Online Supplement for

“Model counting of monotone CNF formulas with Spectra”

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Proof of Lemma 2:

Define the event

$$A_j(G) = \{\pi = i_1, \ldots, i_n \in \Pi \mid v_{i_1}, \ldots, v_{i_n} \text{ first form a vertex cover in } G \text{ at index } i_j\}.$$

From this and similarly to Definitions 2 and 3 from Section 2 we define the Construction Spectrum

$$S_p = \{f_0, f_1, \cdots, f_n\}$$

where

$$f_i = \mathbb{P}_\Pi\left(A_i(G)\right) = \mathbb{P}(\pi \in A_i(G)) = \frac{|A_i(G)|}{n!}$$

and the Cumulative Spectrum

$$F(k) = \sum_{i=0}^{k} f_i, \;\; k = 0, \ldots, n.$$

respectively. We continue with

$$\begin{aligned}\mathbb{E}_\Pi(2^n X) &= 2^n \mathbb{E}_\Pi(X) = 2^n \sum_{j=0}^{n} \left(\mathbb{P}(A_j(G)) 2^{-n} \sum_{i=j}^{n} \binom{n}{i} \right) = \\ &= 2^n \sum_{j=0}^{n} \left(f_j 2^{-n} \sum_{i=j}^{n} \binom{n}{i} \right) = \sum_{j=0}^{n} \left(f_j \sum_{i=j}^{n} \binom{n}{i} \right) = \\ &= \sum_{j=0}^{n} \left(\binom{n}{j} \sum_{i=0}^{j} f_i \right) = \sum_{j=0}^{n} \binom{n}{j} F(j) = |\mathcal{X}|.\end{aligned}$$

$\square$

Proof of Lemma 5:

The intuition behind this lemma is as follows. When one choose a graph uniformly at random from $G(n, \frac{1}{2})$, the actual number of cliques, independent sets and vertex covers will be very close to the average over all graphs, or, more formally, if G was chosen u.a.r. from $G(n, \frac{1}{2})$, then

$$|\mathcal{X}_G^{clq}| = |\mathcal{X}_G^{is}| = |\mathcal{X}_G| \approx \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2}^{\binom{i}{2}}, \quad n \to \infty.$$

Let Y_i^{clq}, Y_i^{is} and Y_i^{vc} be random variables representing the number of cliques, independent sets and vertex covers of size i respectively. Rasmussen (1997) pp. $407-408$ proves that

$$\frac{\mathbb{E}_\Omega\left(\left(Y_e^{clq}\right)^2\right)}{\mathbb{E}_\Omega\left(Y_e^{clq}\right)^2} \le 1+o(1) \text{ for } e \approx \log_2(n)-\log_2\log_2(n), \quad n\to\infty. \tag{1}$$

Combining (1) with Chebyshev's inequality for any $\xi > 10^{-8}$ and for any constant C one can write

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4}C^{-1}Y_e^{clq} < \mathbb{E}_\Omega(Y_e^{clq})\right) = 0.$$

Now, taking C to be

$$C \approx 2^{\frac{5}{8}}\sqrt{\frac{\pi}{\ln(2)}},$$

and combining with Lemma 11 from (Rasmussen 1997) we get that

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4}Y_e^{clq} < C\mathbb{E}(Y_e^{clq})\right) = 0 \overset{Y_e^{clq}\le|\mathcal{X}_G^{clq}|}{\Rightarrow} \tag{2}$$
$$\overset{Y_e^{clq}\le|\mathcal{X}_G^{clq}|}{\Rightarrow} \lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4}|\mathcal{X}_G^{clq}| < \sum_{i=0}^{n}\binom{n}{i}\frac{1}{2}^{\binom{i}{2}}\right) = 0.$$

Recall that in graph theory, the complement or inverse of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. Having in mind that we are working under $G(n,\frac{1}{2})$ random graph model, the generation of G and H happens with the same probability. Moreover, every clique of size i in G corresponds to an independent set of size i in H and as a consequence this clique corresponds to the specific vertex cover in H. From this symmetry we can conclude that the random variables $Y_i^{(clq)}, Y_i^{(is)}$ and $Y_{n-i}^{(vc)}$ have the same distribution under Ω because for any $y \in \mathbb{N}$

$$\mathbb{P}_\Omega\left(Y_i^{(clq)} = y\right) = \mathbb{P}_\Omega\left(Y_i^{(is)} = y\right) = \mathbb{P}_\Omega\left(Y_{n-i}^{(vc)} = y\right).$$

Let us mark this equality in distribution by

$$Y_i^{(clq)} \sim Y_i^{(is)} \sim Y_{n-i}^{(vc)}.$$

Now, (2) can be rewritten as

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4}Y_e^{clq} < C\mathbb{E}_\Omega(Y_e^{clq})\right) = 0 \overset{Y_i^{(clq)}\sim Y_i^{(is)}}{\Rightarrow}$$

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4} Y_e^{is} < C\mathbb{E}_\Omega(Y_e^{clq})\right) = 0 \overset{Y_i^{(is)} \sim Y_{n-i}^{(vc)}}{\Rightarrow}$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4} Y_{n-e}^{vc} < C\mathbb{E}_\Omega(Y_e^{clq})\right) = 0 \overset{Y_{n-e}^{vc} \le |\mathcal{X}_G|}{\Rightarrow}$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4} |\mathcal{X}_G| < \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2}^{\binom{i}{2}}\right) = 0,$$

completing the proof. □

Proof of Lemma 6:

By Markov inequality we have that

$$\mathbb{P}(Z > a) \le \frac{\mathbb{E}(Z)}{a}, \quad \text{taking } Z = \mathbb{E}_\Pi\left(X^2\right) \text{ and } a = n^{\frac{\xi}{4}} \mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right) \Rightarrow$$

$$\Rightarrow \mathbb{P}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right) \ge n^{\frac{\xi}{4}} \mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right)\right) \le \frac{1}{n^{\frac{\xi}{4}}} \to 0, \quad n \to \infty. \tag{3}$$

Recall that

$$\mathbb{E}_\Pi(2^n X) = |\mathcal{X}_{G_\Omega}| \text{ and } \mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(2^n X\right)\right) = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2}^{\binom{i}{2}},$$

so, using the concentration result from Lemma 5 we conclude that when $n \to \infty$ we also have

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4} |\mathcal{X}_{G_\Omega}| < \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2}^{\binom{i}{2}}\right) = 0 \quad \Rightarrow \tag{4}$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(n^{\xi/4} \mathbb{E}_\Pi(2^n X) < \mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(2^n X\right)\right)\right) = 0 \quad \overset{\div 2^n}{\Rightarrow}$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(\mathbb{E}_\Pi(X) < \frac{\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X\right)\right)}{n^{\xi/4}}\right) = 0 \quad \overset{\text{power}\{2-\varepsilon\}}{\Rightarrow}$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(\left[\mathbb{E}_\Pi(X)\right]^{2-\varepsilon} < \left[\frac{\left[\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X\right)\right)\right]}{n^{\frac{\xi}{4}}}\right]^{2-\varepsilon}\right) = 0 \quad \Rightarrow$$
$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(\left[\mathbb{E}_\Pi(X)\right]^{2-\varepsilon} < \frac{\left[\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X\right)\right)\right]^{2-\varepsilon}}{n^{\frac{(2-\varepsilon)\xi}{4}}}\right) = 0.$$

From (3) and (4) we conclude that when $n \to \infty$, the following expressions holds:

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right) \le n^{\frac{\xi}{4}} \mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right)\right) = 1 \tag{5}$$

$$\lim_{n\to\infty} \mathbb{P}_\Omega\left(\left[\mathbb{E}_\Pi(X)\right]^{2-\varepsilon} \ge \frac{\left[\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X\right)\right)\right]^{2-\varepsilon}}{n^{\frac{(2-\varepsilon)\xi}{4}}}\right) = 1. \tag{6}$$

Combining (5) and (6) results in

$$\lim_{n\to\infty} \mathbb{P}_\Omega \left(\frac{\mathbb{E}_\Pi\left(X^2\right)}{\mathbb{E}_\Pi\left(X\right)^{2-\varepsilon}} \leq n^{\frac{1}{4}\xi} \frac{\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right)}{\frac{\mathbb{E}_\Omega(\mathbb{E}_\Pi(X))^{2-\varepsilon}}{n^{\frac{(2-\varepsilon)\xi}{4}}}} \leq n^{\frac{3-\varepsilon}{4}\xi} \frac{\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right)}{\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X\right)\right)^{2-\varepsilon}} \right) = 1$$

thus completing the proof. □

Proof of Lemma 7: Consider the continues counterpart of $b(n,k)$. Denote the latter by $b'(n,x)$, where $x \in \{y \in \mathbb{R} \mid 0 \leq y \leq n\}$. Let $x^* \in \mathbb{R}$ be the value such that $\ln\left(\sqrt{b'(n,x^*)}\right)$ is maximal. Note that since both square root and natural logarithm are monotonic, x^* brings $b'(n,x^*)$ to its maximum. Lemma 1 yields the concavity of $\ln\left(\sqrt{b'(n,x^*)}\right)$; that is, x^* is the function's global maximum (note that x^* is not necessary unique). We will prove that all optimal points are located in the $[2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2}, 2(\log_2(n) - \log_2\log_2(n)) + \frac{1}{2}]$ neighbourhood. Hence, the natural number k^* that brings $b(n,k)$ to its greatest value satisfies:

$$\left\lfloor 2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2} \right\rfloor \leq k^* \leq \left\lceil 2(\log_2(n) - \log_2\log_2(n)) + \frac{1}{2} \right\rceil .$$

We are interested in asymptotic behavior, that is, our analysis is for $n \to \infty$, $x \sim 2(\log_2(n) - \log_2\log_2(n)) \to \infty$ and $n - x \sim n - 2(\log_2(n) - \log_2\log_2(n)) \to \infty$. From the Stirling's asymptotic formula $\lim_{z\to\infty} z!/\sqrt{2\pi z}\,(z/e)^z = 1$, we arrive at

$$\ln(z!) \sim z\ln(z) - z + \frac{1}{2}\ln(z) + \frac{1}{2}\ln(2\pi), \quad z \to \infty. \tag{7}$$

Applying (7) on $\ln\left(\sqrt{b(n,x)}\right)$ results in

$$\begin{aligned}
\ln\left(\sqrt{b(n,x)}\right) &= \ln\left(\binom{n}{x}\frac{1}{2}^{\frac{1}{2}\binom{x}{2}}\right) = \ln\left(\frac{n!}{x!(n-x)!}\frac{1}{2}^{\frac{1}{2}\binom{x}{2}}\right) = \\
&= \ln(n!) - (\ln(x!) + \ln((n-x)!)) + \ln\left(\frac{1}{2}^{\frac{1}{2}\binom{x}{2}}\right) \sim \\
&\underbrace{\sim}_{(7)} n\ln(n) - n - (x\ln(x) - x + (n-x)\ln(n-x) - (n-x)) + \\
&+ \frac{1}{2}(\ln(n) - \ln(x) - \ln(n-x) - \ln(2\pi)) + \frac{1}{2}\binom{x}{2}\ln\left(\frac{1}{2}\right) = \\
&= n\ln(n) - x\ln(x) - (n-x)\ln(n-x) + \frac{1}{2}\left(\ln(n) - \ln(x) - \ln(n-x) - \right. \\
&- \left. \ln(2\pi)\right) + \frac{x(x-1)}{4}\ln\left(\frac{1}{2}\right), \quad n\to\infty,\ x\to\infty,\ n-x\to\infty.
\end{aligned}$$

We continue with the derivative for $n \to \infty$, $x \to \infty$, $n - x \to \infty$:

$$\begin{aligned} \frac{\partial \ln\left(\sqrt{b'(n,x)}\right)}{\partial x} &\sim -(\ln(x)+1) - (-\ln(n-x)-1) + \tfrac{1}{2}\left(\tfrac{1}{n-x} - \tfrac{1}{x}\right) + \ln\left(\tfrac{1}{2}\right)\left(\tfrac{x}{2} - \tfrac{1}{4}\right) \\ &= \ln(n-x) - \ln(x) + \tfrac{1}{2}\left(\tfrac{1}{n-x} - \tfrac{1}{x}\right) + \ln\left(\tfrac{1}{2}\right)\left(\tfrac{x}{2} - \tfrac{1}{4}\right). \end{aligned} \tag{8}$$

By letting the derivative from (8) to be equal to zero we immediately derive

$$\begin{aligned} \frac{\partial \ln\left(\sqrt{b'(n,x)}\right)}{\partial x} &= 0 \Rightarrow \ln(n-x) - \ln(x) = -\ln\left(\frac{1}{2}\right)\left(\frac{x}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{n-x} - \frac{1}{x}\right) \Rightarrow \\ &\Rightarrow \ln(n-x) - \ln(x) = -\ln\left(\frac{1}{2}\right)\left(\frac{x}{2} - \frac{1}{4}\right) - \ln\left(\frac{1}{2}\right)\left(\frac{\frac{1}{2}\left(\frac{1}{n-x} - \frac{1}{x}\right)}{\ln\left(\frac{1}{2}\right)}\right) \Rightarrow \\ &\Rightarrow e^{\ln(n-x)-\ln(x)} = e^{-\ln\left(\frac{1}{2}\right)\left(\frac{x}{2} - \frac{1}{4} + \frac{\frac{1}{2}\left(\frac{1}{n-x} - \frac{1}{x}\right)}{\ln\left(\frac{1}{2}\right)}\right)} \Rightarrow \\ &\Rightarrow \left(\frac{n-x}{x}\right) = 2^{\frac{x}{2} - \frac{1}{4} - \frac{1}{2\ln(2)}\left(\frac{1}{n-x} - \frac{1}{x}\right)} \Rightarrow \\ &\Rightarrow n - x\left(2^{\frac{x}{2} - \frac{1}{4} - \frac{1}{2\ln(2)}\left(\frac{1}{n-x} - \frac{1}{x}\right)} + 1\right) = 0. \end{aligned}$$

Note that when $n \to \infty$ and $x \to 2\left(\log_2(n) - \log_2\log_2(n)\right)$, $\frac{1}{2\ln(2)}\left(\frac{1}{n-x} - \frac{1}{x}\right) \to 0$, so

$$n - x\left(2^{\frac{x}{2} - \frac{1}{4} - \frac{1}{2\ln(2)}\left(\frac{1}{n-x} - \frac{1}{x}\right)} + 1\right) \sim n - x\left(2^{\frac{x}{2} - \frac{1}{4}} + 1\right).$$

Consider now the continuous function $g(n,x) = n - x\left(2^{\frac{x}{2} - \frac{1}{4}} + 1\right)$. We will show that for $x = 2\left(\log_2(n) - \log_2\log_2(n)\right) - 1\frac{1}{2}$ and $x = 2(\log_2(n) - \log_2\log_2(n)) + \frac{1}{2}$, $\lim_{n\to\infty} g(n,x)$ is ∞ and $-\infty$ respectively.

- For the case of $x = 2\left(\log_2(n) - \log_2\log_2(n)\right) - 1\frac{1}{2}$:

$$\begin{aligned} &\lim_{n\to\infty} g(n,x) = \\ &= n - \left(2\left(\log_2(n) - \log_2\log_2(n)\right) - 1\frac{1}{2}\right)\left(2^{\frac{2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2}}{2} - \frac{1}{4}} + 1\right) = \\ &= n - \left(2\left(\log_2(n) - \log_2\log_2(n)\right) - 1\frac{1}{2}\right)\left(\frac{n}{2\log_2(n)} + 1\right) = \\ &= n - n + \frac{n\log_2\log_2(n)}{\log_2(n)} + \frac{\frac{3}{4}n}{\log_2(n)} - 2\log_2(n) + 2\log_2\log_2(n) + 1\frac{1}{2} = \\ &= \underbrace{\frac{n\log_2\log_2(n)}{\log_2(n)} + 2\log_2\log_2(n) + 1\frac{1}{2}}_{\to\infty} + \frac{\frac{3}{4}n}{\log_2(n)} - 2\log_2(n) = \\ &= \underbrace{\frac{n\log_2\log_2(n)}{\log_2(n)} + 2\log_2\log_2(n) + 1\frac{1}{2}}_{\to\infty} + \frac{3}{4}\log_2(n)\left(\frac{n}{\log_2^2(n)} - \frac{2}{\frac{3}{4}}\right) = \end{aligned} \tag{9}$$

$$= \underbrace{\frac{n\log_2\log_2(n)}{\log_2(n)} + 2\log_2\log_2(n) + 1\frac{1}{2}}_{\to\infty} + \frac{3}{4}\log_2(n)\left(\left(\frac{\sqrt{n}}{\log_2(n)}\right)^2 - \frac{8}{3}\right) \overset{n\to\infty}{=} \infty.$$

The last equality result from

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{\log_2(n)}\right) = \infty.$$

- For the case $x = 2(\log_2(n) - \log_2\log_2(n)) + \frac{1}{2}$:

$$\begin{aligned}
&\lim_{n\to\infty} g(n,x) = \qquad (10)\\
&= n - \left(2\left(\log_2(n) - \log_2\log_2(n)\right) + \frac{1}{2}\right)\left(2^{\frac{2(\log_2(n)-\log_2\log_2(n))+\frac{1}{2}}{2}-\frac{1}{4}} + 1\right) =\\
&= n - \left(2\left(\log_2(n) - \log_2\log_2(n)\right) + \frac{1}{2}\right)\left(\frac{n}{\log_2(n)} + 1\right) =\\
&= n - 2n + \frac{2n\log_2\log_2(n)}{\log_2(n)} - \frac{n}{2\log_2(n)} - 2\log_2(n) + 2\log_2\log_2(n) - \frac{1}{2} =\\
&= n\left(1 - 2 + \frac{2\log_2\log_2(n)}{\log_2(n)} - \frac{1}{2\log_2(n)} - \frac{2\log_2(n)}{n} + \frac{2\log_2\log_2(n)}{n} - \frac{1}{2n}\right) \overset{n\to\infty}{=} -\infty.
\end{aligned}$$

From (9), (10) and the fact that due to concavity the first derivative should be non increasing, we conclude that x^* (for which $g(n,x^*) = 0$) satisfies

$$2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2} \leq x^* \leq 2(\log_2(n) - \log_2\log_2(n)) + \frac{1}{2},$$

thus completing the proof. □

Proof of Lemma 8:

For (12) we have that:

$$\begin{aligned}
&\mathbb{E}_\Omega\left(\mathbb{E}_\Pi\left(X^2\right)\right) = \mathbb{E}_\Omega\left(\sum_{j=0}^{n}\left(f_j\left(2^{-n}\sum_{i=j}^{n}\binom{n}{i}\right)^2\right)\right) \overset{\left(\sum_{i=1}^{n} x_i\right)^2 \leq n\left(\sum_{i=1}^{n} x_i^2\right)\ \text{(Chandler 1987)}}{\leq}\\
&\leq 2^{-2n}n\mathbb{E}_\Omega\left(\sum_{j=0}^{n} f_j\left(\sum_{i=j}^{n}\binom{n}{i}^2\right)\right) = 2^{-2n}n\mathbb{E}_\Omega\left(\sum_{j=0}^{n}\left(\binom{n}{j}^2\sum_{i=0}^{j} f_i\right)\right) =\\
&= 2^{-2n}n\left(\sum_{j=0}^{n}\binom{n}{j}^2\mathbb{E}_\Omega\left(\sum_{i=0}^{j} f_i\right)\right) \overset{(10)}{=} 2^{-2n}n\left(\sum_{j=0}^{n}\binom{n}{j}^2\frac{1}{2}^{\binom{n-j}{2}}\right) = 2^{-2n}n\left(\sum_{j=0}^{n}\binom{n}{j}^2\frac{1}{2}^{\binom{j}{2}}\right).
\end{aligned}$$

For (13) we have that:

$$\dot{\mathrm{B}} = \left\lceil 2\left(\log_2(n) - \log_2\log_2(n)\right) + \frac{1}{2}\right\rceil \leq$$

$$\leq \left\lceil 2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2} + 2 \right\rceil \leq$$
$$\leq \left\lfloor 2(\log_2(n) - \log_2\log_2(n)) - 1\frac{1}{2} \right\rfloor + 3 = \underset{\cdot}{\mathrm{B}} + 3\,.$$

For (14) we have that:

$$\lim_{n\to\infty} \underset{\cdot}{\mathrm{B}} = \lim_{n\to\infty} \left\lfloor 2\left(\log_2(n) - \log_2\log_2(n)\right) - 1\frac{1}{2} \right\rfloor \leq$$
$$\leq \lfloor 2\log_2(n) \rfloor \leq \lim_{n\to\infty} 2\log_2(n)\,.$$

For (15) we have that:

$$\binom{n}{\dot{\mathrm{B}}}^2 \overset{(13)}{\leq} \binom{n}{\underset{\cdot}{\mathrm{B}}+3}^2 \overset{(*)}{\leq} \left(\frac{n}{\underset{\cdot}{\mathrm{B}}}\right)^{3^2} \binom{n}{\underset{\cdot}{\mathrm{B}}}^2 \leq n^9 \binom{n}{\underset{\cdot}{\mathrm{B}}}^2 .$$

The second inequality $(*)$ follows from the fact that

$$\lim_{n\to\infty} \underset{\cdot}{\mathrm{B}} \overset{(14)}{\leq} \lim_{n\to\infty} 2\log_2(n) \ll \lim_{n\to\infty} \frac{n}{2}$$

and that for any $0 \leq c \leq \frac{n}{2} - 3$

$$\frac{\binom{n}{c+3}}{\binom{n}{c}} = \frac{\frac{n!}{(c+3)!(n-c-3)!}}{\frac{n!}{c!(n-c)!}} = \frac{\frac{n!(n-c)(n-c-1)(n-c-2)}{c!(c+1)(c+2)(c+3)(n-c)!}}{\frac{n!}{c!(n-c)!}} = \frac{(n-c)(n-c-1)(n-c-2)}{(c+1)(c+2)(c+3)} \frac{\frac{n!}{c!(n-c)!}}{\frac{n!}{c!(n-c)!}} =$$
$$= \frac{(n-c)(n-c-1)(n-c-2)}{(c+1)(c+2)(c+3)} \leq \left(\frac{n}{c}\right)^3 .$$

For (16) we have that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 \frac{1}{2}^{\binom{i}{2}} \overset{\text{Lemma 7}}{\leq} n \binom{n}{\dot{\mathrm{B}}}^2 \frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}} \overset{(15)}{\leq} nn^9 \binom{n}{\underset{\cdot}{\mathrm{B}}}^2 \frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}} \leq n^{10} \binom{n}{\underset{\cdot}{\mathrm{B}}}^2 \frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}} .$$

For (17) we have that:

$$\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2}^{\binom{i}{2}} = \binom{n}{0} \frac{1}{2}^{\binom{0}{2}} + \cdots + \binom{n}{\underset{\cdot}{\mathrm{B}}} \frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}} + \cdots + \binom{n}{n} \frac{1}{2}^{\binom{n}{2}} \geq \binom{n}{\underset{\cdot}{\mathrm{B}}} \frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}} .$$

To see why (18) holds, note that

$$\log_c(c^k) = k \text{ and } \log_c\left(n^{\frac{k}{\log_c(n)}}\right) = \frac{k}{\log_c(n)} \log_c(n) = k \text{ so}$$
$$c^k = n^{\frac{k}{\log_c(n)}}\,.$$

□

Proof of Lemma 9:

$$\begin{aligned}
& n^{\frac{3-\varepsilon}{4}\xi}\frac{\mathbb{E}_{\Omega}\left(\mathbb{E}_{\Pi}\left(X^2\right)\right)}{\mathbb{E}_{\Omega}\left(\mathbb{E}_{\Pi}\left(X\right)\right)^{2-\varepsilon}} \overset{(12,11)}{\leq} n^{1+\frac{3-\varepsilon}{4}\xi}\frac{\left(2^{-2n}\sum_{i=0}^{n}\binom{n}{i}^2\frac{1}{2}^{\binom{i}{2}}\right)}{\left(2^{-n}\sum_{i=0}^{n}\binom{n}{i}\frac{1}{2}^{\binom{i}{2}}\right)^{2-\varepsilon}} \overset{(16,17)}{\leq} \\
& \leq n^{11+\frac{3-\varepsilon}{4}\xi}\frac{\left(2^{-2n}\binom{n}{\underset{\cdot}{\mathrm{B}}}^2\frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}}\right)}{\left(2^{-n}\binom{n}{\underset{\cdot}{\mathrm{B}}}\frac{1}{2}^{\binom{\underset{\cdot}{\mathrm{B}}}{2}}\right)^{2-\varepsilon}} \overset{b=11+\frac{3-\varepsilon}{4}\xi}{\leq} n^{b}\frac{2^{-2n}\binom{n}{\underset{\cdot}{\mathrm{B}}}^2 2^{-\binom{\underset{\cdot}{\mathrm{B}}}{2}}}{\left(2^{-n}\binom{n}{\underset{\cdot}{\mathrm{B}}}2^{-\binom{\underset{\cdot}{\mathrm{B}}}{2}}\right)^{2-\varepsilon}} \leq \\
& \leq n^{b}\binom{n}{\underset{\cdot}{\mathrm{B}}}^{\varepsilon} 2^{\binom{\underset{\cdot}{\mathrm{B}}}{2}(1-\varepsilon)-\varepsilon n} \overset{\binom{n}{k}\leq\left(\frac{en}{k}\right)^k}{\leq} n^{b}\left(\frac{ne}{\underset{\cdot}{\mathrm{B}}}\right)^{\underset{\cdot}{\mathrm{B}}\varepsilon} 2^{\binom{\underset{\cdot}{\mathrm{B}}}{2}(1-\varepsilon)-\varepsilon n} \overset{(14)}{\leq} \\
& \leq n^{b}e^{\varepsilon 2\log_2(n)}n^{\varepsilon 2\log_2(n)}2^{\binom{\log_2(n)}{2}(1-\varepsilon)-\varepsilon n} \overset{(18)}{\leq} \\
& \leq n^{b}n^{\frac{2\varepsilon\log_2(n)}{\ln(n)}}n^{\varepsilon 2 log_2(n)}n^{\frac{\binom{\log_2(n)}{2}(1-\varepsilon)-\varepsilon n}{\log_2(n)}} \overset{\binom{\log_2(n)}{2}\leq\log_2^2(n)}{\leq} \\
& \leq n^{b+\frac{2\varepsilon\log_2(n)}{\ln(n)}+2\varepsilon\log_2(n)+\frac{(1-\varepsilon)\log_2^2(n)-\varepsilon n}{\log_2(n)}} = n^{b+\frac{2\varepsilon\log_2(n)}{\ln(n)}+(1+\varepsilon)\log_2(n)-\frac{\varepsilon n}{\log_2(n)}} = \\
& = n^{\frac{n}{\log_2(n)}\left(\frac{\left(11+\frac{3-\varepsilon}{4}\xi\right)\log_2(n)}{n}+2\varepsilon\left(\frac{\log_2^2(n)}{n\ln(n)}\right)+(1+\varepsilon)\frac{\log_2^2(n)}{n}-\varepsilon\right)} \overset{n\to\infty}{\to} \\
& \overset{n\to\infty}{\to} n^{\frac{-\varepsilon n}{\log_2(n)}} \overset{n\to\infty}{=} 0.
\end{aligned}$$

□

Lemma 1 **(Concave function)**. *For a real numbers n and x such that $n > 0$ is constant and $0 \leq x \leq n$, the following function is concave:*

$$f(x) = \ln\left(\binom{n}{x}\frac{1}{2}^{\frac{x(x-1)}{4}}\right).$$

Proof. Adopting the gamma function we arrive at:

$$\begin{aligned}
f(x) &= \ln(n!) - (\ln(x!) + \ln((n-x)!)) + \ln(1/2)\frac{x(x-1)}{4} \\
&= \underbrace{\ln(\Gamma(n+1))}_{\text{(a)}} + \underbrace{[-\ln(\Gamma(x+1))]}_{\text{(b)}} + \underbrace{[-\ln(\Gamma(n-x+1))]}_{\text{(c)}} + \underbrace{\ln(1/2)\frac{x(x-1)}{4}}_{\text{(d)}}.
\end{aligned}$$

Having in mind that

$$\frac{d^2}{d^2x}\ln(1/2)\frac{x(x-1)}{4} = \frac{d^2}{d^2x}\frac{\ln(1/2)}{4}(x^2-x) = \frac{\ln(1/2)}{4}2 < 0,$$

and that $\ln(\Gamma(n+1))$ is constant, we conclude that (a) and (d) are concave. From Bohr-Mollerup Theorem (Krantz 1999), $\ln(\Gamma(z))$, $z \in (0,\infty)$ is convex so $[-\ln(\Gamma(z))]$ is concave.

Combining this with the fact that $x + 1$ and $n - x + 1$ are both affine, we conclude that (b) and (c) are concave since the composition of concave with affine function is concave. The theorem follows by combining the concavity of (a),(b),(c) and (d) with the fact that the sum of concave functions is also concave. □

References

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