Electronic Companion to "Estimating Sensitivities of Portfolio Credit Risk using Monte Carlo"

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EC.1. Technical Remarks

Proof of Corollary 1. Let $X(\theta)=\max_{i=1,\dots,k} X_i(\theta)$. Then $X(\theta)$ is non-differentiable when there exist some $i\neq j$ such that $X(\theta)=X_i(\theta)=X_j(\theta)$ or $X_\ell(\theta)$ is non-differentiable for $\ell=1,2,\dots,k$. Because $X_\ell(\theta)$ is differentiable w.p.1 and $\Pr\{X_i(\theta)=X_j(\theta)\}=0$, $X(\theta)$ is also differentiable w.p.1 at any $\theta\in\mathcal{N}(\theta_0)$. Let $\mathcal{K}=\sum_{i=1}^k \mathcal{K}_i$. Then, $\mathrm{E}[\mathcal{K}]<\infty$ and $|X(\theta_0+\Delta\theta)-X(\theta_0)|\le \mathcal{K}\cdot|\Delta\theta|$ for any $\Delta\theta$ that is close enough to 0.

Let $i^*=\mathrm{argmax}_{i=1,\dots,k}X_i(\theta)$. Then, $X'(\theta)=X'_{i^*}(\theta)$ w.p.1. Note that

$$\begin{aligned}\psi(\theta,t) &= \mathrm{E}\left[X'(\theta);X(\theta)=t\right] \;=\; \mathrm{E}\left[X'_{i^*}(\theta);X_{i^*}(\theta)=t\right] \;=\; \sum_{i=1}^k \mathrm{E}\left[X'_i(\theta)\mathbf{1}_{\{i^*=i\}};X_i(\theta)=t\right]\\ &= \sum_{i=1}^k \mathrm{E}\left[X'_i(\theta)\prod_{j=1,j\neq i}^k \mathbf{1}_{\{X_j(\theta)<X_i(\theta)\}};X_i(\theta)=t\right] \;=\; \sum_{i=1}^k \psi_i(\theta,t).\end{aligned}$$

Therefore, $\psi(\theta,t)$ is continuous at $(\theta_0,0)$ because $\psi_i(\theta,t)$ are continuous at $(\theta_0,0)$. Then, the conclusion of this corollary follows directly from Lemma 1.

Verification of Conditions (a1)–(a3). We next verify that Conditions (a1)–(a3) in Section 3.1 imply the conditions of Theorem 1. Note that $X_i(\theta)=\epsilon_i-\Upsilon_i(\theta)$ and ϵ_i is a continuous random variable that is independent of all other variable factors, so Conditions 1 and 2 of Theorem 1 are satisfied. Furthermore, note that

$$\psi_i(\theta,y)=-\mathrm{E}\left[\Upsilon'_i(\theta)\prod_{j=1,j\neq i}^m \mathbf{1}_{\{(-1)^{a_j}[\epsilon_j-\Upsilon_j(\theta)]<y\}};\epsilon_i=\Upsilon_i(\theta)+y\right].$$

Let $\boldsymbol{\xi}_i$ denote all random factors other than ϵ_i. Note that $\boldsymbol{\xi}_i$ and ϵ_i are mutually independent, and $\Upsilon_i(\theta)$ and $\Upsilon'_i(\theta)$ are fully determined when $\boldsymbol{\xi}_i$ is given. By conditioning on $\boldsymbol{\xi}_i$, we have

$$\psi_i(\theta,y) = -\mathrm{E}\left\{\mathrm{E}\left[\Upsilon'_i(\theta)\prod_{j=1,j\neq i}^m \mathbf{1}_{\{(-1)^{a_j}[\epsilon_j-\Upsilon_j(\theta)]<y\}};\epsilon_i=\Upsilon_i(\theta)+y\;\Big|\;\boldsymbol{\xi}_i\right]\right\}$$

$$= -\mathrm{E}\left[\Upsilon_i'(\theta)\prod_{j=1,j\neq i}^{m}\mathbf{1}_{\{(-1)^{a_j}[\epsilon_j-\Upsilon_j(\theta)]<y\}}\cdot \mathrm{E}\left[1;\epsilon_i=\Upsilon_i(\theta)+y \;\middle|\; \Upsilon_i(\theta)\right]\right]$$

$$= -\mathrm{E}\left[\Upsilon_i'(\theta)\prod_{j=1,j\neq i}^{m}\mathbf{1}_{\{(-1)^{a_j}[\epsilon_j-\Upsilon_j(\theta)]<y\}}\cdot f_{\epsilon_i}\left(\Upsilon_i(\theta)+y\right)\right], \quad \text{(EC.1)}$$

where Equation (EC.1) follows from Equation (2) and the independence between ϵ_i and $\Upsilon_i(\theta)$. Given the conditions of the theorem, we can easily check that the term inside of the expectation of Equation (EC.1) is continuous w.p.1 with respect to (θ, y) and is dominated by a random variable K_iB_i with a finite first moment. Then, by the dominated convergence theorem (Durrett 2005), $\psi_i(\theta, y)$ is continuous in (θ, y), i.e., Condition 3 of Theorem 1 also holds. Therefore, the conclusion of Theorem 1 holds.

Then, by Theorem 1,

$$p'(\theta) = \sum_{i=1}^{m} \mathrm{E}\left\{\left[g\left(L_{-i}+l_i\right)-g\left(L_{-i}\right)\right]\cdot\Upsilon_i'(\theta)\,;\,\epsilon_i=\Upsilon_i\right\}$$

$$= \sum_{i=1}^{m} \mathrm{E}\left\{\left[g\left(L_{-i}+l_i\right)-g\left(L_{-i}\right)\right]\cdot\Upsilon_i'(\theta)\cdot f_{\epsilon_i}(\Upsilon_i)\right\},$$

where the last equation can be derived similar to Equation (EC.1).

Derivation of the LR estimator in Section 5.3. Since the parameter $S_1(0)$ can be viewed as a distributional parameter under the discretization scheme in Equation (31), we now apply the conditioning technique in Hong and Liu (2010) to derive the LR estimator in Equation (32).

Let $f_i(\cdot|s_{i-1})$ denote the conditional density function of $\widehat{S_1}(t_i)$ given that $\widehat{S_1}(t_{i-1}) = s_{i-1}$, $i = 1, 2, \ldots, k$. By Equation (24),

$$f_1\left(x|S_1(0)\right) = \frac{1}{\sigma_1\left(0,S_1(0)\right)\sqrt{\Delta t}}\cdot\phi\left(\frac{x-S_1(0)-\mu_1\left(0,S_1(0)\right)\Delta t}{\sigma_1\left(0,S_1(0)\right)\sqrt{\Delta t}}\right),$$

where $\phi(\cdot)$ is the density function of standard normal distribution. Then, the SF can be expressed as

$$\mathrm{SF} = \frac{d}{dS_1(0)}\log\left(f_1\left(\widehat{S_1}(t_1)|S_1(0)\right)\right)$$

$$= \frac{\widehat{S_1}(t_1)-S_1(0)-\mu_1\left(0,S_1(0)\right)\Delta t}{\sigma_1\left(0,S_1(0)\right)\sqrt{\Delta t}}\cdot\left[\frac{1}{\sigma_1\left(0,S_1(0)\right)\sqrt{\Delta t}}\left(1+\frac{d\mu_1\left(0,S_1(0)\right)}{dS_1(0)}\right)\right.$$

$$\left.+\frac{\widehat{S_1}(t_1)-S_1(0)-\mu_1\left(0,S_1(0)\right)\Delta t}{\sigma_1^2\left(0,S_1(0)\right)\sqrt{\Delta t}}\cdot\frac{d\sigma_1\left(0,S_1(0)\right)}{dS_1(0)}\right]-\frac{1}{\sigma_1\left(0,S_1(0)\right)}\frac{d\sigma_1\left(0,S_1(0)\right)}{dS_1(0)}$$

$$= \frac{\left(\widehat{S_1}(t_1)-\kappa_1\mu_1\Delta t\right)^2-\sigma_1^2S_1(0)\Delta t-\left(1-\kappa_1\Delta t\right)^2S_1^2(0)}{2\sigma_1^2S_1^2(0)\Delta t}, \quad \text{(EC.2)}$$

where Equation (EC.2) is by plugging Equation (30) with $i=1$. Then, the LR estimator is

$$p'(S_1(0)) = \mathrm{E}\{g(L)\cdot \mathrm{SF}\} = \mathrm{E}\left\{ g(L)\cdot \frac{\left(\widehat{S_1}(t_1)-\kappa_1\mu_1\Delta t\right)^2 - \sigma_1^2 S_1(0)\Delta t - (1-\kappa_1\Delta t)^2 S_1^2(0)}{2\sigma_1^2 S_1^2(0)\Delta t} \right\}.$$

EC.2. SPA Estimators in A Latent Variable Model

We now use Example 3 (The model of Bassamboo et al. (2008)) to derive SPA estimators. We consider only the case that the performance function is $g(L)=\mathbf{1}_{\{L>y\}}$. In fact, it is not clear how to derive SPA estimators for general performance functions. Conditioning on $\{Z,\mathcal{E}\}$,

$$\begin{aligned}
\mathrm{E}[g(L)|Z,\mathcal{E}] &= \Pr\left\{ \sum_{i=1}^m l_i \mathbf{1}_{\{Y_i\le d_i\}} > y \Big| Z,\mathcal{E} \right\} \\
&= \Pr\left\{ \epsilon_1 \le \frac{\theta d_1\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}} \Big| Z,\mathcal{E}\right\}\cdot \Pr\left\{ \sum_{i=2}^m l_i \mathbf{1}_{\{Y_i\le d_i\}} > y - l_1 \Big| Z,\mathcal{E} \right\} \\
&\quad + \Pr\left\{ \epsilon_1 > \frac{\theta d_1\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}} \Big| Z,\mathcal{E}\right\}\cdot \Pr\left\{ \sum_{i=2}^m l_i \mathbf{1}_{\{Y_i\le d_i\}} > y \Big| Z,\mathcal{E} \right\}.
\end{aligned}$$

We can recursively use the same approach to deal with $\Pr\left\{\sum_{i=2}^m l_i\mathbf{1}_{\{Y_i\le d_i\}} > y-l_1 \Big| Z,\mathcal{E}\right\}$ and $\Pr\left\{\sum_{i=2}^m l_i\mathbf{1}_{\{Y_i\le d_i\}} > y \Big| Z,\mathcal{E}\right\}$. Then, after m iterations, we have

$$\mathrm{E}[g(L)|Z,\mathcal{E}] = \sum_{s\in\mathscr{S}(m)} \prod_{i\in s^1} F_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \prod_{i\in s^0} \bar{F}_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \mathbf{1}_{\{\sum_{i\in s^1} l_i > y\}}, \tag{EC.3}$$

where $F_{\epsilon_i}(\cdot)$ is the cumulative distribution function (cdf) of ϵ_i, $\bar{F}_{\epsilon_i}(\cdot)=1-F_{\epsilon_i}(\cdot)$, and $\mathscr{S}(m)=\{0,1\}^m$ with s^1 denoting the set of default obligors and s^0 denoting the set of non-default obligors. Because the closed-form expresson in Equation (EC.3) is Lipschitz continuous, then we can apply SPA method to obtain a SPA estimator,

$$\begin{aligned}
p'(\theta) &= \frac{\partial}{\partial\theta}\mathrm{E}\{\mathrm{E}[g(L)|Z,\mathcal{E}]\} = \mathrm{E}\left\{\frac{\partial}{\partial\theta}\mathrm{E}[g(L)|Z,\mathcal{E}]\right\} \\
&= \mathrm{E}\left\{ \sum_{s\in\mathscr{S}(m)} \left[\sum_{j\in s^1} f_{\epsilon_j}\left(\frac{\theta d_j\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \frac{d_j\mathcal{E}}{\sqrt{1-\rho^2}}\cdot \prod_{i\in s^1, i\ne j} F_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right) \right.\right. \\
&\quad \cdot \prod_{i\in s^0} \bar{F}_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \mathbf{1}_{\{\sum_{i\in s^1} l_i>y\}} - \sum_{j\in s^0} f_{\epsilon_j}\left(\frac{\theta d_j\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \frac{d_j\mathcal{E}}{\sqrt{1-\rho^2}} \\
&\quad \left.\left. \cdot \prod_{i\in s^1} F_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \prod_{i\in s^0, i\ne j} \bar{F}_{\epsilon_i}\left(\frac{\theta d_i\mathcal{E}-\rho Z}{\sqrt{1-\rho^2}}\right)\cdot \mathbf{1}_{\{\sum_{i\in s^1} l_i>y\}} \right]\right\}.
\end{aligned} \tag{EC.4}$$

Similarly, by conditioning on $\{\mathcal{E},\epsilon_1,\ldots,\epsilon_m\}$, we know that

$$\begin{aligned}
\mathrm{E}\left[g(L)|\mathcal{E},\epsilon_1,\ldots,\epsilon_m\right] &= \sum_{s\in\mathscr{S}(m)} \Pr\left\{\bigcup_{i\in s^1}\left\{Z\le\frac{\theta d_i\mathcal{E}-\sqrt{1-\rho^2}\epsilon_i}{\rho}\right\}\right.\\
&\qquad \left.\bigcup_{i\in s^0}\left\{Z>\frac{\theta d_i\mathcal{E}-\sqrt{1-\rho^2}\epsilon_i}{\rho}\right\}\Big|\mathcal{E},\epsilon_1,\ldots,\epsilon_m\right\}\cdot\mathbf{1}_{\{\sum_{i\in s^1} l_i>y\}}\\
&= \sum_{s\in\mathscr{S}(m)}\left[F_Z\left(\frac{\theta d_{i_*^1}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^1}}{\rho}\right)-F_Z\left(\frac{\theta d_{i_*^0}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^0}}{\rho}\right)\right]\\
&\qquad\cdot\mathbf{1}_{\{\theta d_{i_*^1}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^1}>\theta d_{i_*^0}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^0}\}}\cdot\mathbf{1}_{\{\sum_{i\in s^1} l_i>y\}},
\end{aligned}$$

where $i_*^1=\mathrm{argmin}_{i\in s^1}\left\{\frac{\theta d_i\mathcal{E}-\sqrt{1-\rho^2}\epsilon_i}{\rho}\right\}$ and $i_*^0=\mathrm{argmax}_{i\in s^0}\left\{\frac{\theta d_i\mathcal{E}-\sqrt{1-\rho^2}\epsilon_i}{\rho}\right\}$. Then,

$$\begin{aligned}
p'(\theta) &= \mathrm{E}\left\{\sum_{s\in\mathscr{S}(m)}\left[f_Z\left(\frac{\theta d_{i_*^1}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^1}}{\rho}\right)\cdot\frac{d_{i_*^1}\mathcal{E}}{\rho}-f_Z\left(\frac{\theta d_{i_*^0}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^0}}{\rho}\right)\cdot\frac{d_{i_*^0}\mathcal{E}}{\rho}\right]\right.\\
&\qquad\left.\cdot\mathbf{1}_{\{\theta d_{i_*^1}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^1}>\theta d_{i_*^0}\mathcal{E}-\sqrt{1-\rho^2}\epsilon_{i_*^0}\}}\cdot\mathbf{1}_{\{\sum_{i\in s^1} l_i>y\}}\right\}
\end{aligned}\tag{EC.5}$$

There are several things we would like to point out. First, the reason we can easily derive multiple SPA estimators is because of the general closed-form expression in Equation (3) (also Equations (7) and (8) in Section 3), which implies the possibility of deriving SPA estimators under some transformations. In fact, if we write Equations (11) and (13) into the combinatorial form, we find that Equation (EC.4) is the same as Equation (11) and Equation (EC.5) is the same as Equation (13). Second, however, without the general form of Equation (3), it is not easy to convert Equations (EC.4) and (EC.5) into the neat form as in (11) and (13). Therefore, the computational complexity of both (EC.4) and (EC.5) are at least $O(2^m)$, which is much higher compared with their counterpart in (11) and (13) in the paper.

EC.3. Extension to Sensitivities of VaR and CVaR

Our method can be extended to compute the sensitivity of VaR and Conditional VaR (CVaR) when l_i are mutually independent continuous random variables and independent with X_j for all $i,j=1,2,\ldots,m$. Then the cdf $F_L(\cdot)$ of the loss function L is continuous except at the point 0. In general, when computing the sensitivity of VaR or CVaR with respect L, we are interested in the event that the loss is beyond some large threshold y (which is typically greater than 0), so the discontinuity at 0 will not cause any problem. Note that if we restrict l_i to be constant, then L become a discrete random variable with cdf $F_L(\cdot)$ be a step function. Then the sensitivity of the VaR or the CVaR may become hard to analyze since the perturbation of θ may lead to a dramatic change or no change of the value of VaR depending on whether the level α (defined later on) is at an exact probability mass point or not. Interested readers may be referred to Rockafellar and Uryasev (2002) for representing the CVaR for general loss functions as an expectation of properly modified tail distribution.

Considering l_i, $i=1,2,\ldots,m$, as continuous random variables, then L is a continuous random variable except at the point 0, where $\Pr(L=0)=\Pr\left(X_i\geq 0,\ \text{for all } i=1,2,\ldots,m\right)>0$. Generally, when computing the sensitivity of VaR with respect to L, we are interested in the event that the loss is beyond some large threshold y, which is typically greater than 0.

EC.3.1. The sensitivity of VaR

Let $F_L(y)=\Pr(L\leq y)$ be the cdf of L. Define the VaR at level α (α-VaR) of L as $v_\alpha=\inf\{y: F_L(y)\geq\alpha\}$. Then for $v_\alpha\neq 0$, the equality can be achieved with $F_L(v_\alpha)=\alpha$. To estimate $v'_\alpha(\theta)$, we write $F_L(v_\alpha)=\alpha$ as $F_L(v_\alpha(\theta),\theta)=\alpha$ and take derivative with respect to θ on both sides, which yields

$$\partial_y F_L(y,\theta)|_{y=v_\alpha(\theta)}\cdot v'_\alpha(\theta)+\partial_\theta F_L(y,\theta)|_{y=v_\alpha(\theta)}=0.$$

Then,

$$v'_\alpha(\theta)=\left.\frac{\partial_\theta F_L(y,\theta)}{\partial_y F_L(y,\theta)}\right|_{y=v_\alpha(\theta)}=\left.\frac{\partial_\theta F_L(y,\theta)}{f_L(y,\theta)}\right|_{y=v_\alpha(\theta)}, \tag{EC.6}$$

where $f_L(y,\theta)$ denote the pdf of L. Then our goal is to calculate $\partial_\theta F_L(v_\alpha(\theta),\theta)$ and $f_L(v_\alpha(\theta),\theta)$ respectively. The numerator on the right-hand-side of Equation (EC.6)

$$\partial_\theta F_L(v_\alpha(\theta),\theta)=\frac{d}{d\theta}\mathrm{E}\left[\mathbf{1}_{\{L(\theta)\leq y\}}\right]\big|_{y=v_\alpha(\theta)}. \tag{EC.7}$$

Letting $g(L)=\mathbf{1}_{\{L(\theta)\leq y\}}$, then Equation (EC.7) can be handled by the conditional technique in Section 2. We just summarize the result as follows,

$$\frac{d}{d\theta}\mathrm{E}\left[\mathbf{1}_{\{L(\theta)\leq y\}}\right]\Big|_{y=v_\alpha(\theta)}=-\sum_{i=1}^{m}\ \mathrm{E}\left\{\left[\mathbf{1}_{\{L_{-i}+l_i\leq y\}}-\mathbf{1}_{\{L_{-i}\leq y\}}\right]\cdot X'_i(\theta)\,;\,X_i=0\right\}\Big|_{y=v_\alpha(\theta)},$$

where $L_{-i}=\sum_{j=1,j\neq i}^m l_j\cdot\mathbf{1}_{\{X_j<0\}}$. The denominator on the right-hand-side of Equation (EC.6)

$$\begin{aligned}f_L(y,\theta)&=\lim_{\Delta y\to 0}\frac{F_L(y+\Delta y)-F_L(y)}{\Delta y}\\&=\lim_{\Delta y\to 0}\frac{\mathrm{E}\left[\mathbf{1}_{\{L(\theta)\leq y+\Delta y\}}-\mathbf{1}_{\{L(\theta)\leq y\}}\right]}{\Delta y}.\end{aligned}$$

Then, we may choose a proper Δy to approximate $f_L(y,\theta)$ at $y=v_\alpha(\theta)$, which can be viewed as the kernel method. Another way is to apply the conditional Monte Carlo derived in Fu et al. (2009). Basically, it requires to find some random variable $Y(\theta)$ such that

$$F_L(y,\theta)=\mathrm{E}[\Pr\{L(\theta)\leq y|Y(\theta)\}]=\mathrm{E}\left[G(y,Y(\theta),\theta)\right],$$

where $G(y,Y(\theta),\theta)$ is differential w.p.1 with respect to y and

$$|G(y+\Delta y,Y(\theta),\theta)-G(y,Y(\theta),\theta)|\leq K|\Delta y|, \tag{EC.8}$$

for some random variable K with $\mathrm{E}[K]<\infty$. Then,

$$f_L(y,\theta)|_{y=v_\alpha(\theta)}=\mathrm{E}\left[\partial_y G(y,Y(\theta),\theta)\right]\big|_{y=v_\alpha(\theta)}. \tag{EC.9}$$

The closed-form expression of G may be complicated, but the idea behind is straightforward. By conditioning on some random variable $Y(\theta)$, we can write F_L as an expectation of a function of cdf's and pdf's with closed-form. In addition, the condition of (EC.8) can be easily verified after giving the closed-form of G (which should be satisfied due to the differentiability of F_L at $y=v_\alpha(\theta)$). Equations (EC.7) and (EC.9) together provide the estimator $v'_\alpha(\theta)$ in Equation (EC.6). Now we derive a general closed-form expression of G given the form of loss function L in our paper. Suppose we may write $X_i=\eta_i-A_i$, where η_i is an idiosyncratic random variable which is independent of all other random variables. Let $H_i(\cdot)$ and $h_i(\cdot)$ be the cdf and pdf of η_i, respectively. Let $F_i(\cdot)$ be the cdf of l_i and $\bar{F}_i(\cdot)=1-F_i(\cdot)$. Then,

$$F_L(y,\theta)=\mathrm{E}\left[\Pr\left\{\sum_{i=1}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_1,A_2,\ldots,A_m\right\}\right]. \tag{EC.10}$$

For any $y>0$,

$$\begin{aligned}
&\Pr\left\{\sum_{i=1}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_1,A_2,\ldots,A_m\right\}\\
&=\Pr\{\eta_1<A_1|A_1\}\cdot\Pr\left\{l_1\leq y-\sum_{i=2}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\middle|A_2,A_3,\ldots,A_m\right\}\\
&\quad+\Pr\{\eta_1\geq A_1|A_1\}\cdot\Pr\left\{\sum_{i=2}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_2,A_3,\ldots,A_m\right\}\\
&=F_1(A_1)\cdot\mathrm{E}\left[H_1\left(y-\sum_{i=2}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\right)\middle|A_2,A_3,\ldots,A_m\right]\\
&\quad+\bar{F}_1(A_1)\cdot\Pr\left\{\sum_{i=2}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_2,A_3,\ldots,A_m\right\}.
\end{aligned}$$

Recursively, we can use the same approach to analyze $\Pr\left\{\sum_{i=2}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_2,A_3,\ldots,A_m\right\}$. Then, after m iterations, we obtain that

$$\begin{aligned}
&\Pr\left\{\sum_{i=1}^m l_i\cdot\mathbf{1}_{\{\eta_i<A_i\}}\leq y\middle|A_1,A_2,\ldots,A_m\right\}\\
&=\sum_{i=1}^m\left\{F_i(A_i)\cdot\mathrm{E}\left[H_i\left(y-\sum_{j=i+1}^m l_j\cdot\mathbf{1}_{\{\eta_j<A_j\}}\right)\middle|A_{i+1},A_{i+2},\ldots,A_m\right]\cdot\prod_{j=1}^{i-1}\bar{F}_j(A_j)\right\}\\
&\quad+\prod_{i=1}^m\bar{F}_i(A_i).
\end{aligned}$$

By Equation (EC.10),

$$F_L(y) = \sum_{i=1}^{m} \mathrm{E}\left[F_i(A_i) \cdot H_i\left(y - \sum_{j=i+1}^{m} l_j \cdot \mathbf{1}_{\{\eta_j < A_j\}}\right) \cdot \prod_{j=1}^{i-1} \bar{F}_j(A_j)\right] + \mathrm{E}\left[\prod_{i=1}^{m} \bar{F}_i(A_i)\right].$$

Differentiating $F_L(y)$ with respect to y yields

$$f_L(y) = \sum_{i=1}^{m} \mathrm{E}\left[F_i(A_i) \cdot h_i\left(y - \sum_{j=i+1}^{m} l_j \cdot \mathbf{1}_{\{\eta_j < A_j\}}\right) \cdot \prod_{j=1}^{i-1} \bar{F}_j(A_j)\right].$$

EC.3.2. The sensitivity of CVaR

According to Acerbi and Tasche (2002), the CVaR is equivalent to the expected shortfall (ES) when L is a real integrable random variable (i.e., $\mathrm{E}[|L|] < \infty$). In addition, L is continuous in the neighborhood of v_α (which is the α-VaR), then CVaR at level α (denoted by α-CVaR) of L is

$$u_\alpha = v_\alpha + \frac{1}{1-\alpha}\mathrm{E}[(L - v_\alpha); L \geq v_\alpha],$$

which is also known as the tail conditional expectation. We are interested to calculate $u'_\alpha(\theta)$. Note that $\Pr\{L(\theta) = v_\alpha(\theta)\} = 0$. If we want to use the Monte Carlo method in this paper, we need to re-derive $p'(\theta)$ in Equation (6) in the paper since $g(\cdot)$ is a function of θ now. Moreover, we define $g(x,\theta) = (x - v_\alpha(\theta)) \cdot \mathbf{1}_{\{x \geq v_\alpha(\theta)\}}$, then $\partial_\theta g(x,\theta) = -v'_\alpha(\theta) \cdot \mathbf{1}_{\{x \geq v_\alpha(\theta)\}}$ when $v_\alpha(\theta) \neq x$. We know that

$$p(\theta) = \sum_{s \in \mathscr{S}(m)} \mathrm{E}\left[g\left(\sum_{i \in s^1} l_i\right)\right] \cdot \mathrm{E}\left[\prod_{i \in s^1} \mathbf{1}_{\{X_i < 0\}} \prod_{i \in s^0} \mathbf{1}_{\{X_i \geq 0\}}\right].$$

Then,

$$\begin{aligned}
p'(\theta) = & \sum_{s \in \mathscr{S}(m)} \mathrm{E}\left[g'\left(\sum_{i \in s^1} l_i\right)\right] \cdot \mathrm{E}\left[\prod_{i \in s^1} \mathbf{1}_{\{X_i < 0\}} \prod_{i \in s^0} \mathbf{1}_{\{X_i \geq 0\}}\right] \\
& + \sum_{s \in \mathscr{S}(m)} \mathrm{E}\left[g\left(\sum_{i \in s^1} l_i\right)\right] \cdot \frac{d}{d\theta}\mathrm{E}\left[\prod_{i \in s^1} \mathbf{1}_{\{X_i(\theta) < 0\}} \prod_{i \in s^0} \mathbf{1}_{\{X_i(\theta) \geq 0\}}\right]. \\
= & -v'_\alpha(\theta) \sum_{s \in \mathscr{S}(m)} \mathrm{E}\left[\mathbf{1}_{\{\sum_{i \in s^1} l_i \geq v_\alpha(\theta)\}}\right] \cdot \mathrm{E}\left[\prod_{i \in s^1} \mathbf{1}_{\{X_i < 0\}} \prod_{i \in s^0} \mathbf{1}_{\{X_i \geq 0\}}\right] \\
& + \sum_{s \in \mathscr{S}(m)} \mathrm{E}\left[g\left(\sum_{i \in s^1} l_i\right)\right] \cdot \frac{d}{d\theta}\mathrm{E}\left[\prod_{i \in s^1} \mathbf{1}_{\{X_i(\theta) < 0\}} \prod_{i \in s^0} \mathbf{1}_{\{X_i(\theta) \geq 0\}}\right] \\
= & -v'_\alpha(\theta)\mathrm{E}\left[\mathbf{1}_{\{\sum_{i=1}^m l_i \mathbf{1}_{\{X_i < 0\}} \geq v_\alpha(\theta)\}}\right] - \sum_{i=1}^{m} \mathrm{E}\left\{\left[g\left(L_{-i} + l_i\right) - g\left(L_{-i}\right)\right] \cdot X'_i(\theta)\,;\, X_i = 0\right\} \\
= & -(1-\alpha)v'_\alpha(\theta) - \sum_{i=1}^{m} \mathrm{E}\left\{\left[g\left(L_{-i} + l_i\right) - g\left(L_{-i}\right)\right] \cdot X'_i(\theta)\,;\, X_i = 0\right\}.
\end{aligned}$$

The sensitivity of α-CVaR is

$$\begin{aligned} u'_\alpha(\theta) &= v'_\alpha(\theta) + \frac{1}{1-\alpha}\cdot\frac{d}{d\theta}\mathrm{E}[(L(\theta)-v_\alpha(\theta));L(\theta)\geq v_\alpha(\theta)] \\ &= \frac{-1}{1-\alpha}\sum_{i=1}^{m}\ \mathrm{E}\left\{\left[(L_{-i}+l_i)\cdot\mathbf{1}_{\{L_{-i}+l_i\geq v_\alpha(\theta)\}} - L_{-i}\cdot\mathbf{1}_{\{L_{-i}\geq v_\alpha(\theta)\}}\right]\cdot X'_i(\theta)\,;\,X_i=0\right\}. \end{aligned}$$

If Assumptions 1–3 in Hong and Liu (2009) hold, then we can directly apply the result in Hong and Liu (2009) to obtain that

$$\begin{aligned} u'_\alpha(\theta) &= \frac{1}{1-\alpha}\mathrm{E}\left[L'(\theta)\cdot\mathbf{1}_{\{L(\theta)\geq v_\alpha(\theta)\}}\right] \\ &= \frac{1}{1-\alpha}\mathrm{E}\left[\left(\sum_{i=1}^{m} l_i\cdot\mathbf{1}_{\{X_i(\theta)<0\}}\right)'\cdot\mathbf{1}_{\{L(\theta)\geq v_\alpha(\theta)\}}\right]=0, \end{aligned}$$

which is obviously wrong. This is because the Assumption 1 of Hong and Liu (2009) fails with respect the loss function L considered in our paper.

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