# Online Supplement for “Optimal Budget Allocation across Search Advertising Markets”

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## Appendix A: Proof of Theorem 1 and Theorem 2

*Proof.* In order to solve the constrained optimization problem with constraint $\sum_i \int_0^T e^{-rt} b_i(t)\mathrm{d}t \leqslant B$, we introduce a Lagrange multiplier $\lambda \geq 0$. Suppose $C_i(t)$, $q_i(t)$ and $\alpha_i(t)$ are all continuous differentiable functions. Starting with a pair of initial market shares with random values $(y_1, y_2) \in [0,1] \times [0,1]$ at time $t$, we define the value function $V_\lambda(t, y_1, y_2)$ to be the maximal payoff we can obtain during time interval $[t, T]$. That is,

$$
\begin{aligned}
&V_\lambda(t, y_1, y_2) := \\
&\sup_{b_1(\cdot)\geq 0, b_2(\cdot)\geq 0} \quad \int_t^T e^{-rs}\{C_1(s)\theta_1(s) + C_2(s)\theta_2(s) \\
&\qquad\qquad -b_1(s) - b_2(s) - \lambda(b_1(s) + b_2(s))\}\mathrm{d}s,
\end{aligned}
$$

where we set $C_i(t) = v_i(t)m_i(t)c_i(t)$ for simplicity.

Notice that for any pair of $(y_1, y_2) \in [0,1] \times [0,1]$, $V_\lambda(T, y_1, y_2) = 0$. Following the principle of dynamic programming, we have the Hamilton-Jacobi-Bellman equation:

$$
\begin{aligned}
0 = \quad V_{\lambda,t} + \quad & \max_{\{b_1 \geq 0, b_2 \geq 0\}} \{\rho_1 q_1 (b_1)^{\alpha_1} \sqrt{1-\theta_1} \cdot V_{\lambda,\theta_1} \\
& + \rho_2 q_2 (b_2)^{\alpha_2} \sqrt{1-\theta_2} \cdot V_{\lambda,\theta_2} \\
& + e^{-rt}(C_1\theta_1 + C_2\theta_2 - (1+\lambda)(b_1+b_2))\}
\end{aligned} \tag{1}
$$

Differentiating (1), we obtain the optimal feedback advertising decision:

$$
\begin{aligned}
b_1 &= \left(\tfrac{(1+\lambda)e^{-rt}}{\rho_1 q_1 \alpha_1 \sqrt{1-\theta_1} V_{\lambda,\theta_1}}\right)^{1/(\alpha_1-1)}, \\
b_2 &= \left(\tfrac{(1+\lambda)e^{-rt}}{\rho_2 q_2 \alpha_2 \sqrt{1-\theta_2} V_{\lambda,\theta_2}}\right)^{1/(\alpha_2-1)}
\end{aligned} \tag{2}
$$

Substituting (2) into (1), we obtain the Hamilton-Jacobi equation:

$$
\begin{aligned}
& V_{\lambda,t} \\
+ \quad & (1-\alpha_1)(e^{rt}\frac{\alpha_1}{1+\lambda})^{\frac{\alpha_1}{1-\alpha_1}} (\rho_1 q_1 \sqrt{1-\theta_1} V_{\lambda,\theta_1})^{\frac{1}{1-\alpha_1}} \\
+ \quad & (1-\alpha_2)(e^{rt}\frac{\alpha_2}{1+\lambda})^{\frac{\alpha_2}{1-\alpha_2}} (\rho_2 q_2 \sqrt{1-\theta_2} V_{\lambda,\theta_2})^{\frac{1}{1-\alpha_2}} \\
+ \quad & e^{-rt}(C_1\theta_1 + C_2\theta_2) = 0.
\end{aligned} \tag{3}
$$

Solve (3) with the terminal value condition $V_\lambda(T, \theta_1, \theta_2) = 0$, we get $V_\lambda(t, \theta_1, \theta_2)$. Put it into (2), then we can obtain the optimal budget allocation strategy: $b^*_{\lambda,1}(t, \theta_1, \theta_2), b^*_{\lambda,2}(t, \theta_1, \theta_2)$.

Now let us determine the constant $\lambda$. Because

$$
\lambda \left( B - \int_0^T e^{-rt}(b^*_{\lambda,1}(t,\theta_1,\theta_2) + b^*_{\lambda,2}(t,\theta_1,\theta_2))dt \right) = 0 \tag{4}
$$

If

$$
\int_0^T e^{-rt}(b^*_{0,1}(t,\theta_1,\theta_2) + b^*_{0,2}(t,\theta_1,\theta_2))dt \leq B,
$$

where we set $\lambda = 0$. Note that this is the case when the advertising budget is sufficient or unlimited. Then let us consider the case with the limited budget. That is,

$$\int_0^T e^{-rt}(b^*_{0,1}(t,\theta_1,\theta_2)+b^*_{0,2}(t,\theta_1,\theta_2))dt > B,$$

We set the minimal $\lambda > 0$, then

$$\int_0^T e^{-rt}(b^*_{\lambda,1}(t,\theta_1,\theta_2)+b^*_{\lambda,2}(t,\theta_1,\theta_2))dt = B. \tag{5}$$

The choice of the control variables $b_1$ and $b_2$ depends on the value of $\lambda$. Given a fixed $\lambda$ we denote the optimal control solution with $b^*_{\lambda,1}$ and $b^*_{\lambda,2}$. Therefore, the choice of $\lambda$ together with correspondent optimal budget allocation strategy $(b^*_{\lambda,1}(t,\theta_1,\theta_2), b^*_{\lambda,2}(t,\theta_1,\theta_2))$ provides a theoretical solution for model (3). □

# Appendix B: Proof of Corollary 1

*Proof.* From (2), we always have the following equality:

$$\begin{aligned} & \rho_1 q_1 \alpha_1 (b_1^*)^{\alpha_1 - 1}\sqrt{1-\theta_1^*}V_{\theta_1^*} \\ = \;& \rho_2 q_2 \alpha_2 (b_2^*)^{\alpha_2 - 1}\sqrt{1-\theta_2^*}V_{\theta_2^*} \\ = \;& e^{-rt}(1+\lambda). \end{aligned}$$

Rewrite the above equality as:

$$V_{\theta_1^*}\frac{\mathrm{d}\theta_1^*}{\mathrm{d}t}\frac{\alpha_1(t)}{b_1^*(t)} = V_{\theta_2^*}\frac{\mathrm{d}\theta_2^*}{\mathrm{d}t}\frac{\alpha_2(t)}{b_2^*(t)} = e^{-rt}(1+\lambda).$$

□

# Appendix C: Proof of Theorem 3

*Proof.* Let $(b_1^*(t), b_2^*(t))$ be the optimal budget allocation strategy for model (5). Thus,

$$\int_0^T e^{-rt}(b_1^*(t)+b_2^*(t))\mathrm{d}t = R^*(0) - R^*(T) \le B,$$

$$\mathrm{d}\theta_1^*/\mathrm{d}t = \rho_1 q_1 (b_1^*)^{\alpha_1(t)} \sqrt{1-\theta_1^*},$$

$$\mathrm{d}\theta_2^*/\mathrm{d}t = \rho_2 q_2 (b_2^*)^{\alpha_2(t)} \sqrt{1-\theta_2^*},$$

and $b_i^*(t) \geqslant 0$.

Let $(\tilde{b}_1(t), \tilde{b}_2(t))$ be the optimal budget allocation strategy for model (3), and $\tilde{\lambda}$ be the corresponding Lagrange multiplier. By the definition of "optimum", we get

$$\begin{aligned} &\int_0^T e^{-rt}\{C_1(t)\tilde{\theta}_1(t) - \tilde{b}_1(t) \\ &\quad + C_2(t)\tilde{\theta}_2(t) - \tilde{b}_2(t) - \tilde{\lambda}(\tilde{b}_1(t) + \tilde{b}_2(t))\}\mathrm{d}t \\ &\geq \int_0^T e^{-rt}\{C_1(t)\theta_1^*(t) - b_1^*(t) \\ &\quad + C_2(t)\theta_2^*(t) - b_2^*(t) - \tilde{\lambda}(b_1^*(t) + b_2^*(t))\}\mathrm{d}t, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\int_0^T e^{-rt}\{C_1(t)\tilde{\theta}_1(t) - \tilde{b}_1(t) + C_2(t)\tilde{\theta}_2(t) - \tilde{b}_2(t)\}\mathrm{d}t \\ \leq\ &\int_0^T e^{-rt}\{C_1(t)\theta_1^*(t) - b_1^*(t) + C_2(t)\theta_2^*(t) - b_2^*(t)\}\mathrm{d}t. \end{aligned} \tag{7}$$

$(i)$ On the one hand, comparing (6) with (7), we have

$$\begin{aligned} &\tilde{\lambda}\int_0^T e^{-rt}(\tilde{b}_1(t) + \tilde{b}_2(t))\}\mathrm{d}t \\ \leq\ &\tilde{\lambda}\int_0^T e^{-rt}(b_1^*(t) + b_2^*(t))\}\mathrm{d}t \leq \tilde{\lambda}B. \end{aligned} \tag{8}$$

If $\tilde{\lambda} = 0$, then

$$\begin{aligned} &\textstyle\int_0^T e^{-rt}\{C_1(t)\tilde{\theta}_1(t) - \tilde{b}_1(t) + C_2(t)\tilde{\theta}_2(t) - \tilde{b}_2(t)\}\mathrm{d}t \\ =\ &\textstyle\int_0^T e^{-rt}\{C_1(t)\theta_1^*(t) - b_1^*(t) + C_2(t)\theta_2^*(t) - b_2^*(t)\}\mathrm{d}t. \end{aligned}$$

This proves that $(b_1^*(t), b_2^*(t))$ is also the optimal budget allocation strategy for model (3).

If $\tilde{\lambda} \neq 0$, from (8), we get $\int_0^T e^{-rt}(\tilde{b}_1(t) + \tilde{b}_2(t))\}\mathrm{d}t \leq B$. Following the method for determining $\lambda$ in (5), $\tilde{\lambda} = 0$, which contradicts $\tilde{\lambda} \neq 0$. Therefore, the optimal budget allocation strategy of model (5) is also the optimal budget allocation strategy of model (3).

$(ii)$ On the other hand, we can show that $(\tilde{b}_1(t), \tilde{b}_2(t))$ is also the optimal budget allocation strategy for model (5) as well.

If $\tilde{\lambda} = 0$, then

$$\begin{aligned} & \int_0^T e^{-rt}\{C_1(t)\tilde{\theta}_1(t) - \tilde{b}_1(t) + C_2(t)\tilde{\theta}_2(t) - \tilde{b}_2(t)\}\mathrm{d}t \\ = & \int_0^T e^{-rt}\{C_1(t)\theta_1^*(t) - b_1^*(t) + C_2(t)\theta_2^*(t) - b_2^*(t)\}\mathrm{d}t. \end{aligned}$$

Thus, $(\tilde{b}_1(t), \tilde{b}_2(t))$ is also the optimal budget allocation strategy for model (5).

If $\tilde{\lambda} > 0$, then $\int_0^T e^{-rt}(\tilde{b}_1(t) + \tilde{b}_2(t))\}\mathrm{d}t = B$.

$$\begin{aligned} & -\tilde{\lambda}\int_0^T e^{-rt}(\tilde{b}_1(t) + \tilde{b}_2(t))\}\mathrm{d}t \\ \leq & -\tilde{\lambda}\int_0^T e^{-rt}(b_1^*(t) + b_2^*(t))\}\mathrm{d}t. \end{aligned}$$

Using (6), we get

$$\begin{aligned} & \int_0^T e^{-rt}\{C_1(t)\tilde{\theta}_1(t) - \tilde{b}_1(t) + C_2(t)\tilde{\theta}_2(t) - \tilde{b}_2(t)\}\mathrm{d}t \\ \geq & \int_0^T e^{-rt}\{C_1(t)\theta_1^*(t) - b_1^*(t) + C_2(t)\theta_2^*(t) - b_2^*(t)\}\mathrm{d}t, \end{aligned}$$

which implies that $(\tilde{b}_1(t), \tilde{b}_2(t))$ is also the optimal budget allocation strategy for model (5).

Therefore, the optimization problem (3) is equivalent to the optimization problem (5). $\square$