Online supplement to: "On bounding the bandwidth of graphs with symmetry"

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Lemma 3 (see Section 4.3). *The action of $\mathcal{H}(B(\beta))$ has 12 orbitals.*

Proof. In the following, we will consider an edge (s_1, s_2) in the cut graph (without loss of generality $s_1 \in S_1$ and $s_2 \in S_2$). Because of the simple structure of the cut graph, $\mathcal{H}(B(\beta))$ can be easily described. Indeed, because s_1 and s_2 are fixed by $P \in \text{stab}((s_1, s_2), B)$, the sets $S_1 \setminus \{s_1\}$ and $S_2 \setminus \{s_2\}$ are fixed (as sets) by $P(\beta) \in \mathcal{H}(B(\beta))$. This implies that $\mathcal{H}(B(\beta))$ is the direct product of the symmetric groups on $S_1 \setminus \{s_1\}$, $S_2 \setminus \{s_2\}$, and S_3. In fact, this is the full automorphism group of $B(\beta)$ in case $m_1 \neq m_2$ (in the case that $m_1 = m_2$ it is an index 2 subgroup of $\text{aut}(B(\beta))$ since the 'swapping' of $S_1 \setminus \{s_1\}$ and $S_2 \setminus \{s_2\}$ is not allowed). Therefore, the action of $\mathcal{H}(B(\beta))$ on β has 12 orbitals, similar as described in Example 1. □

Proposition 3 (see Section 4.3). *Let $r_1, r_2, s_1, s_2 \in \{1, \ldots, n\}$, $\alpha = \{1, \ldots, n\} \setminus \{r_1, r_2\}$, $\beta = \{1, \ldots, n\} \setminus \{s_1, s_2\}$, and $\hat{C}(\alpha, \beta) = 2A(\alpha, r_1)B(s_1, \beta) + 2A(\alpha, r_2)B(s_2, \beta)$. Then*

$$\mathcal{H}(B(\beta)) \otimes \mathcal{H}(A(\alpha))$$

is a subgroup of the automorphism group of $B(\beta) \otimes A(\alpha) + \text{Diag}(\text{vec}(\hat{C}(\alpha, \beta)))$.

Proof. Let $P_B \in \mathcal{H}(B(\beta))$ and $P_A \in \mathcal{H}(A(\alpha))$. It is clear that $P_B \otimes P_A$ is an automorphism of $B(\beta) \otimes A(\alpha)$, so we may restrict to showing that is also an automorphism of $\text{Diag}(\text{vec}(\hat{C}(\alpha, \beta)))$. In order to show this, we will use that for $i = 1, 2$ we have that

$$P_A^{\mathrm{T}} \text{Diag}(A(\alpha, r_i)) P_A = \text{Diag}(A(\alpha, r_i)) \quad \text{and} \quad P_B^{\mathrm{T}} \text{Diag}(B(\beta, s_i)) P_B = \text{Diag}(B(\beta, s_i)).$$

Indeed, the first equation is equivalent to the (valid) property that $a_{\pi(j)r_i} = a_{jr_i}$ for all $j \neq r_1, r_2$ and all automorphisms π of A that fix both r_1 and r_2, and the second equation is similar. Because $\text{vec}(\hat{C}(\alpha, \beta)) = 2B(\beta, s_1) \otimes A(\alpha, r_1) + 2B(\beta, s_2) \otimes A(\alpha, r_2)$, the result now follows from

$$\begin{aligned}
& (P_B \otimes P_A)^{\mathrm{T}} \text{Diag}\left[B(\beta, s_1) \otimes A(\alpha, r_1) + B(\beta, s_2) \otimes A(\alpha, r_2)\right](P_B \otimes P_A) \\
= \; & P_B^{\mathrm{T}} \text{Diag}(B(\beta, s_1)) P_B \otimes P_A^{\mathrm{T}} \text{Diag}(A(\alpha, r_1)) P_A \\
& + P_B^{\mathrm{T}} \text{Diag}(B(\beta, s_2)) P_B \otimes P_A^{\mathrm{T}} \text{Diag}(A(\alpha, r_2)) P_A \\
= \; & \text{Diag}[B(\beta, s_1) \otimes A(\alpha, r_1) + B(\beta, s_2) \otimes A(\alpha, r_2)],
\end{aligned}$$

where we have used the properties in (1) of the Kronecker product. □

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Appendices

A $*$-isomorphism from Example 1

The associated $*$-isomorphism φ satisfies:

$$\varphi(B_1) = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix}, \quad \varphi(B_2) = \begin{pmatrix} -1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & m_1-1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(B_3) = \sqrt{m_1 m_2} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix}, \quad \varphi(B_4) = \sqrt{m_1 m_3} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(B_5) = \sqrt{m_1 m_2} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix}, \quad \varphi(B_6) = \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(B_7) = \begin{pmatrix} 0 & & & & & \\ & -1 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & m_2-1 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix}, \quad \varphi(B_8) = \sqrt{m_2 m_3} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(B_9) = \sqrt{m_1 m_3} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \end{pmatrix}, \quad \varphi(B_{10}) = \sqrt{m_2 m_3} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix},$$

$$\varphi(B_{11}) = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \end{pmatrix}, \quad \varphi(B_{12}) = \begin{pmatrix} -1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & m_3-1 \end{pmatrix}.$$

B Symmetry reduction of $\mathrm{MC}^h_{\mathrm{fix}}$

Let G be an undirected graph on n vertices with adjacency matrix A and t orbitals $\mathcal{O}_h$ ($h = 1, 2, \ldots, t$). Let (s_1, s_2) be an arbitrary edge in the cut graph G_{m_1,m_2,m_3} with the adjacency matrix B, and (r_{h1}, r_{h2}) be an arbitrary pair of vertices in $\mathcal{O}_h$ ($h = 1, 2, \ldots, t$). We let $\alpha^h = \{1, \ldots, n\} \backslash \{r_{h1}, r_{h2}\}$ and $\beta = \{1, \ldots, n\} \backslash \{s_1, s_2\}$. Now, the relaxation $\mathrm{MC}^h_{\mathrm{fix}}$

(see Section 4.2) reduces to

$$\begin{aligned}\min \quad & \tfrac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{12} p_i^{-1}\operatorname{tr}(A(\alpha^h)A_i)x_j^{(i)} \\ & + \sum_{i\in\mathcal{I}_\mathcal{A}}\sum_{j\in\{1,6,11\}}(q_jp_i)^{-1}B(\beta,s_1)^{\mathrm{T}}\operatorname{diag}(B_j)A(\alpha^h,r_{h1})^{\mathrm{T}}\operatorname{diag}(A_i)x_j^{(i)} \\ & \quad + \sum_{i\in\mathcal{I}_\mathcal{A}}\sum_{j\in\{1,6,11\}}(q_jp_i)^{-1}B(\beta,s_2)^{\mathrm{T}}\operatorname{diag}(B_j)A(\alpha^h,r_{h2})^{\mathrm{T}}\operatorname{diag}(A_i)x_j^{(i)} + \tfrac{1}{2}d^h \\ \text{s.t.} \quad & \sum_{i\in\mathcal{I}_\mathcal{A}} x_1^{(i)} = q_1, \quad \sum_{i\in\mathcal{I}_\mathcal{A}} x_6^{(i)} = q_6 \quad \sum_{i\in\mathcal{I}_\mathcal{A}} x_{11}^{(i)} = q_{11} \\ & \sum_{i=1}^{d}\sum_{j=1}^{12} q_j^{-1}x_j^{(i)}B_j = J_{n-2} \\ & \sum_{j=1}^{12} x_j^{(i)} = p_i, \quad i=1,\ldots,d \\ & \sum_{i=1}^{d}\sum_{j=1}^{12}\frac{1}{q_jp_i}x_j^{(i)}(B_j\otimes A_i)\succeq 0 \\ & x_j^{(i)}\geq 0,\ x_{j^*}^{(i)} = x_j^{(i^*)}, \quad i=1,\ldots,d,\ j=1,\ldots,12,\end{aligned} \tag{1}$$

where B_j ($j=1,\ldots,12$) is defined in Example 1, and $\{A_i : i=1,\ldots,d\}$ spans $\mathcal{H}(A(\alpha^h))$. The set $\mathcal{I}_\mathcal{A} := \mathcal{I}_{\mathcal{H}(A(\alpha^h))}$ is as in Definition 1, $p_i = \operatorname{tr}(J_{n-2}A_i)$, $i=1,\ldots,d$, $q_j = \operatorname{tr}(J_{n-2}B_j)$, $j=1,\ldots,12$. The constraint $x_{j^*}^{(i)} = x_j^{(i^*)}$ requires that the variables $x_j^{(i)}$ form complementary pairs. The SDP relaxation (1) can be further simplified by exploiting the $*$-isomorphism associated to $\mathcal{H}(B(\beta))$, see Appendix A.

C Orbitals in stabilizer subgroups

Table 11: Number of orbitals in $\mathcal{H}(Q_d(\alpha))$.

d	♯ orbitals				
4	80	100	80	35	–
5	140	200	200	140	56

Table 12: Number of orbitals in $\mathcal{H}([H(d,q)](\alpha))$.

d	q	♯ orbitals			
3	3	135	225	165	–
3	4	150	275	220	–
3	5	150	275	220	–
3	6	150	275	220	–
4	3	315	675	825	495

Table 13: Number of orbitals in $\mathcal{H}(H_{q_1,q_2,q_3}(\alpha))$

q_1	q_2	q_3	♯ orbitals						
2	3	3	180	180	60	180	180	–	–
2	3	4	200	180	360	100	200	180	360
2	3	5	200	180	360	100	200	180	360
2	4	4	200	220	60	200	220	–	–
3	3	4	150	225	450	225	450	–	–
3	3	5	150	225	450	225	450	–	–
3	4	4	250	275	135	450	495	–	–
3	4	5	250	250	500	225	450	450	900

Table 14: Number of orbitals in $\mathcal{H}([J(v,3)](\alpha))$ and $\mathcal{H}([J(v,4)](\alpha))$.

v	d	♯ orbitals			
6	3	88	88	24	–
7	3	195	257	90	–
8	3	220	333	158	–
9	3	227	361	203	–
10	3	228	368	220	–
11	3	228	369	225	–
8	4	220	358	220	46
9	4	484	916	742	195