Algorithmic approach for improved mixed-integer reformulations of convex Generalized Disjunctive Programs: On-line supplement

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1 Example of GDP reformulation

To illustrate a linear GDP formulation and its corresponding (BM) and (HR) reformulations, consider the optimization of a process superstructure illustrated in Figure1. In this process, either reactor 1 or reactor 2 have to be selected. For reactor 1 there are three different alternatives (R_{11}, R_{12}, R_{13}). If reactor 2 (R_2) is selected, then a choice needs to be made between using separation process 1 or 2 (S_1, S_2). A GDP representation of this problem is shown in (1).

$$
\begin{aligned}
& max\ z = P_{11}F_{11} - P_1F_1 - C_R - C_S \\
s.t.\quad & F_1 = F_2 + F_4 \\
& F_5 = F_6 + F_7 \\
& F_{10} = F_8 + F_9 \\
& F_{11} = F_3 + F_{10} \\
& \begin{bmatrix} Y_{R11} \\ F_3 = \beta_{R11}F_2 \\ F_{4,5,6,7,8,9,10} = 0 \\ C_R = \gamma_{R11} \end{bmatrix} \vee \begin{bmatrix} Y_{R12} \\ F_3 = \beta_{R12}F_2 \\ F_{4,5,6,7,8,9,10} = 0 \\ C_R = \gamma_{R12} \end{bmatrix} \vee \begin{bmatrix} Y_{R13} \\ F_3 = \beta_{R13}F_2 \\ F_{4,5,6,7,8,9,10} = 0 \\ C_R = \gamma_{R13} \end{bmatrix} \vee \begin{bmatrix} Y_{R2} \\ F_5 = \beta_{R2}F_4 \\ F_{2,3} = 0 \\ C_R = \gamma_{R2} \end{bmatrix} \\
& \begin{bmatrix} Y_{S1} \\ F_8 = \beta_{S1}F_6 \\ C_S = \gamma_{S1} \end{bmatrix} \vee \begin{bmatrix} Y_{S2} \\ F_9 = \beta_{S1}F_7 \\ C_S = \gamma_{S2} \end{bmatrix} \vee \begin{bmatrix} Y_{notR2} \\ C_S = 0 \end{bmatrix} \\
& Y_{R11} \veebar Y_{R12} \veebar Y_{R13} \veebar Y_{R2} \\
& Y_{S1} \veebar Y_{S2} \veebar Y_{notR2} \\
& Y_{R2} \Leftrightarrow Y_{S1} \vee Y_{S2} \\
& 0 \leq F_i \leq F_i^{up} \qquad\qquad i = 1, ..., 11 \\
& 0 \leq C_{R,S} \leq C^{up} \\
& Y_k \in \{True, False\} \qquad k \in \{R11, R12, R13, R2, S1, S2, notR2\}
\end{aligned} \tag{1}
$$

In (1) F_i represents the flows in the superstructure. Y_k represents the selection of an equipment. C_R and C_S represent the cost of the reactor and the cost of the separation unit respectively. The global constraints represent the material balances in the nodes of the flowsheet. The first disjunction allows the selection of any of the three alternatives of reactor 1 or the selection of reactor 2. When the alternative of reactor 1 is selected, F_3 becomes a function of F_2, the flows associated with R_2 become 0, and the reactor cost takes a value that corresponds to that selection. When R_2 is selected, F_5 becomes proportional to F_4, the flows associated with rector 1 (F_2, F_3) become 0, and C_R takes a value that corresponds to the cost of reactor 2. The second disjunction represents the selection of either one of the two separation processes or no separation process (when reactor 2 is not selected). The flows after the separation process and the cost of separation unit are associated with this decision. The logic constraint forces that if and only if Y_{R2} is selected then either Y_{S1} or Y_{S2} has to be selected. Note that this constraint implies that if Y_{R11} or Y_{R12} or Y_{R13} is selected, then Y_{notR2} is selected.

The (BM) and (HR) reformulations of this example are shown in (2) and (3) respectively. Note that in (2) we are additionally relaxing some of the equality constraints as inequalities, in order to avoid additional

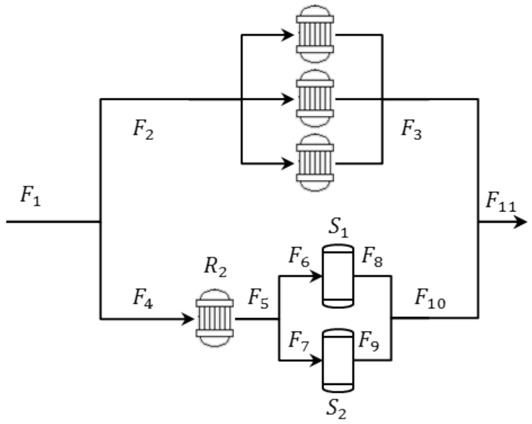

Figure 1: Process flowsheet example

constraints. It is easy to show that the problem with inequality relaxations is equivalent to the original problem.

$$
\begin{aligned}
& max\ z = P_{11}F_{11} - P_1F_1 - C_R - C_S \\
s.t. \quad & F_1 = F_2 + F_4 \\
& F_5 = F_6 + F_7 \\
& F_{10} = F_8 + F_9 \\
& F_{11} = F_3 + F_{10} \\
& F_3 \leq \beta_{R11}F_2 + M(1 - y_{R11}) \\
& F_{4,5,6,7,8,9,10} \leq M(1 - y_{R11}) \\
& C_R \geq \gamma_{R11} - M(1 - y_{R11}) \\
& F_3 \leq \beta_{R12}F_2 + M(1 - y_{R12}) \\
& F_{4,5,6,7,8,9,10} \leq M(1 - y_{R12}) \\
& C_R \geq \gamma_{R12} - M(1 - y_{R12}) \\
& F_3 \leq \beta_{R13}F_2 + M(1 - y_{R13}) \\
& F_{4,5,6,7,8,9,10} \leq M(1 - y_{R13}) \\
& C_R \geq \gamma_{R13} - M(1 - y_{R13}) \\
& F_5 \leq \beta_{R2}F_4 + M(1 - y_{R2}) \\
& F_{2,3} \leq M(1 - y_{R2}) \\
& C_R \geq \gamma_{R2} - M(1 - y_{R2}) \\
& F_8 \leq \beta_{S1}F_6 + M(1 - y_{S1}) \\
& C_S \geq \gamma_{S1} - M(1 - y_{S1}) \\
& F_9 \leq \beta_{S2}F_7 + M(1 - y_{S2}) \\
& C_S \geq \gamma_{S2} - M(1 - y_{S2}) \\
& C_S \leq M(1 - y_{notR2}) \\
& y_{R11} + y_{R12} + y_{R13} + y_{R2} = 1 \\
& y_{S1} + y_{S2} + y_{notR2} = 1 \\
& y_{R2} = y_{S1} + y_{S2} \\
& 0 \leq F_i \leq F_i^{up} \qquad\qquad i = 1, ..., 11 \\
& 0 \leq C_{R,S} \leq C^{up} \\
& y_k \in \{0, 1\} \qquad\qquad k \in \{R11, R12, R13, R2, S1, S2, notR2\}
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
& max\ z = P_{11}F_{11} - P_1F_1 - C_R - C_S \\
s.t.\quad & F_1 = F_2 + F_4 \\
& F_5 = F_6 + F_7 \\
& F_{10} = F_8 + F_9 \\
& F_{11} = F_3 + F_{10} \\
& F_i = F_i^{R11} + F_i^{R12} + F_i^{R13} + F_i^{R2} \qquad i = 1, ..., 11 \\
& F_i = F_i^{S1} + F_i^{S2} + F_i^{notR2} \qquad i = 1, ..., 11 \\
& C_R = C_R^{R11} + C_R^{R12} + C_R^{R13} + C_R^{R2} \\
& C_S = C_S^{S1} + C_S^{S2} + C_S^{notR2} \\
& F_3^{R11} = \beta_{R11}F_2^{R11} \\
& F_{4,5,6,7,8,9,10}^{R11} = 0 \\
& C_R^{R11} = \gamma_{R11} * y_{R11} \\
& F_3^{R12} = \beta_{R12}F_2^{R12} \\
& F_{4,5,6,7,8,9,10}^{R12} = 0 \\
& C_R^{R12} = \gamma_{R12} * y_{R12} \\
& F_3^{R13} = \beta_{R13}F_2^{R13} \\
& F_{4,5,6,7,8,9,10}^{R13} = 0 \\
& C_R^{R13} = \gamma_{R13} * y_{R13} \\
& F_5^{R2} = \beta_{R2}F_4^R2 \\
& F_{2,3}^{R2} = 0 \\
& C_R^{R2} = \gamma_{R2} * y_{R2} \\
& F_8^{S1} = \beta_{S1}F_6^{S1} \\
& C_S^{S1} = \gamma_{S1} * y_{S1} \\
& F_9^{S2} = \beta_{S2}F_7^{S2} \\
& C_S^{S2} = \gamma_{S2} * y_{S2} \\
& C_S^{notR2} = 0 \\
& y_{R11} + y_{R12} + y_{R13} + y_{R2} = 1 \\
& y_{S1} + y_{S2} + y_{notR2} = 1 \\
& y_{R2} = y_{S1} + y_{S2} \\
& 0 \le F_i^k \le F_i^{up} * y_k \qquad i = 1, ..., 11, k \in \{R11, R12, R13, R2, S1, S2, notR2\} \\
& 0 \le C_{R,S}^k \le C^{up} * y_k \qquad k \in \{R11, R12, R13, R2, S1, S2, notR2\} \\
& y_k \in \{0, 1\} \qquad k \in \{R11, R12, R13, R2, S1, S2, notR2\}
\end{aligned} \tag{3}
$$

2 Hybrid reformulation

$$
\begin{aligned}
& min\ lt \\
s.t.\quad & lt \ge x_4 + 3 \\
& x_j = x_j^{1,1} + x_j^{1,2} + x_j^{2,1} + x_j^{2,2} \qquad j = 1, 2, 3 \\
& lt = lt^{1,1} + lt^{1,2} + lt^{2,1} + lt^{2,2} \\
& lt^{1,1} \ge x_1^{1,1} + 6 * \hat{y}_{1,1} \\
& lt^{1,1} \ge x_2^{1,1} + 5 * \hat{y}_{1,1} \\
& lt^{1,1} \ge x_3^{1,1} + 4 * \hat{y}_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
&x_1^{1,1} + 6 * \hat{y}_{1,1} \leq x_2^{1,1} \\
&x_1^{1,1} + 6 * \hat{y}_{1,1} \leq x_3^{1,1} \\
&lt^{1,2} \geq x_1^{1,2} + 6 * \hat{y}_{1,2} \\
&lt^{1,2} \geq x_2^{1,2} + 5 * \hat{y}_{1,2} \\
&lt^{1,2} \geq x_3^{1,2} + 4 * \hat{y}_{1,2} \\
&x_1^{1,2} + 6 * \hat{y}_{1,2} \leq x_2^{1,2} \\
&x_3^{1,2} + 4 * \hat{y}_{1,2} \leq x_1^{1,2} \\
&lt^{2,1} \geq x_1^{2,1} + 6 * \hat{y}_{2,1} \\
&lt^{2,1} \geq x_2^{2,1} + 5 * \hat{y}_{2,1} \\
&lt^{2,1} \geq x_3^{2,1} + 4 * \hat{y}_{2,1} \\
&x_2^{2,1} + 5 * \hat{y}_{2,1} \leq x_1^{2,1} \\
&x_1^{2,1} + 6 * \hat{y}_{2,1} \leq x_3^{2,1} \\
&lt^{2,2} \geq x_1^{2,2} + 6 * \hat{y}_{2,2} \\
&lt^{2,2} \geq x_2^{2,2} + 5 * \hat{y}_{2,2} \\
&lt^{2,2} \geq x_3^{2,2} + 4 * \hat{y}_{2,2} \\
&x_2^{2,2} + 5 * \hat{y}_{2,2} \leq x_1^{2,2} \\
&x_3^{2,2} + 4 * \hat{y}_{2,2} \leq x_1^{2,2} && \text{(4H)} \\
&x_1 - x_4 + 6 \leq M * (1 - y_{31}) \\
&x_4 - x_1 + 3 \leq M * (1 - y_{32}) \\
&h_4 - h_1 + 6 \leq M * (1 - y_{33}) \\
&h_1 - h_4 + 3 \leq M * (1 - y_{34}) \\
&x_2 - x_3 + 5 \leq M * (1 - y_{41}) \\
&x_3 - x_2 + 4 \leq M * (1 - y_{42}) \\
&x_2 - x_4 + 5 \leq M * (1 - y_{51}) \\
&x_4 - x_2 + 3 \leq M * (1 - y_{52}) \\
&h_4 - h_2 + 7 \leq M * (1 - y_{53}) \\
&h_2 - h_4 + 3 \leq M * (1 - y_{54}) \\
&x_3 - x_4 + 4 \leq M * (1 - y_{61}) \\
&x_4 - x_3 + 3 \leq M * (1 - y_{62}) \\
&h_4 - h_3 + 5 \leq M * (1 - y_{63}) \\
&h_3 - h_4 + 3 \leq M * (1 - y_{64}) \\
&y_{k1} + y_{k2} + y_{k3} + y_{k4} = 1 && k = 3, 5, 6 \\
&y_{k1} + y_{k2} = 1 && k = 1, 2, 4 \\
&y_{11} = \hat{y}_{1,1} + \hat{y}_{1,2} \\
&y_{12} = \hat{y}_{2,1} + \hat{y}_{2,2} \\
&y_{21} = \hat{y}_{1,1} + \hat{y}_{2,1} \\
&y_{22} = \hat{y}_{1,2} + \hat{y}_{2,2} \\
&0 \leq x_1^{\hat{i}} \leq 12\hat{y}_{\hat{i}} && \hat{i} = \{(1,1), (1,2), (2,1), (2,2)\} \\
&0 \leq x_2^{\hat{i}} \leq 13\hat{y}_{\hat{i}} && \hat{i} = \{(1,1), (1,2), (2,1), (2,2)\} \\
&0 \leq x_3^{\hat{i}} \leq 14\hat{y}_{\hat{i}} && \hat{i} = \{(1,1), (1,2), (2,1), (2,2)\} \\
&0 \leq x_3^{\hat{i}} \leq 15\hat{y}_{\hat{i}} && \hat{i} = \{(1,1), (1,2), (2,1), (2,2)\} \\
&0 \leq lt^{\hat{i}} \leq 20\hat{y}_{\hat{i}} && \hat{i} = \{(1,1), (1,2), (2,1), (2,2)\} \\
&0 \leq lt \leq 20;\ 0 \leq x_1 \leq 12;\ 0 \leq x_2 \leq 13;\ 0 \leq x_3 \leq 14;\ 0 \leq x_4 \leq 15
\end{aligned}
$$

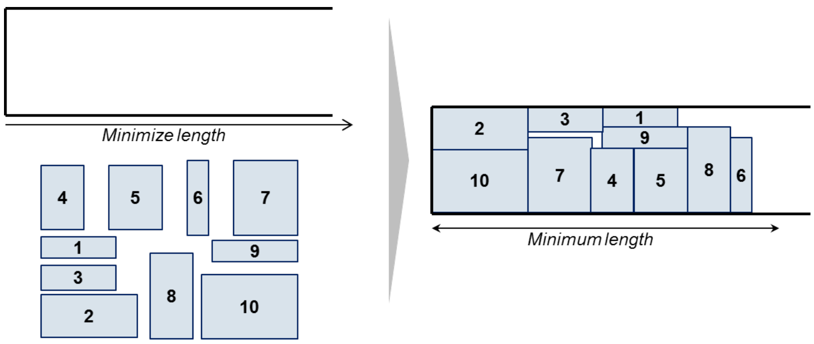

Figure 2: Illustration of strip packing problem

$$
\begin{aligned}
&6 \leq h_1 \leq 10;\ 7 \leq h_2 \leq 10;\ 5 \leq h_3 \leq 10;\ 3 \leq h_4 \leq 10 \\
&y_{ki} \in \{0,1\} && k = 1,...,6,\ i \in D_k \\
&0 \leq \hat{y}_{1,1}, \hat{y}_{1,2}, \hat{y}_{2,1}, \hat{y}_{2,2} \leq 1 \\
&x_j, h_j \in \mathbb{R}^1 && j = 1,2,3,4
\end{aligned}
$$

3 Examples

3.1 Strip Packing (S-Pck)

In the strip packing problem a set of rectangles and a strip with fixed width are given. The objective is to pack these rectangles, without rotation and overlap, within the strip minimizing its length. Figure 2 illustrates this problem. The linear GDP formulation is as follows[1]:

$$
\begin{aligned}
& min\ lt \\
s.t.\quad & lt \geq x_i + L_i && i \in N \\
& \begin{bmatrix} Y_{ij}^1 \\ x_i + L_i \leq x_j \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^2 \\ x_j + L_j \leq x_i \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^3 \\ y_i - H_i \geq y_j \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^4 \\ y_j - H_j \geq y_i \end{bmatrix} && i,j \in N, i < j \\
& Y_{ij}^1 \veebar Y_{ij}^2 \veebar Y_{ij}^3 \veebar Y_{ij}^4 && i,j \in N, i < j \\
& 0 \leq x_i \leq UB - L_i && i \in N \\
& H_i \leq y_i \leq W && i \in N \\
& Y_{ij}^1, Y_{ij}^2, Y_{ij}^3, Y_{ij}^4 \in \{True, False\} && i,j \in N, i < j
\end{aligned} \tag{4}
$$

In (4) x_i and y_i represent the coordinates of the upper-left corner of each of the $i \in N$ rectangles. The global constraints indicate that the total length lt is larger than the x_i coordinate of all the rectangles plus their length L_i. There is a disjunction for each pair of rectangles (i.e. if there are three rectangles there will be three disjunctions representing the pairs: (1,2), (1,3) and (2,3)). Each term in this disjunctions represent the possible relation between two rectangles: rectangle i is either to the left, to the right, above or below rectangle j. These disjunctions ensure that there is no overlap.

3.2 Design of Nontransitive Dice (Dice)

For the design of nontransitive dice problem a number of dice are given ($n \in N$), each with the same number of faces ($f \in F$). The objective is to design a set of dice that maximizes the probability of dice n beating dice $n+1$, and the last dice ($n = |N|$) beating the first one ($n = 1$). For simplicity in the cyclic representation and description of the problem, we will consider the dice $|N| + 1 = 1$. Each dice n must have the same probability of beating dice $n + 1$[2]. More specifically, there are $|F| * |F|$ possible outcomes between each pair of dice n and $n + 1$. The problem seeks to maximize the number of outcomes in which dice n beats dice $n + 1$ (or equivalently, minimize the number of outcomes in which dice $n + 1$ beats dice n). Additionally, in this problem we enforce that each face in each dice has a different integer value[3, 4]. Figure 3 shows an

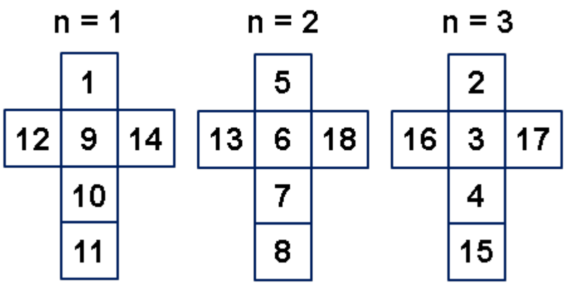

Figure 3: Illustration of 3 Nontransitive Dice with 6 faces

example of 3 nontransitive dice with 6 faces each. In the example the first dice ($n = 1$) beats the second one ($n = 2$) in 21 of the 36 possible outcomes ((9,5); (9,6); (9,7); (9,8); (10,5); (10,6); (10,7); (10,8); (11,5); (11,6); (11,7); (11,8); (12,5); (12,6); (12,7); (12,8); (14,5); (14,6); (14,7); (14,8); (14,13)). This means that dice $n = 1$ has has 58.3% (21/36) probability of beating the second dice. The second one has also 58.3% probability of beating the third dice. Finally, the third dice has also 58.3% probability of beating the first one. The linear GDP formulation of this problem is shown in (5), and it is formulated as minimization of the number of outcomes in which dice $n+1$ beats dice n (instead of maximizing the number of outcomes in which dice n beats dice $n+1$).

$$
\begin{aligned}
& min\ lost \\
s.t.\quad & lost = \sum_{f \in F} \sum_{\hat{f} \in F} w^{lose}_{n,f,\hat{f}} && n \in N \\
& x_{n,f-1} + 1 \le x_{n,f} && n \in N, f \in F \\
& \begin{bmatrix} Y^{win}_{n,f,\hat{f}} \\ w^{lose}_{n,f,\hat{f}} = 0 \\ x_{n,f} \ge x_{n+1,\hat{f}} + 1 \end{bmatrix} \vee \begin{bmatrix} Y^{lose}_{n,f,\hat{f}} \\ w^{lose}_{n,f,\hat{f}} = 1 \\ x_{n,f} \le x_{n+1,\hat{f}} \end{bmatrix} && n \in N, f \in F, \hat{f} \in F \\
& \bigvee_{\hat{n} \in N, \hat{f} \in F} \begin{bmatrix} Z_{n,f,\hat{n},\hat{f}} \\ x_{n,f} = \hat{f} + |F|(\hat{n} - 1) \end{bmatrix} && n \in N, f \in F \qquad (5) \\
& Y^{win}_{n,f,\hat{f}} \veebar Y^{lose}_{n,f,\hat{f}} && n \in N, f \in F, \hat{f} \in F \\
& \underset{\hat{n} \in N, \hat{f} \in F}{\veebar} Z_{n,f,\hat{n},\hat{f}} && n \in N, f \in F \\
& \underset{n \in N, f \in F}{\veebar} Z_{n,f,\hat{n},\hat{f}} && \hat{n} \in N, \hat{f} \in F \\
& 0 \le x_{n,f} \le |N| * |F| && n \in N, f \in F \\
& 0 \le w^{lose}_{|N|,f,\hat{f}} \le 1 && n \in N, f \in F, \hat{f} \in F \\
& Y^{win}_{n,f,\hat{f}}, Y^{lose}_{n,f,\hat{f}}, Z_{n,f,\hat{n},\hat{f}} \in \{True, False\} && n, \hat{n} \in N, f, \hat{f} \in F
\end{aligned}
$$

In (5), variable $w^{lose}_{n,f,\hat{f}}$ represents face f of dice n loosing to face $\hat{f}$ of dice $n+1$. Variable $x_{n,f}$ represents the integer value assigned to face f of dice n (though we represent it with a continuous variable, the second disjunction enforces that $x_{n,f}$ is integer). Problem (5) seeks to minimize the number outcomes in which a dice n loses to dice $n+1$. The first global constraint ensures that all dice n have the same number of losses to dice $n+1$. The second constraint breaks symmetry by assigning the numbers of the faces of a dice in increasing order to the face number. The first disjunction defines if a face f of a dice n wins or looses against a face $\hat{f}$ of the dice $n+1$. The second disjunction enforces that each face in each dice is assigned an integer number. The first two logic constraints correspond to the disjunctions of the GDP formulation. The third one enforces that each number is assigned only once.

3.3 Design of Nontransitive Dice Hybrid GDP-MINLP (DiceH)

Problem "DiceH" is the same problem as "Dice", but using a hybrid MINLP-GDP formulation. In this hybrid reformulation, the disjunctions that assign one number to each face of each dice are formulated as (HR). After a couple of algebraic steps, the number of continuous variables and constraints can be greatly reduced. The hybrid GDP-MINLP formulation is shown in (6).

$$
\begin{aligned}
& min\ lost \\
s.t.\quad & lost = \sum_{f \in F} \sum_{\hat{f} \in F} w^{lose}_{n,f,\hat{f}} && n \in N \\
& x_{n,f-1} + 1 \leq x_{n,f} && n \in N, f \in F \\
& x_{n,f} = \sum_{\hat{n} \in N} \sum_{\hat{f} \in F} (\hat{f} + |F|(\hat{n}-1)) * z_{n,f,\hat{n},\hat{f}} && n \in N, f \in F \\
& \sum_{n \in N} \sum_{f \in F} z_{n,f,\hat{n},\hat{f}} = 1 && \hat{n} \in N, \hat{f} \in F \\
& \begin{bmatrix} Y^{win}_{n,f,\hat{f}} \\ w^{lose}_{n,f,\hat{f}} = 0 \\ x_{n,f} \geq x_{n+1,\hat{f}} + 1 \end{bmatrix} \vee \begin{bmatrix} Y^{lose}_{n,f,\hat{f}} \\ w^{lose}_{n,f,\hat{f}} = 1 \\ x_{n,f} \leq x_{n+1,\hat{f}} \end{bmatrix} && n \in N, f \in F, \hat{f} \in F \\
& Y^{win}_{n,f,\hat{f}} \veebar Y^{lose}_{n,f,\hat{f}} && n \in N, f \in F, \hat{f} \in F \\
& 0 \leq x_{n,f} \leq |N| * |F| && n \in N, f \in F \\
& 0 \leq w^{lose}_{|N|,f,\hat{f}} \leq 1 && n \in N, f \in F, \hat{f} \in F \\
& Y^{win}_{n,f,\hat{f}}, Y^{lose}_{n,f,\hat{f}}, Z_{n,f,\hat{n},\hat{f}} \in \{True, False\} && n, \hat{n} \in N, f, \hat{f} \in F
\end{aligned} \tag{6}
$$

Problem (6) has two main difference with (5). The first one is that the second disjunction of (5) is formulated as an MILP constraint, the third global constraint in (6). The second important difference is that the logic constraint that corresponds to the second disjunction in (5) is removed. Note that in (5) there are two logic constraints that involve the Boolean variables of the second disjunction $\left(\veebar_{\hat{n} \in N, \hat{f} \in F} Z_{n,f,\hat{n},\hat{f}} \text{ and } \veebar_{n \in N, f \in F} Z_{n,f,\hat{n},\hat{f}} \right)$, while in (6) only the second one appears $\left(\sum_{n \in N} \sum_{f \in F} z_{n,f,\hat{n},\hat{f}} = 1 \right)$. This change is valid, because constraint $\left(\sum_{n \in N} \sum_{f \in F} z_{n,f,\hat{n},\hat{f}} = 1 \right)$ enforces that each number appear exactly in one face. Since there are $|N| * |F|$ faces and numbers, this constraint also enforces that each face has exactly one of the different numbers.

3.4 Process Network (Process)

The process network problem "Process" is a classic optimization problem in process design. The model seeks to maximize the profit of selling a set of products taking into account the cost of raw materials and equipment. Figure 4 illustrates the superstructure for a process with potentially 8 units. The model that describes the performance of each unit is normally large and quite complex. In this example, however, the process is simplified to single input-output relations that give rise to a convex GDP[5]. The GDP problem formulation is as follows:

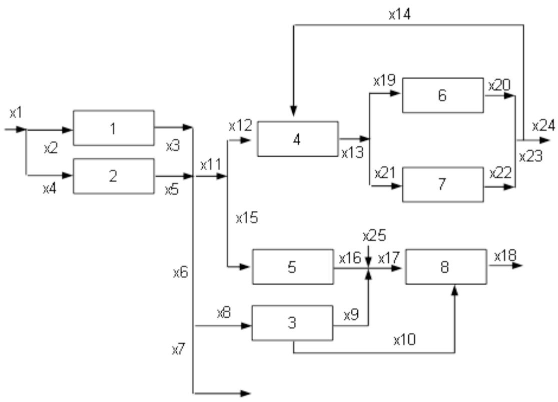

Figure 4: Superstructure illustration of an 8-equipment process network

$$
\begin{aligned}
& min\ Z = \sum_{i \in I} c_i + \sum_{j \in J} p_j x_j + \alpha \\
s.t. \quad & \sum_{j \in J} r_{jn} x_j \leq 0 \qquad\qquad \forall n \in N \\
& \begin{bmatrix} Y_i \\ \sum_{j \in J^i} d_{ij}(e^{x_j/t_{ij}} - 1) - \sum_{j \in J^i} s_{ij} x_j \leq 0 \\ c_i = \gamma_i \end{bmatrix} \vee \begin{bmatrix} \neg Y_i \\ x_j = 0 \quad \forall j \in J^i \\ c_i = 0 \end{bmatrix} \qquad i \in I \qquad (7) \\
& \Omega(Y) = True \\
& c_i, x_j \geq 0 \\
& Y_i \in \{True, False\}
\end{aligned}
$$

In (7) c_i is the cost associated to each equipment $i \in I$. x_j represents each of the flows $j \in J$, and p_j the profit or cost associated to each one. The global constraints represent the mass balance in each of the $n \in N$ nodes, where r_{jn} is the coefficient of the mass balance for flow j. There is a disjunction for each unit i. If a unit is selected ($Y_i = True$) then the corresponding mass balance has to be satisfied, and the cost of the unit c_i takes the value associated to that equipment γ_i. If it is not selected ($Y_i = False$ or, equivalently, $\neg Y_i = True$), then all the flows $j \in J^i$ in and out that equipment become 0, and the cost c_i also becomes 0. Finally $\Omega(Y) = True$ represents the topology of the superstructure.

3.5 Farm Layout (F-Lay)

In the farm layout problem the objective is to determine the width and length of a number of rectangles with fixed area in order to minimize the total perimeter. Figure 5 illustrates this problem, which can be formulated as the following convex GDP[6]:

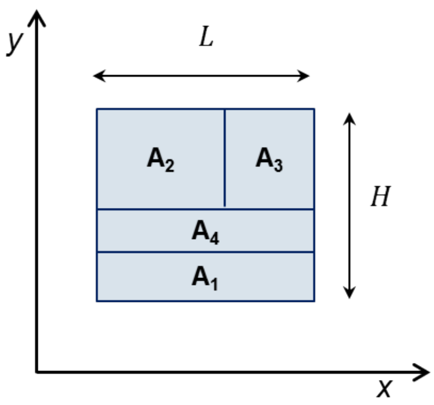

Figure 5: Illustration of farm layout problem

$$
\begin{aligned}
& min\ Z = 2(Length + Width) & \\
s.t.\quad & Length \geq x_i + L_i & i \in N \\
& Width \geq y_i + W_i & i \in N \\
& A_i/W_i - L_i \leq 0 & i \in N \\
& \begin{bmatrix} Y_{ij}^1 \\ x_i + L_i \leq x_j \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^2 \\ x_j + L_j \leq x_i \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^3 \\ y_i + W_i \leq y_j \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^4 \\ y_j + W_j \leq y_i \end{bmatrix} & i,j \in N, i < j \\
& Y_{ij}^1 \veebar Y_{ij}^2 \veebar Y_{ij}^3 \veebar Y_{ij}^4 & i,j \in N, i < j \\
& 0 \leq Length \leq Length^{up};\ \ 0 \leq Width \leq Width^{up} & \\
& L_i^{lo} \leq L_i \leq L_i^{up};\ \ W_i^{lo} \leq W_i \leq L_i^{up} & i \in N \\
& 0 \leq x_i \leq Length^{up} - L_i^{lo};\ \ 0 \leq y_i \leq Width^{up} - L_i^{lo} & i \in N \\
& Y_{ij}^1, Y_{ij}^2, Y_{ij}^3, Y_{ij}^4 \in \{True, False\} & i,j \in N, i < j
\end{aligned} \tag{8}
$$

In formulation (8) the variables x_i and y_i represent the coordinates of lower-left corner of each rectangle $i \in N$, while L_i and W_i represent their corresponding length and width. $Length$ and $Width$ represent the length and width of the total area. A_i is the given area for each rectangle. Similarly to the strip packing problem (4), there is one disjunction for each pair of rectangles. Each term in the disjunction represents the possible relative position between the two rectangles: rectangle i is either to the left, or to the right, or below, or above rectangle j, respectively.

3.6 Constrained Layout (C-Lay)

The constrained layout problem is similar to the strip packing problem, but the rectangles in this case have to be packed inside a set of fixed circles. The objective function is to minimize the distance in x and y axis, with a cost associated to every pair of rectangles. Figure 6 illustrates the constrained layout problem. It can be formulated as the following convex GDP[6]:

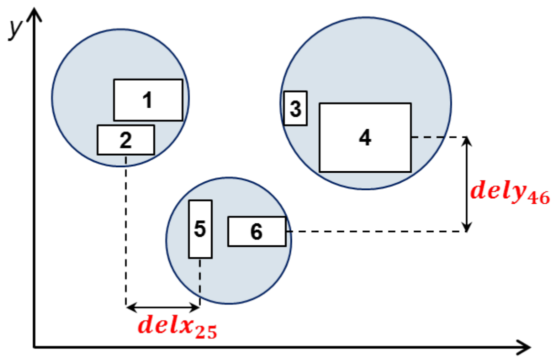

Figure 6: Illustration of constrained layout problem

$$
\begin{aligned}
& min\ Z = \sum_i \sum_j c_{ij}(delx_{ij} + dely_{ij}) && \\
s.t. \quad & delx_{ij} \geq x_i - x_j && i,j \in N, i<j \\
& delx_{ij} \geq x_j - x_i && i,j \in N, i<j \\
& dely_{ij} \geq y_i - y_j && i,j \in N, i<j \\
& dely_{ij} \geq y_j - y_i && i,j \in N, i<j \\
& \begin{bmatrix} Y_{ij}^1 \\ x_i + L_i/2 \leq x_j - L_j/2 \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^2 \\ x_j + L_j/2 \leq x_i - L_i/2 \end{bmatrix} && \\
& \vee \begin{bmatrix} Y_{ij}^3 \\ y_i + H_i/2 \leq y_j - H_j/2 \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^4 \\ y_j + H_j/2 \leq y_i - H_i/2 \end{bmatrix} && i,j \in N, i<j \\
& \bigvee_{t \in T} \begin{bmatrix} W_{it} \\ (x_i + L_i/2 - xc_t)^2 + (y_i + H_i/2 - yc_t)^2 \leq r_t^2 \\ (x_i + L_i/2 - xc_t)^2 + (y_i - H_i/2 - yc_t)^2 \leq r_t^2 \\ (x_i - L_i/2 - xc_t)^2 + (y_i + H_i/2 - yc_t)^2 \leq r_t^2 \\ (x_i - L_i/2 - xc_t)^2 + (y_i - H_i/2 - yc_t)^2 \leq r_t^2 \end{bmatrix} && i \in N \\
& Y_{ij}^1 \veebar Y_{ij}^2 \veebar Y_{ij}^3 \veebar Y_{ij}^4 && i,j \in N, i<j \\
& \underset{t \in T}{\veebar} W_{it} && i \in N \\
& 0 \leq x_i \leq x_i^{up} && i \in N \\
& 0 \leq y_i \leq y_i^{up} && i \in N \\
& Y_{ij}^1, Y_{ij}^2, Y_{ij}^3, Y_{ij}^4 \in \{True, False\} && i,j \in N, i<j \\
& W_{it} \in \{True, False\} && i \in N, t \in T
\end{aligned} \tag{9}
$$

In formulation (9) x_i and y_i represent the coordinates of the centre of the rectangles $i \in N$. $delx_{ij}$ and $dely_{ij}$ represent the distance between two rectangles $i, j \in N, i < j$, and c_{ij} is the cost associated with these. The first disjunctions, similarly to strip packing and farm layout problems, ensures that there is no overlap by expressing the possible relative position between rectangles i and j. The second set of disjunctions ensure that every rectangle i is inside one of the $t \in T$ circles. For a circle t, its coordinates (xc_t,yc_t) and its radius r_t are given.

3.7 Design of multi-product batch plant (Batch)

This problem seeks to minimize the investment cost in the design of a plant with multiple units in parallel and intermediate storage tanks [7]. The design involves selecting the number of parallel units, volume of the equipment, and volume and location of the intermediate storage tanks. This problem can be convexified[8], and the formulation is as follows:

$$
\begin{aligned}
& min\ Z = \alpha_1 \sum_j \exp(n_j + m_j + \beta_1 v_j) + \alpha_2 \sum_{Tj} \exp(\beta_2 v_{Tj}) \\
s.t. \quad & v_j \geq \ln(S_{ij}) + b_{ij} - n_j && \forall i, j \\
& e_i \geq \ln(T_{ij}) - b_{ij} - m_j && \forall i, j \\
& H \geq \sum_i (Q_i e_i) \\
& \begin{bmatrix} YS_j \\ v_{Tj} \geq \ln(S_j^*) + b_{ij+1} \quad \forall i \\ v_{Tj} \geq \ln(S_j^*) + b_{ij} \quad \forall i \\ b_{ij} - b_{ij+1} \leq \ln(S_{ij}^*) \quad \forall i \\ b_{ij} - b_{ij+1} \geq -\ln(S_{ij}^*) \quad \forall i \end{bmatrix} \vee \begin{bmatrix} \neg YS_j \\ v_{Tj} = 0 \\ b_{ij} - b_{ij+1} = 0 \quad \forall i \end{bmatrix} && \forall j < |J| \qquad (10) \\
& \begin{bmatrix} YM_{j,1} \\ m_j = \ln(1) \end{bmatrix} \vee ... \vee \begin{bmatrix} YM_{j,maxp} \\ m_j = \ln(maxp) \end{bmatrix} && \forall j \\
& \begin{bmatrix} YN_{j,1} \\ n_j = \ln(1) \end{bmatrix} \vee ... \vee \begin{bmatrix} YN_{j,maxp} \\ n_j = \ln(maxp) \end{bmatrix} && \forall j \\
& YM_{j,1} \veebar ... \veebar YM_{j,maxp} && \forall j \\
& YN_{j,1} \veebar ... \veebar YN_{j,maxp} && \forall j \\
& YS_j, YM_{j,p}, YN_{j,p} \in \{True, False\} && \forall j, p = 1, ..., maxp
\end{aligned}
$$

3.7.1 Nomenclature for design of a multi-product batch plant example.

Given:

$\alpha_1, \alpha_2, \beta_1, \beta_2$: Coefficients for the capital cost of the units and intermediate storage tanks.
$i \in I$: products.
$j \in J$: stages.
H: horizon time.
Q_i: production rate of product i.
T_{ij}: processing time of product i at stage j.
S_{ij}: size factor of product i at stage j.
S_j^*: size factor for intermediate storage tank.
S_{ij}^*: size factor for stages.

Determine:
B_{ij}: batch size product i at stage j.
E_i: production cycle time / batch size i.
M_j: number of units in parallel out-of-phase at stage j.
N_j: number of units in parallel in phase at stage j.
V_j: Unit size of stage j.
V_{Tj}: size of intermediate storage tank between stage j and $j+1$.

In order to convexify the problem, the following variables are introduced:
$b_{ij} = \ln(B_{ij})$
$e_i = \ln(E_i)$
$m_j = \ln(M_j)$
$n_j = \ln(N_j)$
$v_j = \ln(V_j)$
$v_{Tj} = \ln(V_{Tj})$

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