.50	4.571	14.359	39
	$y^{(5)}$	22.21	-78.22	127.947	28.671	5.28	-9.63	3.468	7.316	37
4	$y^{(2)}$	0.58	1.19	9.273	4.937	0.92	-0.20	0.099	4.898	94
	$y^{(3)}$	2.30	-3.92	15.660	2.730	1.66	-1.33	0.195	2.755	80
	$y^{(4)}$	-1.56	13.82	44.532	1.540	2.22	-1.49	0.578	0.381	77
	$y^{(5)}$	-4.99	27.02	44.132	8.274	2.06	-1.51	0.854	4.137	52
5	$y^{(2)}$	2.55	-1.12	61.960	10.137	2.53	-1.05	0.556	10.135	111
	$y^{(3)}$	10.48	-33.58	51.622	6.794	3.60	-5.72	0.945	2.830	55
	$y^{(4)}$	0.99	5.93	70.907	3.160	3.27	-3.31	0.899	2.959	79
	$y^{(5)}$	-0.84	14.21	156.203	4.238	3.57	-3.63	2.022	3.164	77
6	$y^{(2)}$	-5.69	30.67	45.150	6.161	1.43	1.86	0.789	2.857	57
	$y^{(3)}$	-8.69	48.33	98.478	39.637	2.70	2.18	1.914	32.306	51
	$y^{(4)}$	-10.08	55.31	153.068	46.860	3.32	1.06	2.872	36.722	53
	$y^{(5)}$	-1.05	17.39	182.372	17.166	4.53	-5.20	2.510	16.064	73
7	$y^{(2)}$	-3.69	20.31	32.675	2.011	1.13	0.75	0.536	0.266	61
	$y^{(3)}$	-4.51	27.31	83.276	6.180	2.34	-0.42	1.176	3.368	71
	$y^{(4)}$	-4.16	25.03	103.797	2.990	2.55	-2.13	1.629	0.010	64
	$y^{(5)}$	2.89	-2.56	57.997	0.058	3.56	-5.27	1.029	0.047	56
8	$y^{(2)}$	-1.84	12.80	32.380	0.976	1.14	0.75	0.363	0.238	89
	$y^{(3)}$	6.20	-21.90	21.827	14.631	1.49	-2.83	0.394	13.743	55
	$y^{(4)}$	11.98	-41.93	50.547	7.218	2.52	-3.64	1.150	2.167	44
	$y^{(5)}$	14.03	-49.78	43.680	10.175	3.05	-5.31	1.511	3.278	29
9	$y^{(2)}$	0.27	4.30	29.859	0.310	1.50	-0.68	0.331	0.190	90
	$y^{(3)}$	1.89	-3.49	8.805	7.850	1.50	-1.92	0.118	7.878	75
	$y^{(4)}$	7.80	-25.50	21.995	5.158	2.30	-3.23	0.477	3.688	46
	$y^{(5)}$	6.89	-21.79	25.654	7.815	2.16	-2.62	0.469	7.147	55
10	$y^{(2)}$	-1.05	9.21	14.962	1.194	1.44	-0.86	0.240	0.741	62
	$y^{(3)}$	0.18	5.82	48.831	0.559	1.88	-1.04	0.651	0.298	75
	$y^{(4)}$	0.61	5.85	59.645	0.348	2.52	-1.88	0.823	0.050	72
	$y^{(5)}$	8.66	-23.22	99.961	9.263	4.46	-6.24	1.338	7.637	75
Mean	$y^{(2)}$	0.61	3.32	32.220	3.128	1.55	-0.47	0.430	1.986	82
	$y^{(3)}$	2.83	-4.24	50.471	8.624	2.32	-2.16	0.790	6.497	66
	$y^{(4)}$	4.31	-8.35	79.091	11.589	3.08	-3.35	1.436	6.187	63
	$y^{(5)}$	6.18	-15.49	87.301	8.752	3.63	-5.18	1.520	4.908	58

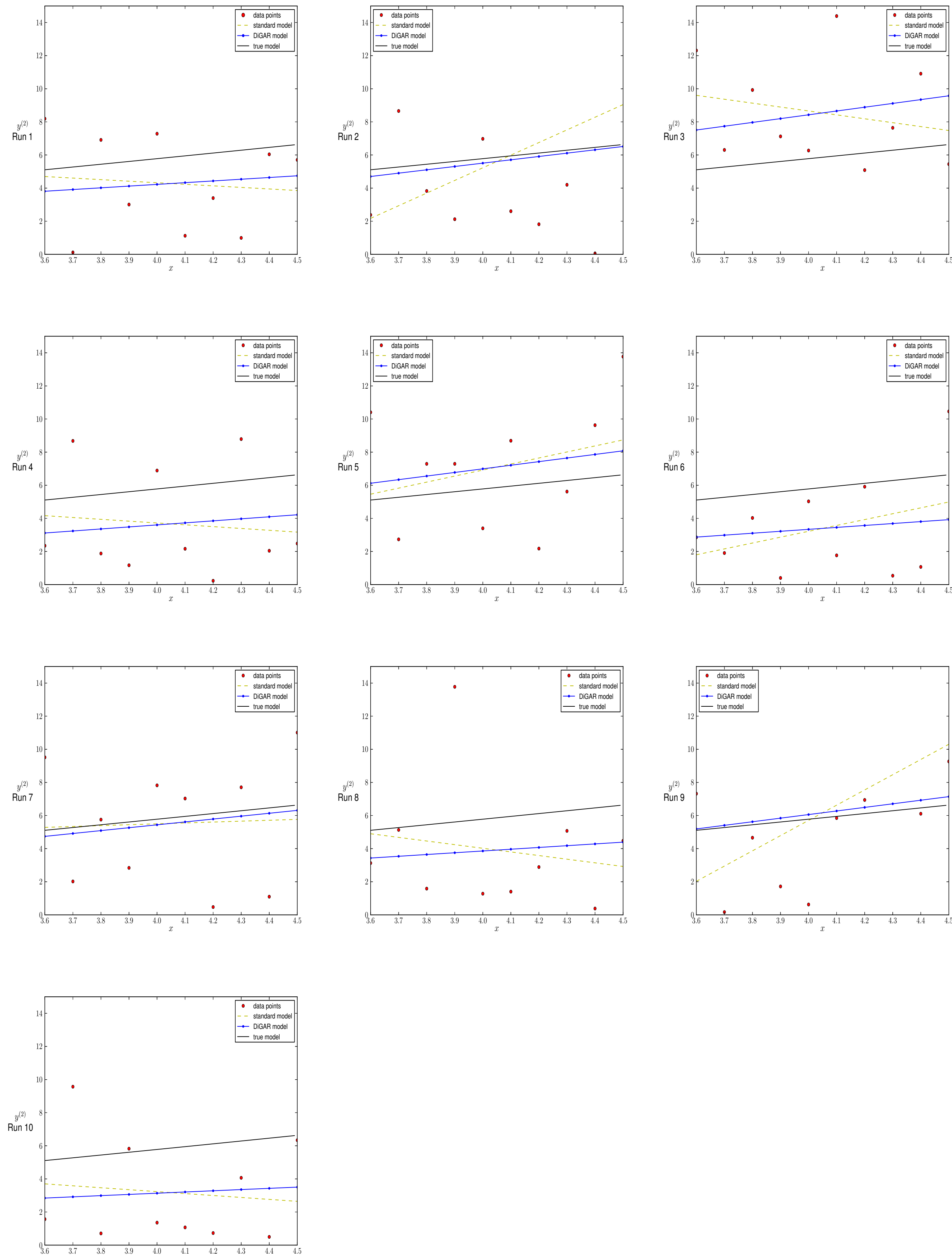

Figure EC.1 $M/M/1$ **queue: fitted models vs. true model for** $y^{(2)}$**, 10 different runs, noting in particular that the standard model has the incorrect slope for graphs 1, 3, 4, 8, 10.**

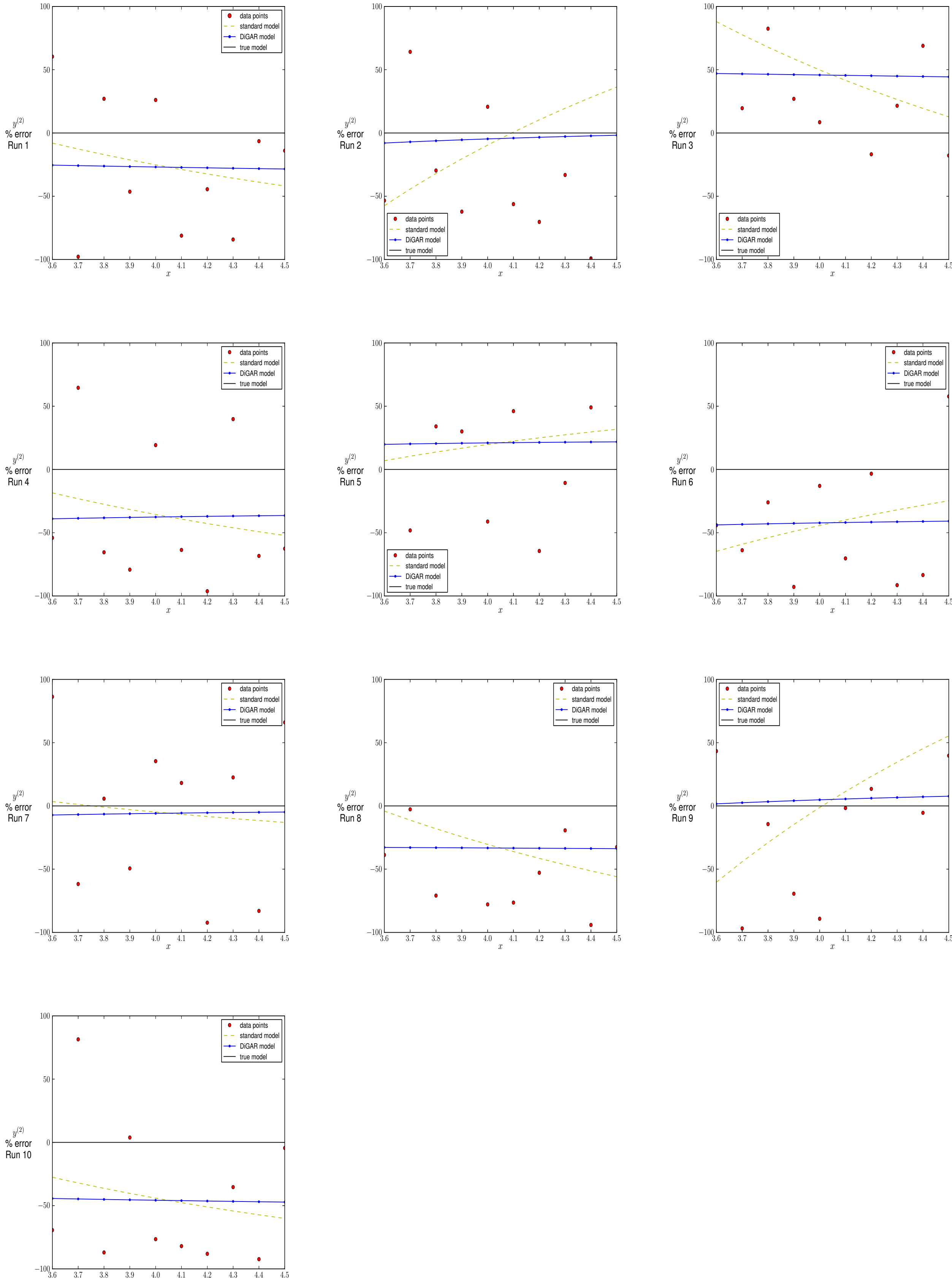

Figure EC.2 Relative error of fitted models for $y^{(2)}$, 10 different runs.

Table EC.4 Optimal value x^* obtained from each of the fitted functions (data set 1)

	$c=1$			$c\approx 27$		
	true	standard	DiGAR	true	standard	DiGAR
$y^{(2)}$	0.88	4.0	2.5	4.00	4.0	4.1
$y^{(3)}$	0.85	4.0	1.7	3.51	4.1	3.8
$y^{(4)}$	0.87	$\boxed{0.4}$	7.8	3.43	$\boxed{-119}$	0.6
$y^{(5)}$	0.84	$\boxed{3.8}$	6.3	3.14	$\boxed{3.5}$	6.9
	$c=100$			$c=500$		
	true	standard	DiGAR	true	standard	DiGAR
$y^{(2)}$	7.38	4.2	5.2	16.04	4.7	5.9
$y^{(3)}$	6.24	4.2	5.1	13.22	4.8	5.8
$y^{(4)}$	5.88	8.2	5.6	11.94	6.4	6.0
$y^{(5)}$	5.30	$\boxed{1.8}$	5.9	10.69	7.8	6.0

boxed entries indicate a maximum rather than a minimum

Table EC.5 Optimal value x^* obtained from each of the fitted functions (data set 2)

	$c=1$			$c\approx 27$		
	true	standard	DiGAR	true	standard	DiGAR
$y^{(2)}$	0.88	$\boxed{5.1}$	1.5	4.00	$\boxed{4.8}$	4.0
$y^{(3)}$	0.85	$\boxed{7.4}$	1.2	3.51	$\boxed{9.9}$	3.7
$y^{(4)}$	0.87	$\boxed{4.5}$	0.8	3.43	$\boxed{4.4}$	3.3
$y^{(5)}$	0.84	$\boxed{7.8}$	2.2	3.14	$\boxed{9.8}$	3.4
	$c=100$			$c=500$		
	true	standard	DiGAR	true	standard	DiGAR
$y^{(2)}$	7.38	21.5	5.2	16.04	6.3	5.9
$y^{(3)}$	6.24	5.1	5.1	13.22	5.9	5.8
$y^{(4)}$	5.88	$\boxed{3.8}$	4.9	11.94	8.8	5.8
$y^{(5)}$	5.30	4.3	4.6	10.69	5.9	5.7

boxed entries indicate a maximum rather than a minimum

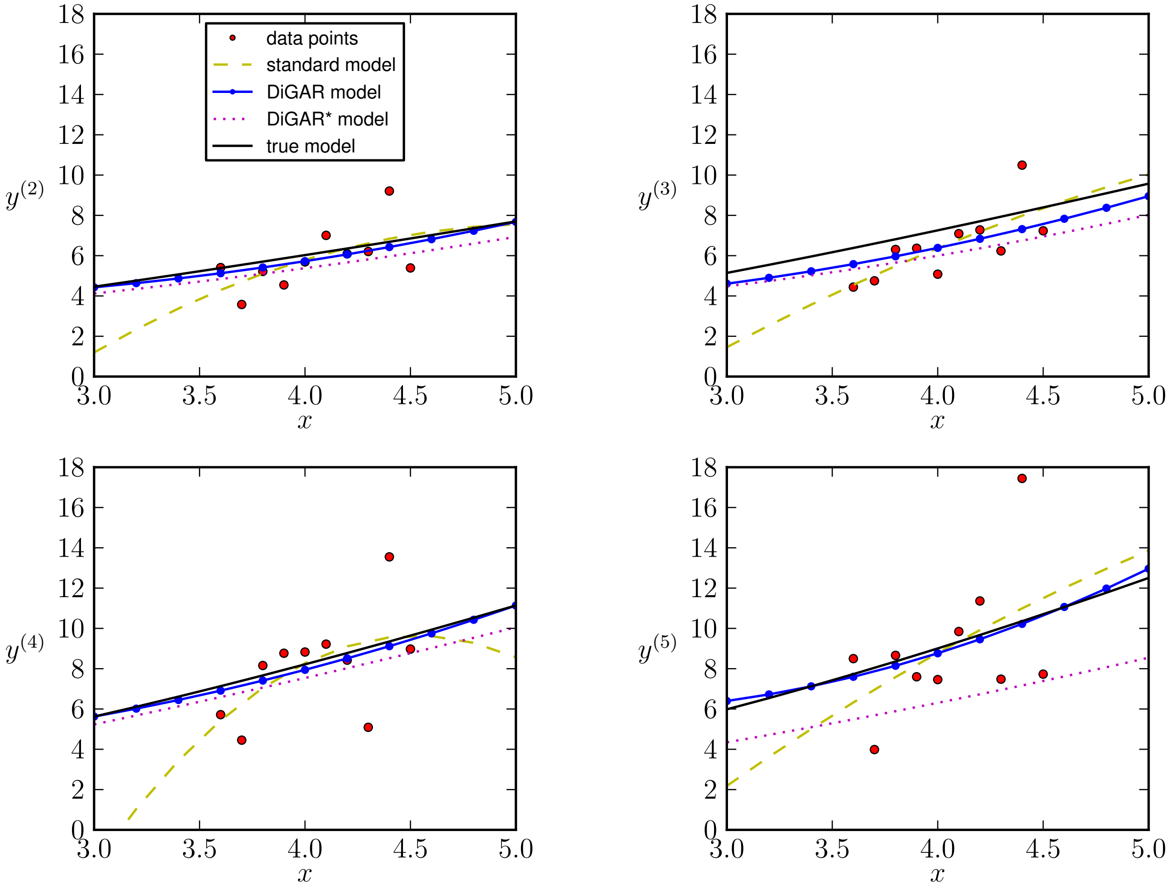

Figure EC.3 $M/M/1$ **queue quadratic fit (**$c = 1$**, 10 replications, data set 1).**

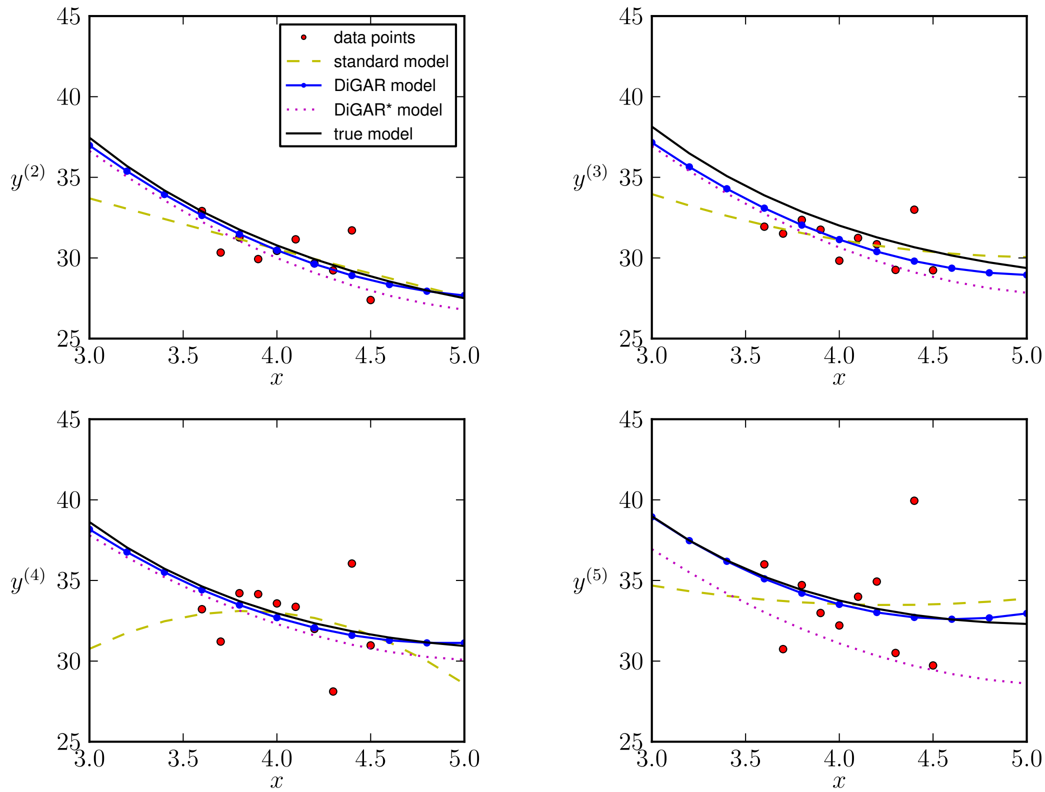

Figure EC.4 $M/M/1$ **queue quadratic fit (**$c = 100$**, 10 replications, data set 1).**

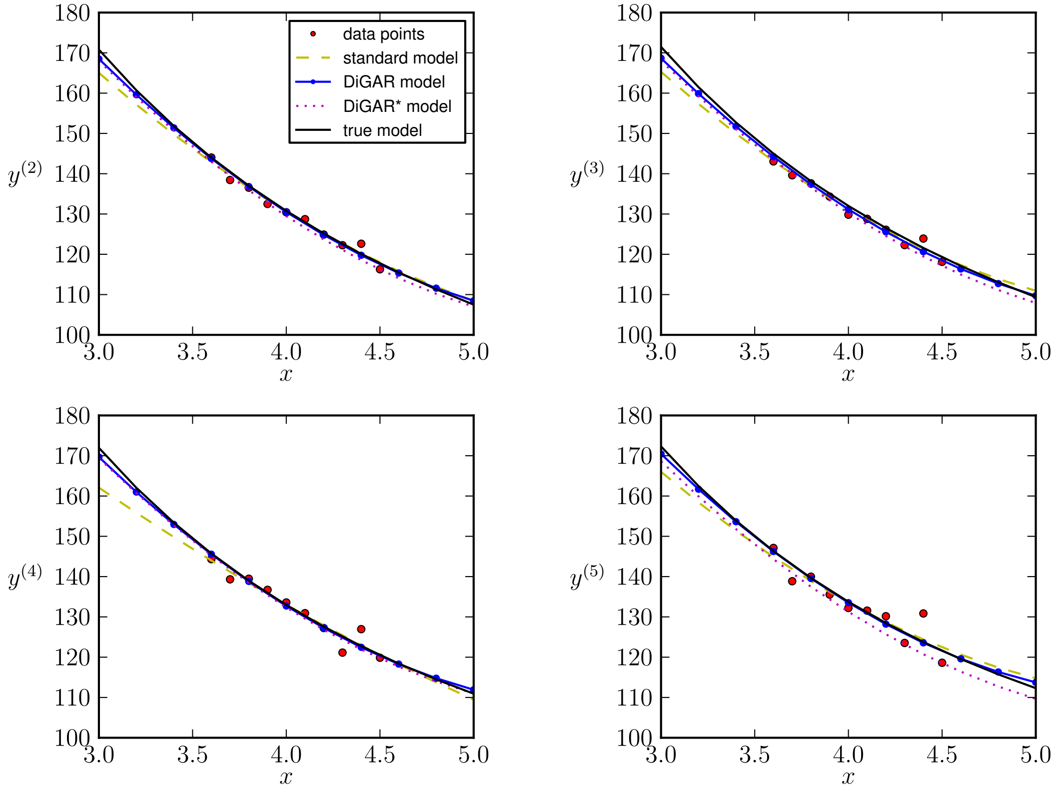

Figure EC.5 $M/M/1$ **queue quadratic fit (**$c = 500$**, 10 replications, data set 1).**

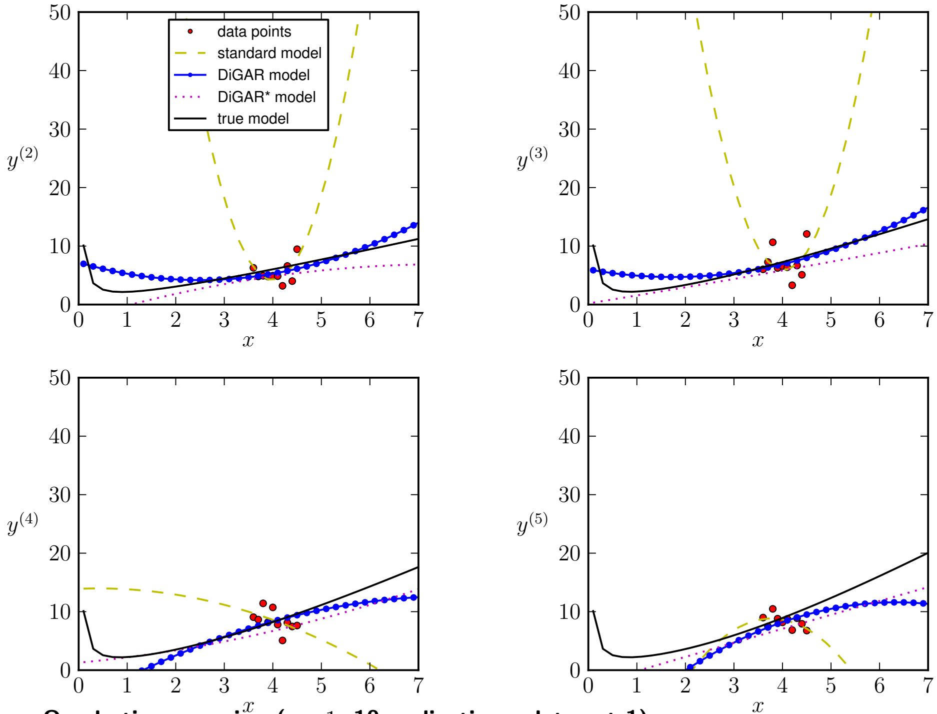

Figure EC.6 Quadratic regression ($c = 1$, 10 replications, data set 1).

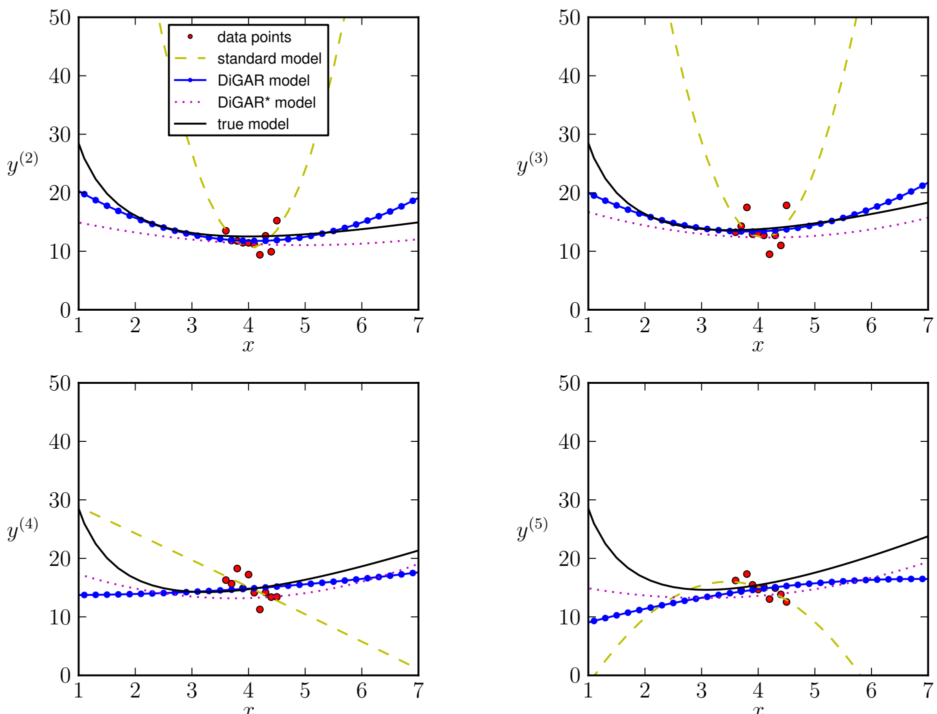

Figure EC.7 Quadratic fit ($c \approx 27$, 10 replications, data set 1).

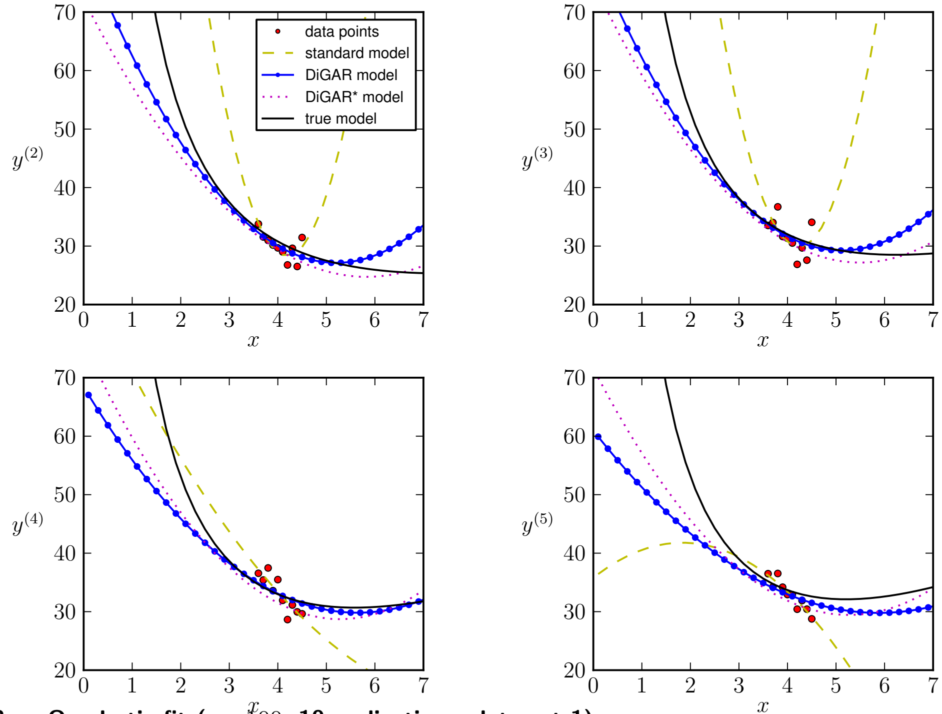

Figure EC.8 Quadratic fit ($c = 100$, 10 replications, data set 1).

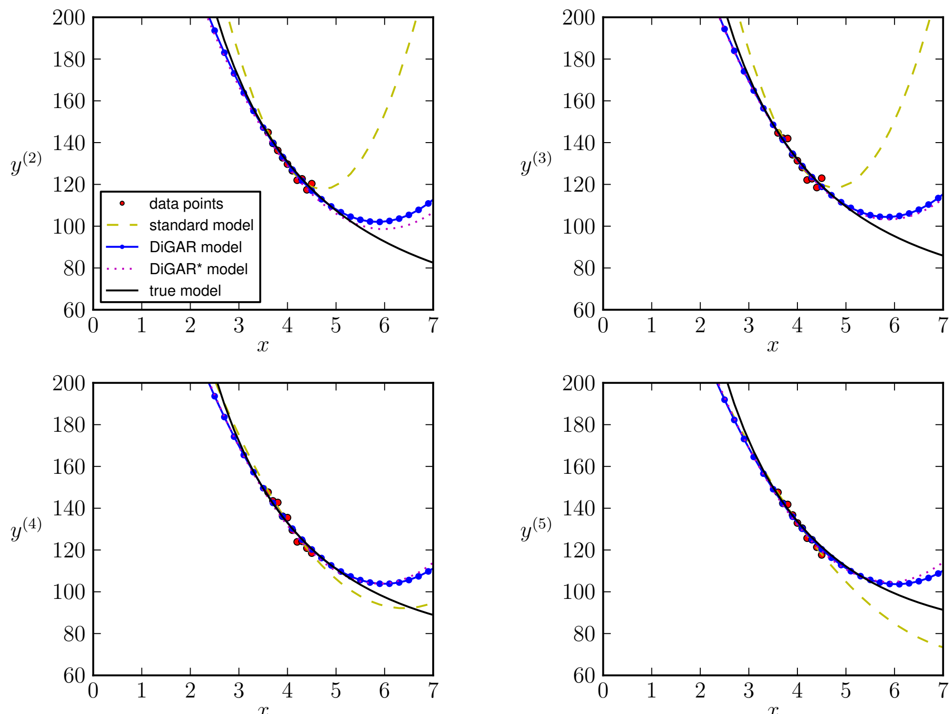

Figure EC.9 Quadratic fit ($c = 500$, 10 replications, data set 1).

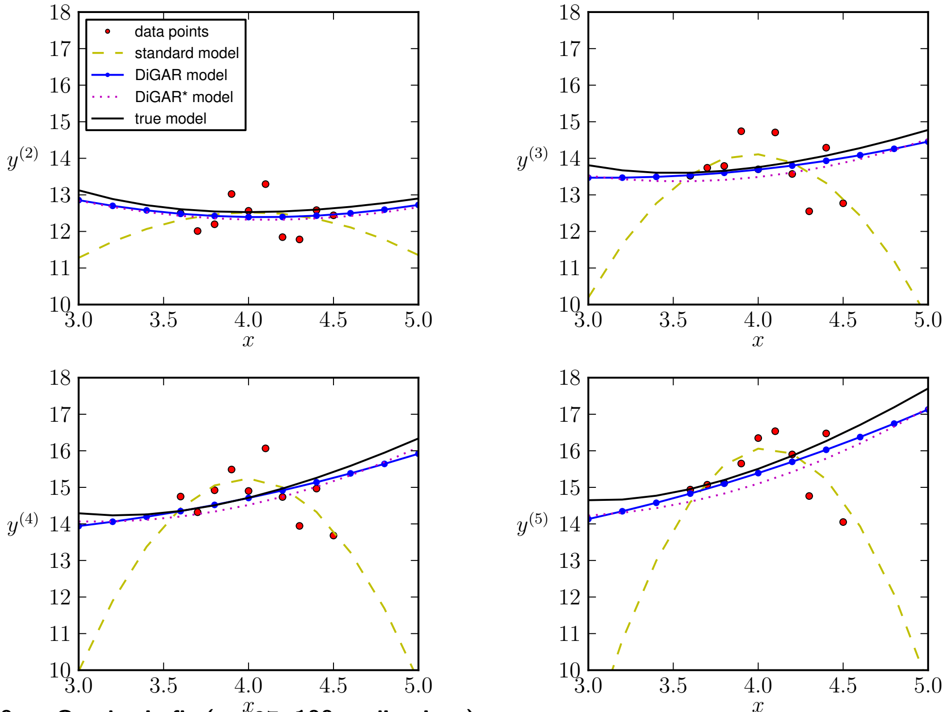

Figure EC.10 **Quadratic fit ($c \approx 27$, 100 replications).**

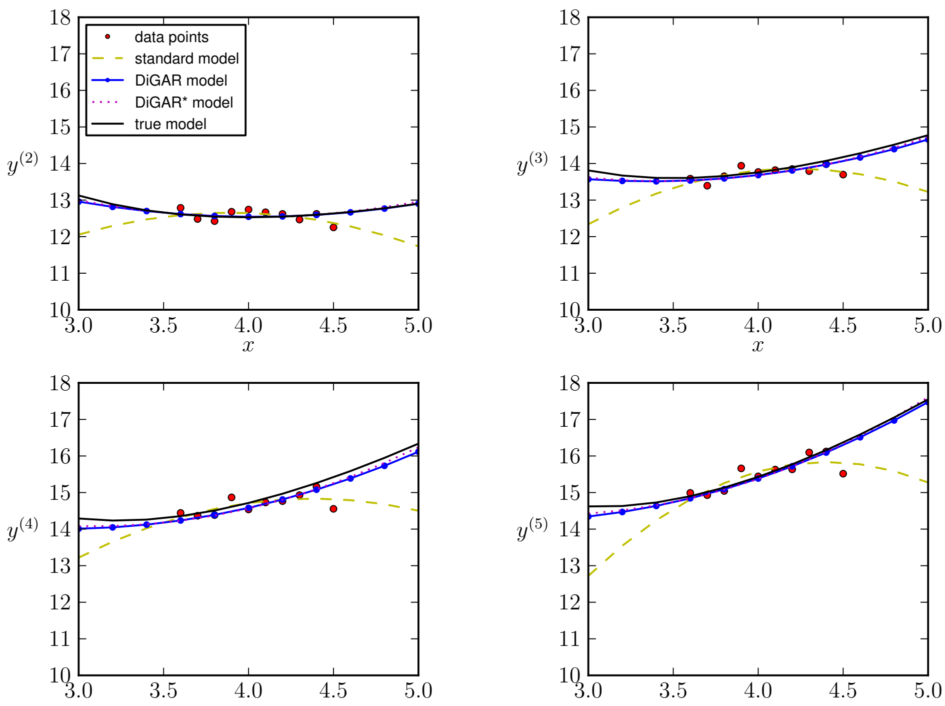

Figure EC.11 **Quadratic fit ($c \approx 27$, 1000 replications).**

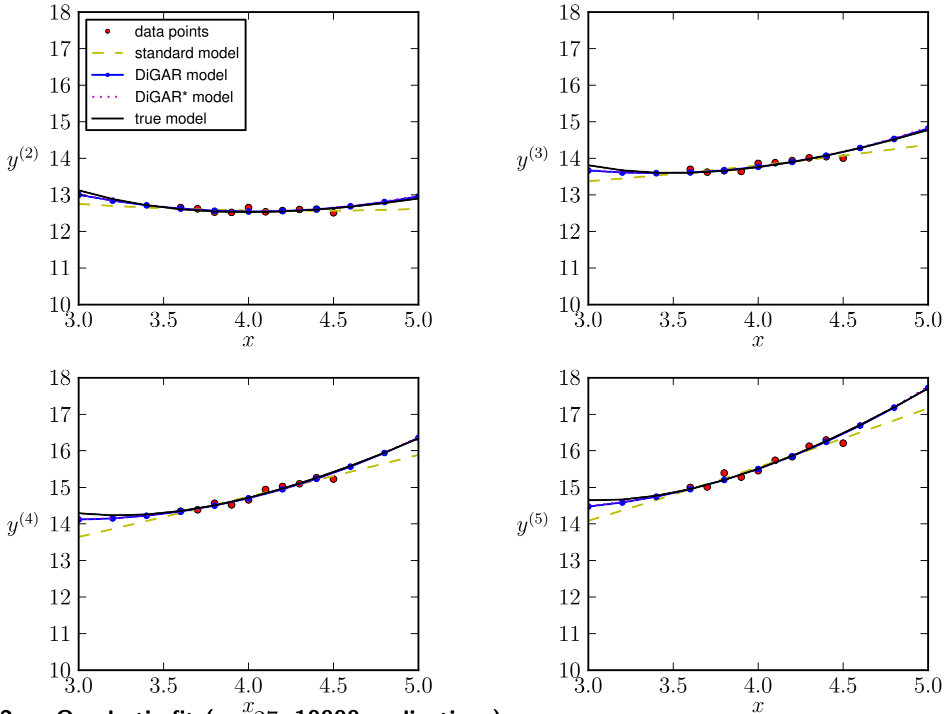

Figure EC.12 Quadratic fit ($c \approx 27$, 10000 replications).

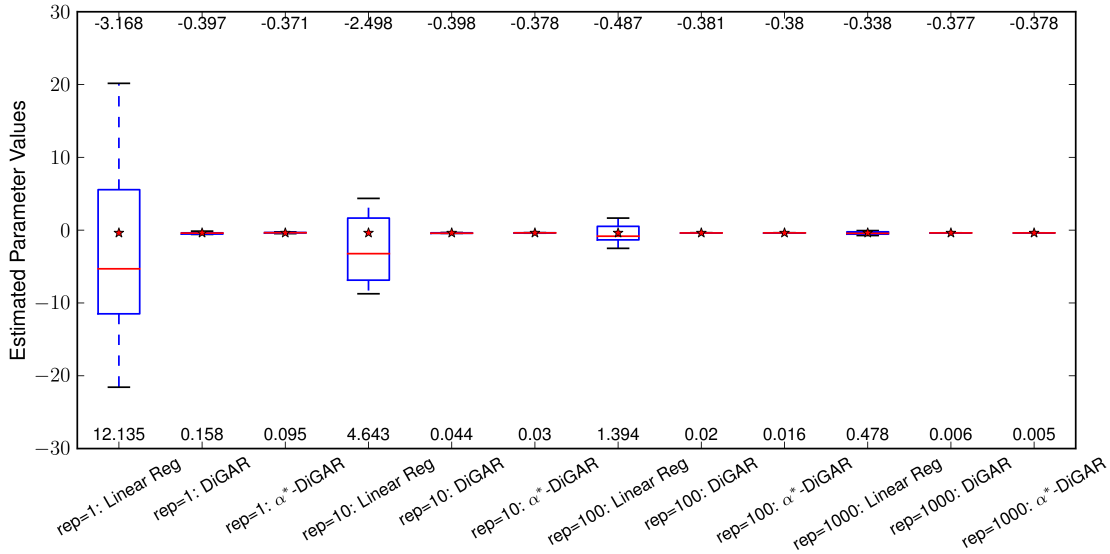

Figure EC.13 $U/U/1$ **queue box plots of** β_1 **for** $y^{(2)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_1 **=-0.375)**

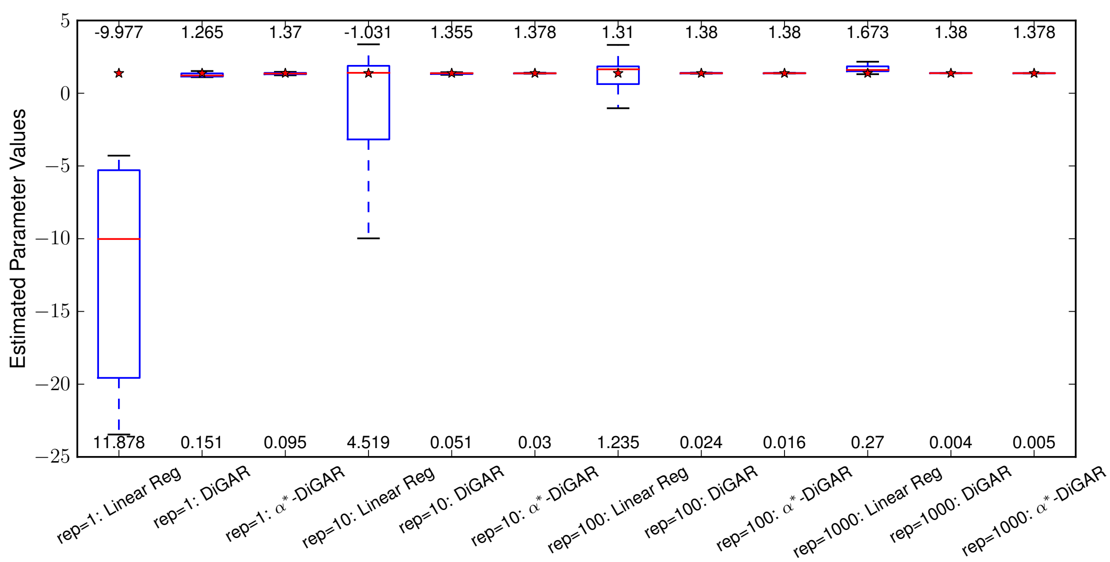

Figure EC.14 $U/U/1$ **queue box plots of** β_2 **for** $y^{(2)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_2 **=1.375)**

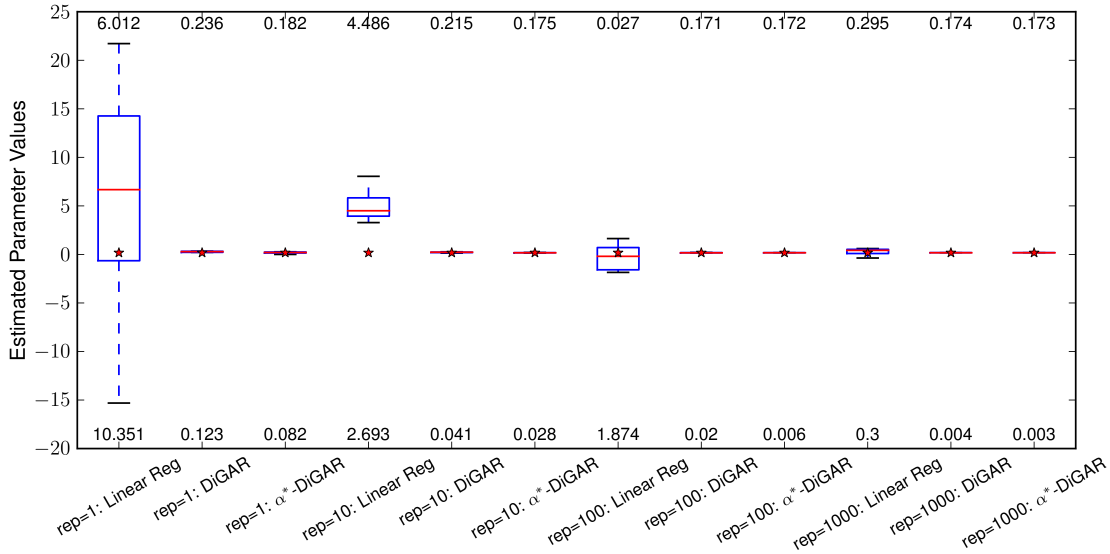

Figure EC.15 $U/U/1$ **queue box plots of** β_3 **for** $y^{(2)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_3 **=0.171)**

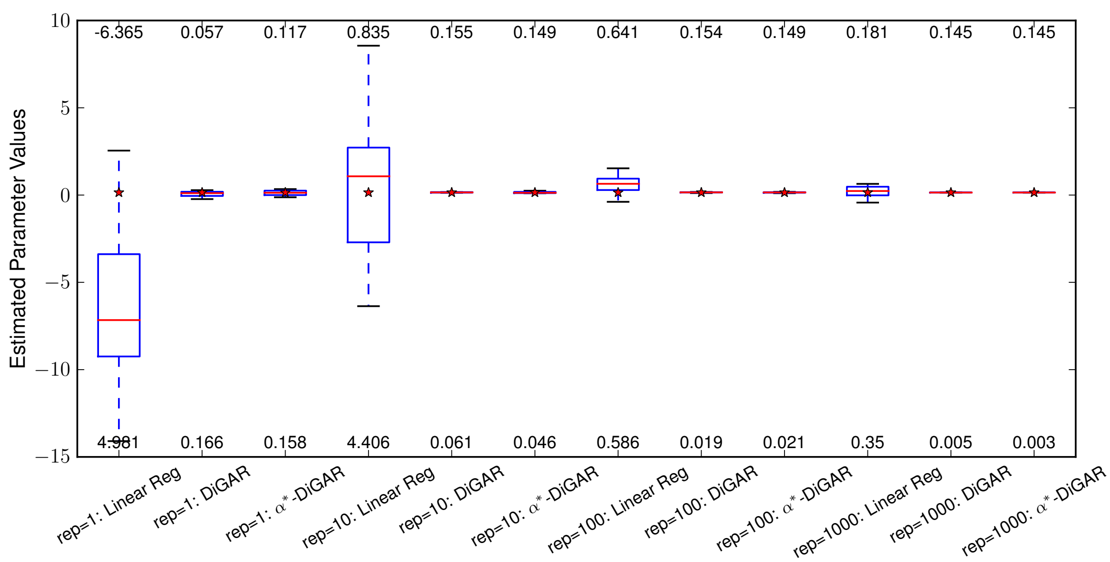

Figure EC.16 $U/U/1$ **queue box plots of** β_4 **for** $y^{(2)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_4 **=0.146)**

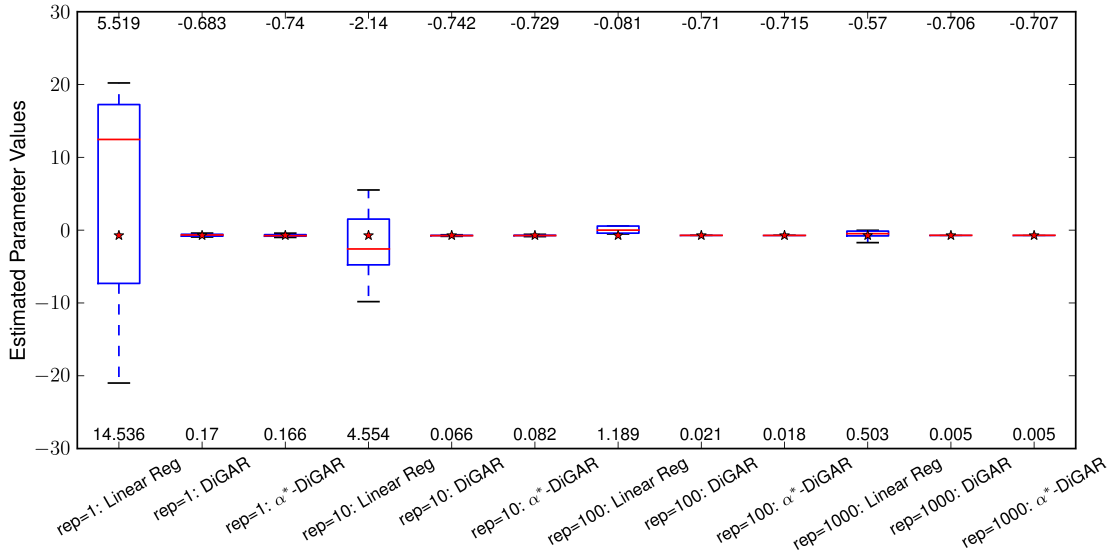

Figure EC.17 $U/U/1$ **queue box plots of** β_1 **for** $y^{(3)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_1 **=-0.720)**

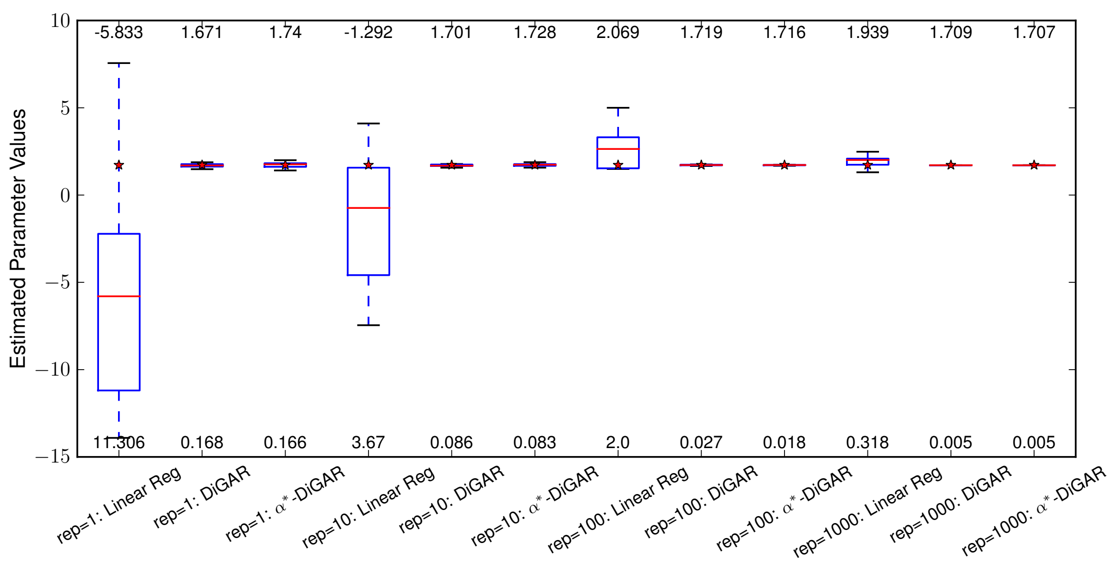

Figure EC.18 $U/U/1$ **queue box plots of** β_2 **for** $y^{(3)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_2 **=1.720)**

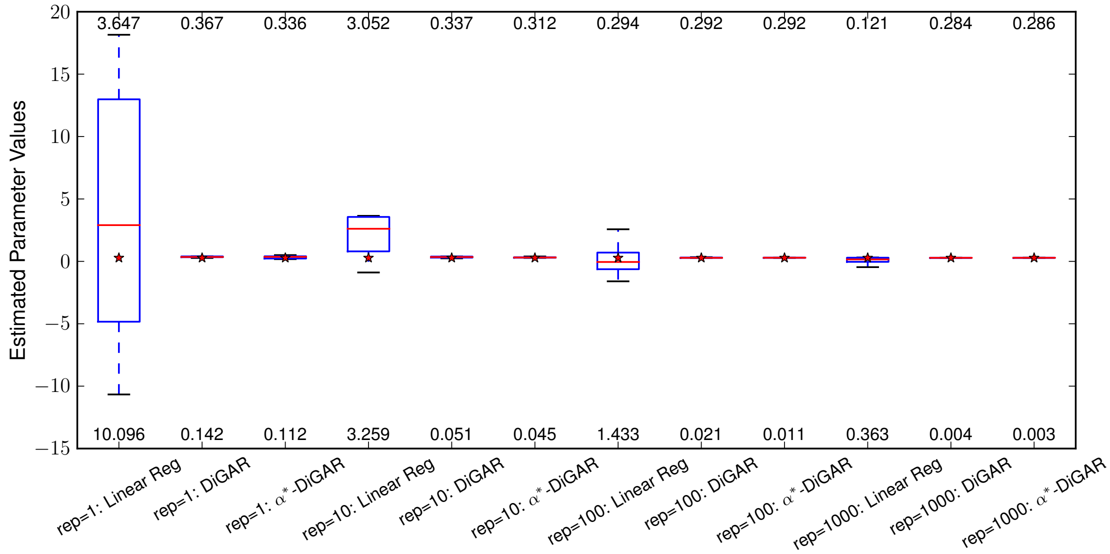

Figure EC.19 $U/U/1$ **queue box plots of** β_3 **for** $y^{(3)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_3 =$**0.279)**

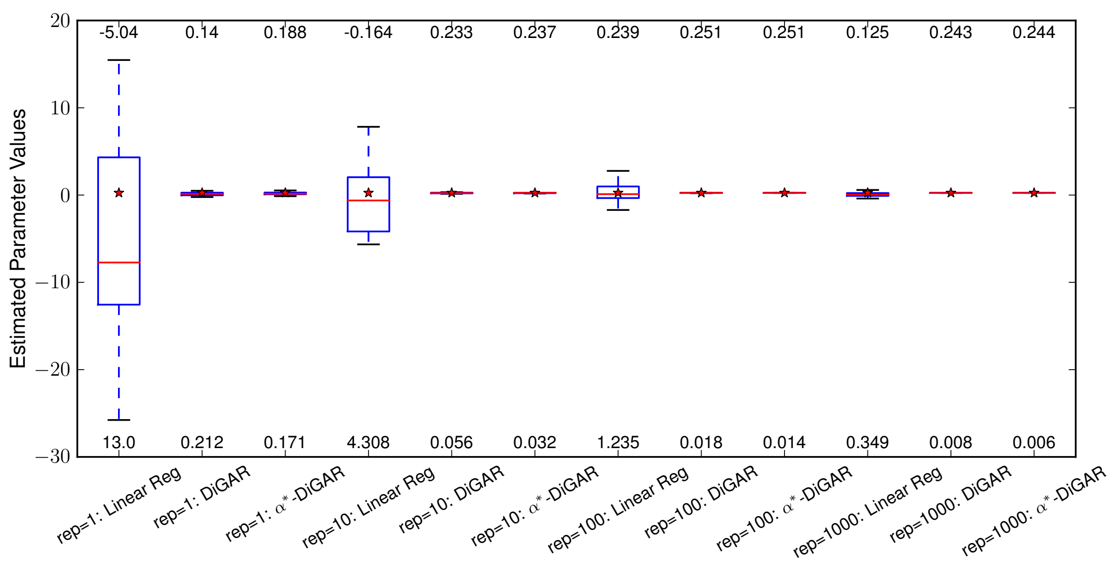

Figure EC.20 $U/U/1$ **queue box plots of** β_4 **for** $y^{(3)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_4 =$**0.255)**

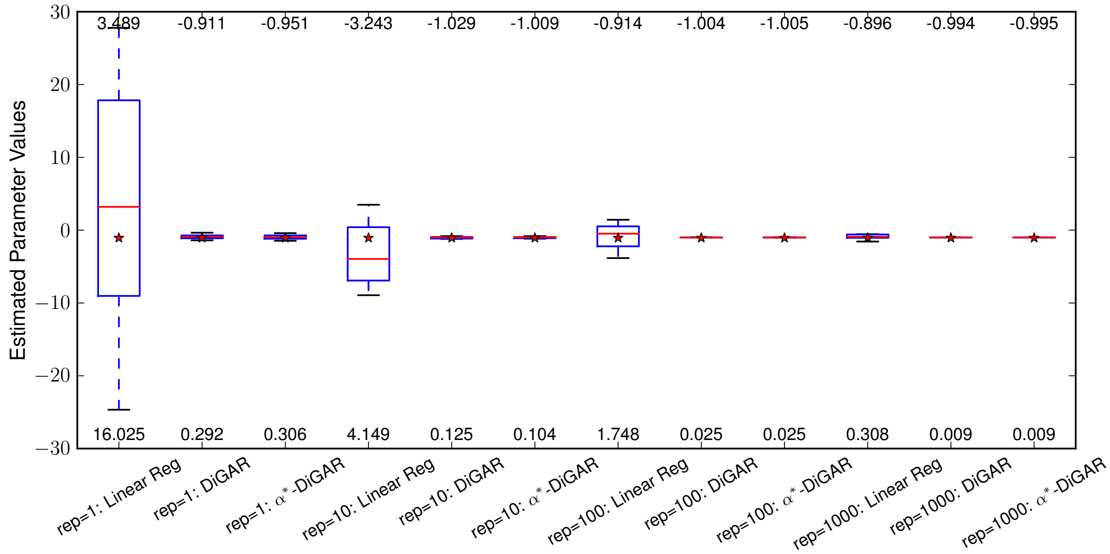

Figure EC.21 $U/U/1$ **queue box plots of** β_1 **for** $y^{(4)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_1 =$**-1.065)**

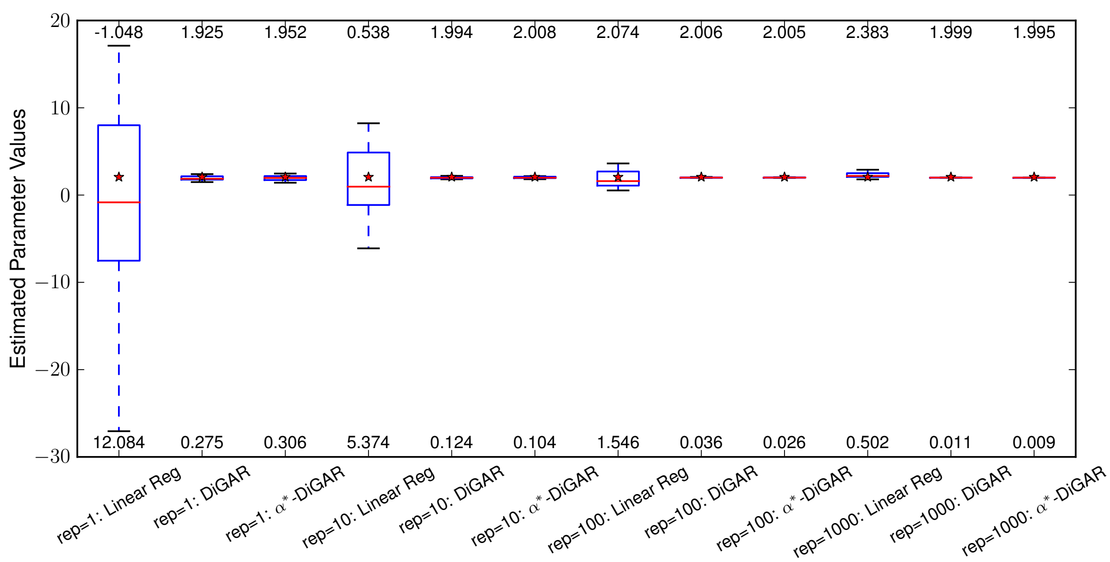

Figure EC.22 $U/U/1$ **queue box plots of** β_2 **for** $y^{(4)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_2 =$**2.065)**

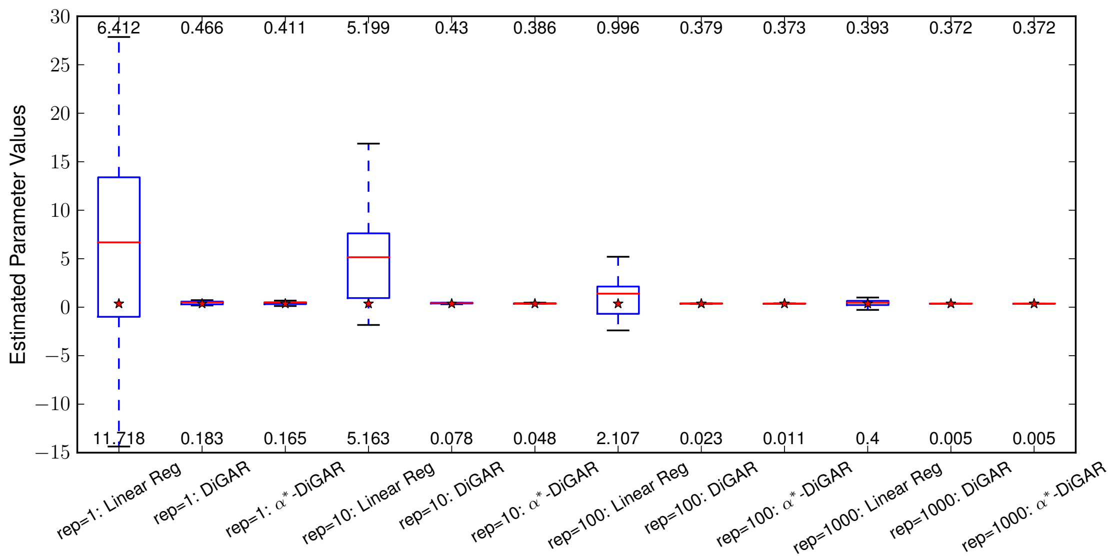

Figure EC.23 $U/U/1$ **queue box plots of** β_3 **for** $y^{(4)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_3 =$**0.362)**

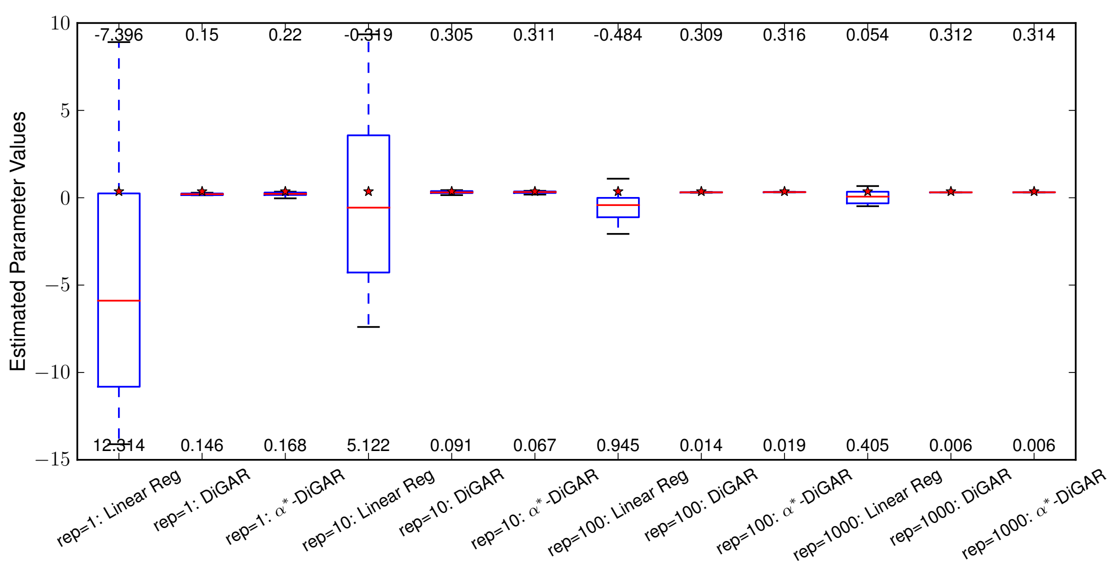

Figure EC.24 $U/U/1$ **queue box plots of** β_4 **for** $y^{(4)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_4 =$**0.360)**

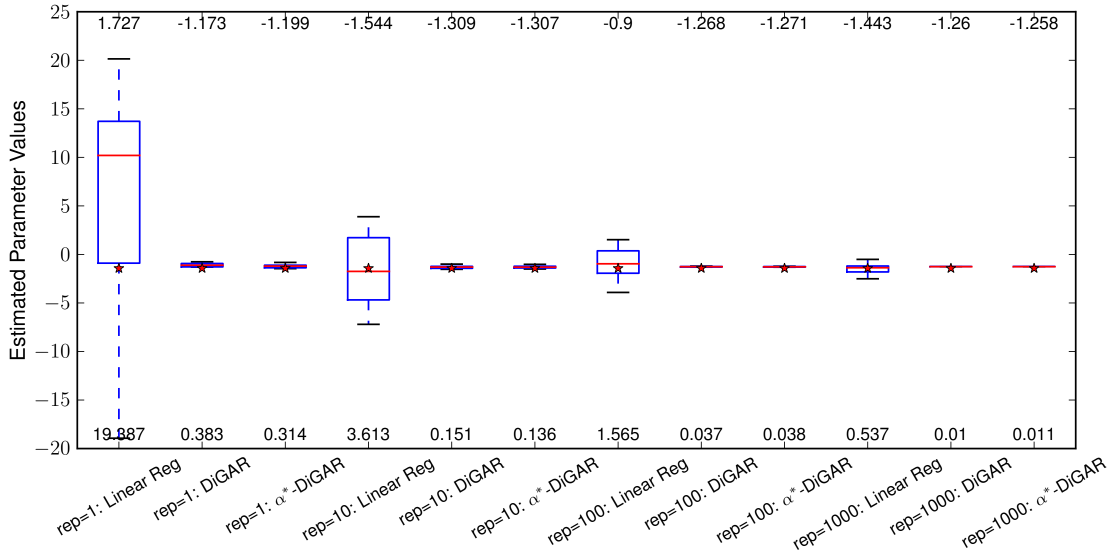

Figure EC.25 $U/U/1$ **queue box plots of** β_1 **for** $y^{(5)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_1 **=-1.427)**

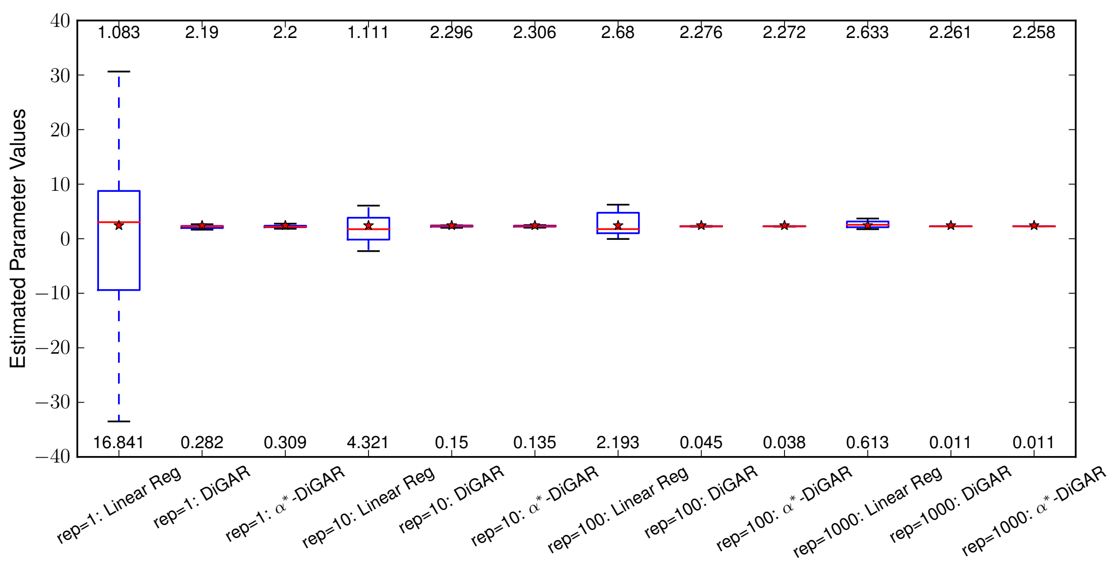

Figure EC.26 $U/U/1$ **queue box plots of** β_2 **for** $y^{(5)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** β_2 **=2.427)**

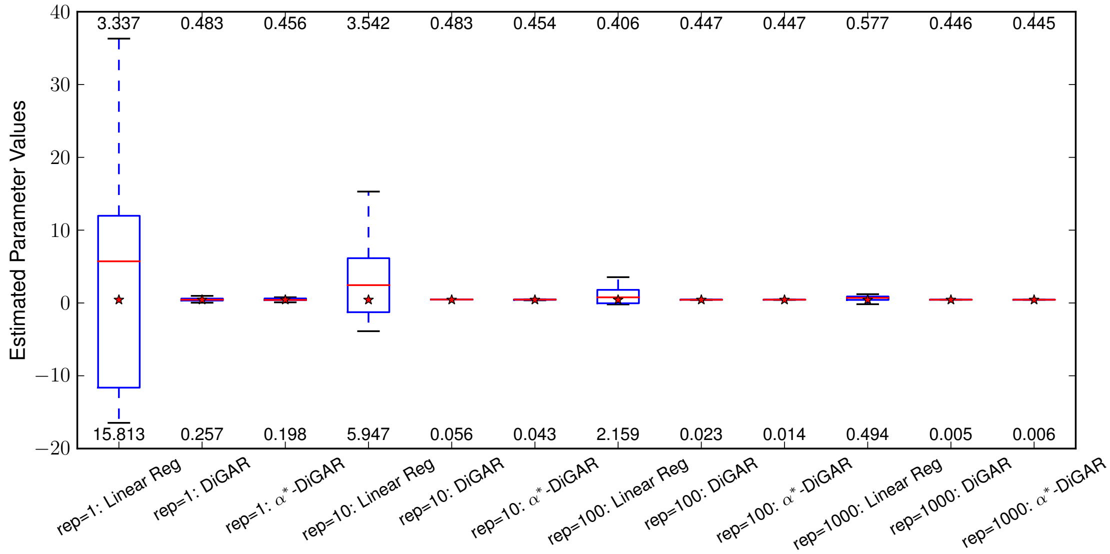

Figure EC.27 $U/U/1$ **queue box plots of** β_3 **for** $y^{(5)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_3 =$**0.431)**

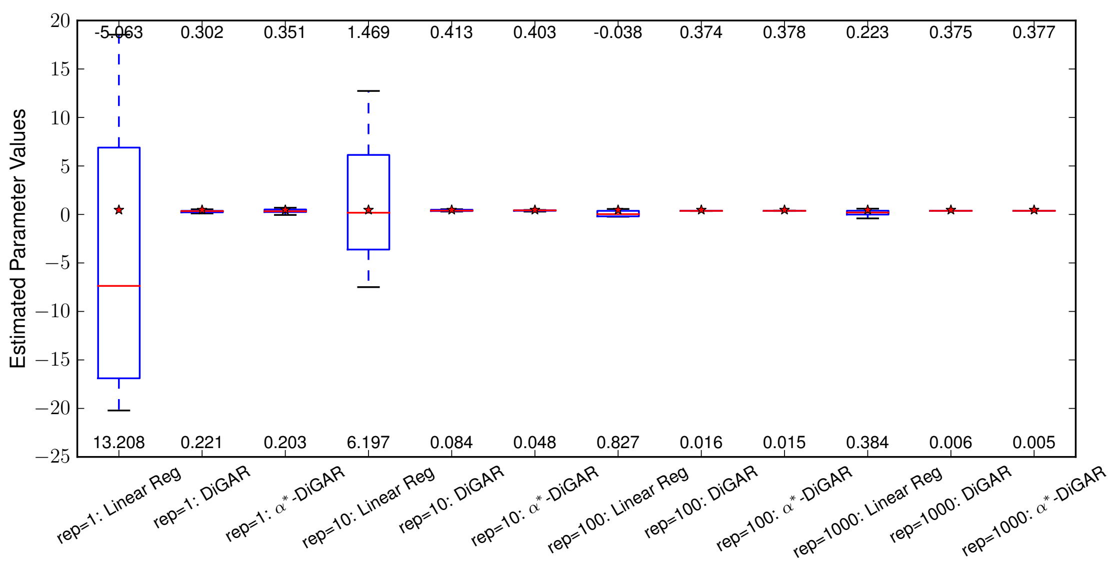

Figure EC.28 $U/U/1$ **queue box plots of** β_4 **for** $y^{(5)}$ **w.r.t. # reps/design point (sample mean at top, sample standard deviation at bottom, true** $\beta_4 =$**0.467)**

EC.6. More Details on the Sphere Function Example

An example of the covariance matrix $\mathbf{\Sigma}$ at the low correlation level $\rho = 0.2$ is the following:

$$\mathbf{\Sigma} = \begin{pmatrix} 10.0 & 2.83 & 3.64 & 4.00 & 4.47 \\ 2.83 & 20.0 & 4.90 & 5.66 & 6.32 \\ 3.64 & 4.90 & 30.0 & 6.93 & 7.75 \\ 4.00 & 5.66 & 6.93 & 40.0 & 8.94 \\ 4.47 & 6.32 & 7.75 & 8.94 & 50.0 \end{pmatrix}.$$

Table EC.6 Parameter estimates and performance metrics for the sphere function for two-level full factorial centered design around $(1, -0.6, 0.8, -0.5)$; 10 replication per design point, gridsize 0.5, 1000 macroreplications.

			Linear Regression		DiGAR		α^*-DiGAR	
correlation		true value	value	MSE	value	MSE	value	MSE
low $\rho = -0.2$	$\hat{\beta}_1$	2.0	2.029	0.263	2.015	0.090	2.017	0.081
	$\hat{\beta}_2$	-1.2	-1.171	0.262	-1.215	0.131	-1.203	0.107
	$\hat{\beta}_3$	1.6	1.588	0.257	1.609	0.165	1.601	0.128
	$\hat{\beta}_4$	-1.0	-0.996	0.250	-0.967	0.216	-0.979	0.139
medium $\rho = -0.5$	$\hat{\beta}_1$	2.0	1.997	0.515	1.998	0.139	1.998	0.138
	$\hat{\beta}_2$	-1.2	-1.227	0.535	-1.198	0.190	-1.206	0.181
	$\hat{\beta}_3$	1.6	1.583	0.516	1.596	0.206	1.591	0.190
	$\hat{\beta}_4$	-1.0	-0.976	0.536	-0.998	0.247	-0.989	0.218
high $\rho = -0.8$	$\hat{\beta}_1$	2.0	2.009	0.656	2.013	0.194	2.012	0.187
	$\hat{\beta}_2$	-1.2	-1.206	0.680	-1.198	0.242	-1.200	0.239
	$\hat{\beta}_3$	1.6	1.609	0.724	1.605	0.261	1.606	0.262
	$\hat{\beta}_4$	-1.0	-0.939	0.695	-0.961	0.325	-0.952	0.309

Table EC.7 Parameter estimates and performance metrics for the sphere function for two-level full factorial centered design around $(1, -0.6, 0.8, -0.5)$; 10 replication per design point, gridsize 0.05, 1000 macroreplications.

			Linear Regression		DiGAR		α^*-DiGAR	
correlation		true value	value	MSE	value	MSE	value	MSE
low $\rho = -0.2$	$\hat{\beta}_1$	2.0	1.953	24.775	2.002	0.111	2.002	0.111
	$\hat{\beta}_2$	-1.2	-1.352	25.365	-1.188	0.173	-1.189	0.172
	$\hat{\beta}_3$	1.6	1.503	26.140	1.589	0.218	1.588	0.218
	$\hat{\beta}_4$	-1.0	-1.068	26.955	-1.009	0.291	-1.010	0.290
medium $\rho = -0.5$	$\hat{\beta}_1$	2.0	1.947	55.213	2.035	0.179	2.035	0.180
	$\hat{\beta}_2$	-1.2	-1.217	54.511	-1.206	0.241	-1.206	0.242
	$\hat{\beta}_3$	1.6	1.642	51.917	1.603	0.291	1.603	0.293
	$\hat{\beta}_4$	-1.0	-0.860	50.657	-1.022	0.334	-1.021	0.337
high $\rho = -0.8$	$\hat{\beta}_1$	2.0	1.714	71.059	1.977	0.270	1.977	0.271
	$\hat{\beta}_2$	-1.2	-0.688	72.366	-1.187	0.310	-1.185	0.310
	$\hat{\beta}_3$	1.6	1.493	70.225	1.589	0.354	1.589	0.356
	$\hat{\beta}_4$	-1.0	-1.302	70.437	-0.996	0.455	-0.999	0.451