Appendices

Optimal Learning in Linear Regression With Combinatorial Feature Selection

Bin Han
Applied Mathematics & Statistics, and Scientific Computation, University of Maryland, College Park, MD, 20742, USA
danielh@math.umd.edu

Ilya O. Ryzhov
Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, USA
iryzhov@rhsmith.umd.edu http://scholar.rhsmith.umd.edu/iryzhov

Boris Defourny
Industrial & Systems Engineering, Lehigh University, Bethlehem, PA 18015, USA
defourny@lehigh.edu http://www.lehigh.edu/defourny

Appendices

Section EC.1 gives the full proofs of all results stated in the main text. Section EC.2 provides a MISOCO reformulation of the A-optimal design policy, which improves the computational efficiency of this method in experiments. Section EC.3 presents tables of optimal quantizations of the standard Student's t-distribution with different degrees of freedom, which may be useful for practitioners looking to implement this approach.

EC.1. Proofs

In this section, we give full proofs of results stated in the main text.

EC.1.1. Proof of Proposition 1

Assume that (β, ρ) follows a multivariate normal-gamma distribution with parameters (θ, Σ, a, b). The joint density is given by

$$\begin{aligned} p(\beta, \rho \,|\, \theta, \Sigma, a, b) &= p(\beta \,|\, \rho, \theta, \Sigma)\, p(\rho \,|\, a, b) \\ &= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma|^{-\frac{1}{2}}\, e^{-\frac{\rho}{2}(\beta-\theta)^\top \Sigma^{-1}(\beta-\theta)} \frac{b^a}{\Gamma(a)} \rho^{a-1} e^{-b\rho} \end{aligned}$$

where Γ is the gamma function. Let $\eta \sim \mathcal{N}\left(\varphi^\top \beta, \frac{1}{\rho}\varphi^\top \Sigma \varphi\right)$ be the observation corresponding to the chosen feature vector φ. From Bayes' rule (DeGroot 1970), we know that $p(\beta, \rho \,|\, \eta)$ is proportional to $p(\beta, \rho)\, q(\eta \,|\, \beta, \rho)$. We then write,

$$\begin{aligned} p(\beta, \rho)\, q(\eta \,|\, \beta, \rho) &= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma|^{-\frac{1}{2}}\, e^{-\frac{\rho}{2}(\beta-\theta)^\top \Sigma^{-1}(\beta-\theta)} \frac{b^a}{\Gamma(a)} \rho^{a-1} e^{-b\rho} \sqrt{\frac{\rho}{2\pi}} e^{-\frac{\rho}{2}\left(\eta - \phi^\top \beta\right)^2} \\ &= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma|^{-\frac{1}{2}}\, e^{-\frac{\rho}{2}\left[(\beta-\theta)^\top \Sigma^{-1}(\beta-\theta) + \left(\eta - \varphi^\top \beta\right)^2\right]} \frac{b^a}{\Gamma(a)} \rho^{a+\frac{1}{2}-1} e^{-b\rho} \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

Define

$$\begin{aligned} \theta' &= \theta + \frac{\eta - \varphi^\top \theta}{1 + \varphi^\top \Sigma \varphi} \Sigma \varphi, \\ \Sigma' &= \Sigma - \frac{\Sigma \varphi \varphi^\top \Sigma}{1 + \varphi^\top \Sigma \varphi}. \end{aligned}$$

By completing the square for β, and using the matrix inversion lemma to observe that $\Sigma' = \left(\Sigma^{-1} + \varphi\varphi^\top\right)^{-1}$, we obtain

$$(\beta - \theta)^\top \Sigma^{-1} (\beta - \theta) + \left(\eta - \varphi^\top \beta\right)^2 = (\beta - \theta')^\top (\Sigma')^{-1} (\beta - \theta') + \frac{\left(\eta - \varphi^\top \theta\right)^2}{1 + \varphi^\top \Sigma \varphi}.$$

It follows that

$$p(\beta,\rho)\,q(\eta\,|\,\beta,\rho)=\left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}}|\Sigma'|^{-\frac{1}{2}}\,e^{-\frac{\rho}{2}(\beta-\theta')^\top(\Sigma')^{-1}(\beta-\theta')}\frac{b^a}{\Gamma(a)}\rho^{a+\frac{1}{2}-1}e^{-\rho\left(b+\frac{\left(\eta-\varphi^\top\theta\right)^2}{2\left(1+\varphi^\top\Sigma\varphi\right)}\right)}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{|\Sigma'|}{|\Sigma|}}.$$

Letting $a'=a+\frac{1}{2}$ and

$$b'=b+\frac{\left(\eta-\varphi^\top\theta\right)^2}{2\left(1+\varphi^\top\Sigma\varphi\right)},$$

we obtain

$$p(\beta,\rho)\,q(\eta\,|\,\beta,\rho)\quad\propto\quad\left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}}|\Sigma'|^{-\frac{1}{2}}\,e^{-\frac{\rho}{2}(\beta-\theta')^\top(\Sigma')^{-1}(\beta-\theta')}\frac{(b')^{a'}}{\Gamma(a')}\rho^{a'-1}e^{-\rho b'},$$

which is precisely the normal-gamma density with parameters (θ',Σ',a',b') calculated according to the desired updating equations.

EC.1.2. Proof of Proposition 2

The characteristic function of η^{n+1} given ρ is

$$\begin{aligned}\mathbb{E}(e^{it\eta^{n+1}}|\rho)=&\mathbb{E}(\mathbb{E}(e^{it\eta^{n+1}}|\rho,\beta)|\rho)\\=&\mathbb{E}(e^{it\psi^\top\beta-\frac{1}{2}\frac{1}{\rho}t^2}|\rho)\\=&e^{-\frac{1}{2}\frac{1}{\rho}t^2}\mathbb{E}(e^{it\psi^\top\beta}|\rho)\\=&e^{-\frac{1}{2}\frac{1}{\rho}t^2}e^{it\psi^\top\theta^n-\frac{1}{2}\psi^\top\frac{1}{\rho}\Sigma^n\psi t^2}\\=&e^{it\psi^\top\theta^n-\frac{1}{2}\frac{1}{\rho}(1+\psi^\top\Sigma^n\psi)t^2}.\end{aligned}$$

Consequently, we can write the density of η^{n+1} as

$$\begin{aligned}p(\eta^{n+1})=&\int_{\mathbb{R}^+}\sqrt{\frac{\rho}{2\pi(1+\psi^\top\Sigma^n\psi)}}\exp\left(-\frac{\rho(\eta^{n+1}-\psi^\top\theta)^2}{2(1+\psi^\top\Sigma^n\psi)}\right)\frac{(b^n)^{a^n}}{\Gamma(a^n)}\rho^{a^n-1}\exp(-b^n\rho)d\rho\\=&\frac{\Gamma(a^n+\frac{1}{2})}{\Gamma(a^n)}\frac{1}{\sqrt{2\pi(1+\psi^\top\Sigma^n\psi)b^n}}\left(1+\frac{\left(\eta^{n+1}-\psi^\top\theta^n\right)^2}{2(1+\psi^\top\Sigma^n\psi)b^n}\right)^{-a^n-\frac{1}{2}}.\end{aligned}$$

Define $\tilde{\eta}^{n+1}=\sqrt{\frac{a^n}{(1+\psi^\top\Sigma^n\psi))b^n}}(\eta^{n+1}-\psi^\top\theta^n)$. Then, the pdf of $\tilde{\eta}^{n+1}$ is given by

$$p\left(\tilde{\eta}^{n+1}\right)=\frac{\Gamma(\frac{2a^n+1}{2})}{\Gamma(\frac{2a^n}{2})(\pi 2a^n)^{\frac{1}{2}}}\left(1+\frac{(\tilde{\eta}^{n+1})^2}{2a^n}\right)^{-\frac{2a^n+1}{2}},$$

which is the pdf of the standard Student's t-distribution with $2a^n$ degrees of freedom. Thus

$$\eta^{n+1}=\sqrt{\frac{b^n(1+\psi^\top\Sigma^n\psi)}{a^n}}\tilde{\eta}^{n+1}+\psi^\top\theta^n$$

follows the desired distribution.

EC.1.3. Proof of Proposition 3

From Proposition 1, we know that $\frac{a^n}{b^n} = \mathbb{E}^n(\rho)$. It follows from Theorem V.4.7 of Çinlar (2011) that the process $\left(\frac{a^n}{b^n}\right)_{n=0}^{\infty}$ is a uniformly integrable martingale and converges a.s. to $\mathbb{E}(\rho \,|\, \mathcal{F}^{\infty})$. Thus, it remains to show that ρ is $\mathcal{F}^{\infty}$-measurable.

Since the set Φ is finite, any policy π must measure at least one alternative $\bar{\varphi}$ infinitely often as $N \to \infty$. Note that $\bar{\varphi}$ may depend on the sample path, but is measurable with respect to $\mathcal{F}^{\infty}$. The sample variance of all η^{n+1} for which $\pi(\theta^n, \Sigma^n, a^n, b^n) = \bar{\varphi}$ converges a.s. to the true variance $\frac{1}{\rho}$. It follows that ρ is measurable with respect to $\mathcal{F}^{\infty}$, whence $\mathbb{E}^n(\rho) \to \rho$ a.s. by Theorem II.2.15 of Çinlar (2011).

EC.1.4. Proof of Proposition 4

We first prove the following technical lemma.

LEMMA EC.1. *Suppose that (p^n, q^n) converges to a finite limit in $\mathbb{R}^K \times \mathbb{R}^K$. Then, the sequence $\left\{\max_{\varphi \in \Phi} p^n_{\varphi} + q^n_{\varphi} T_{s^n}\right\}_{n \geq 0}$ is uniformly integrable.*

Proof. From p. 75 of Çinlar (2011), the componentwise maximum of finitely many uniformly integrable sequences is uniformly integrable. Since both $\{p^n\}$ and $\{q^n\}$ are bounded, it remains to show that $\{T_{s^n}\}$ is uniformly integrable. We choose $s^0 = 2a^0 > 1$ so that each T_{s^n} has finite expectation. Consider the pdf g_{s^n} of the standard Student's t-distribution with s^n degrees of freedom. Because the tails of g_s become lighter with larger s, there exists a value $t^n > 0$ such that $g_{s^n}(t^n) = g_{s^0}(t^n)$ with $g_{s^n}(t) < g_{s^0}(t)$ for $t > t^n$ and $g_{s^n}(t) > g_{s^0}(t)$ for $0 < t < t_n$. Note that $s^n \to \infty$ and g_{∞} is the standard normal pdf since T_{s^n} converges in distribution to a standard normal random variable as $s^n \to \infty$. Consequently, $t^n \to t^{\infty}$ where $g_{\infty}(t^{\infty}) = g_{s^0}(t^{\infty})$.

Thus, $\tilde{t} = \sup_{n \geq 1} t^n$ is finite. For $M > \tilde{t}$, we have

$$\sup_n \mathbb{E}\left(T_{s^n} 1_{\{T_{s^n} > M\}}\right) \leq \mathbb{E}\left(T_{s^0} 1_{\left\{T_{s^0} > M\right\}}\right),$$

whence

$$\begin{aligned}\lim_{M \to \infty} \sup_n \mathbb{E}\left(|T_{s^n}| 1_{\{|T_{s^n}| > M\}}\right) &= 2 \lim_{M \to \infty} \sup_n \mathbb{E}\left(T_{s^n} 1_{\{T_{s^n} > M\}}\right) \\ &\leq 2 \lim_{M \to \infty} \mathbb{E}\left(T_{s^0} 1_{\left\{T_{s^0} > M\right\}}\right).\end{aligned}$$

The limit in the second line is equal to zero since T_{s^0} has finite expectation. □

Now, assume that 1) holds. Since $s^n \to \infty$, we know that T_{s^n} converges in distribution to a standard normal random variable Z. Note, however, that we are only interested in taking expectations over the distribution of T_{s^n} for the purpose of computing h_{s^n}. By Skorokhod's representation theorem (Çinlar 2011, Corollary III.5.9), there exist random variables $\bar{T}_{s^n}$ and $\bar{Z}$ that have the same distribution as T_{s^n} and Z, but with $\bar{T}_{s^n} \to \bar{Z}$ almost surely. Thus,

$$\lim_{n\to\infty} \max_{\varphi\in\Phi} p^n_\varphi + q^n_\varphi \bar{T}_{s^n} = \max_{\varphi\in\Phi} p^\infty_\varphi + q^\infty_\varphi \bar{Z} \tag{EC.1}$$

almost surely. From Proposition EC.1, it follows that the convergence in (EC.1) also holds in L^1, whence

$$\lim_{n\to\infty} \mathbb{E}\left(\max_{\varphi\in\Phi} p^n_\varphi + q^n_\varphi \bar{T}_{s^n}\right) = \mathbb{E}\left(\max_{\varphi\in\Phi} p^\infty_\varphi + q^\infty_\varphi \bar{Z}\right).$$

Since $\bar{T}_{s^n}$ has the same distribution as T_{s^n}, it follows that

$$\begin{aligned}\lim_{n\to\infty} h_{s^n}(p^n, q^n) &= \mathbb{E}\left(\max_{\varphi\in\Phi} p^\infty_\varphi + q^\infty_\varphi \bar{Z}\right) - \max_{\varphi\in\Phi} p^\infty_\varphi \\ &= \sum_{i=1}^{K-1} \left(q^\infty_{i+1} - q^\infty_i\right)\left(f\left(-\left|c^\infty_i\right|\right) - \left|c^\infty_i\right| F\left(-\left|c^\infty_i\right|\right)\right) \\ &= 0, \end{aligned} \tag{EC.2}$$

where the functions f, F are the standard normal pdf and cdf (Frazier et al. 2009), the values q^∞_i are obtained by sorting q^∞_φ in increasing order, and c^∞_i are the breakpoints of the piecewise linear function $t \mapsto \max_i p^\infty_i + q^\infty_i t$ as discussed previously. It can be shown that the function $z \mapsto f(-z) - zF(-z)$ is strictly positive, whence (EC.2) implies $q^\infty_i = q^\infty_{i+1}$ for all i, as required.

Now, assume that 2) holds. In this case,

$$\lim_{n\to\infty} \max_{\varphi\in\Phi} p^n_\varphi + q^n_\varphi \bar{T}_{s^n} = \left(\max_{\varphi\in\Phi} p^\infty_\varphi\right) + \ell \bar{Z}$$

almost surely. Applying Proposition EC.1 again, we obtain $h_{s^n}(p^n, q^n) \to 0$ as required.

EC.1.5. Proof of Proposition 5

For fixed N, let $N_\psi = \sum_{n=0}^N 1_{\{\varphi^n = \psi\}}$ be the number of times ψ is measured by time N. Define $\bar{\Sigma}^N$ by the equation

$$\left(\bar{\Sigma}^N\right)^{-1} = \left(\Sigma^0\right)^{-1} + \sum_{n=0}^N 1_{\{\varphi^n \neq \psi\}} \varphi^n \left(\varphi^n\right)^\top.$$

By the matrix inverse lemma, it follows that

$$\Sigma^N = \left(\left(\bar{\Sigma}^N\right)^{-1} + N_\psi \psi\psi^\top\right)^{-1} = \bar{\Sigma}^N - \frac{N_\psi \bar{\Sigma}^N \psi\psi^\top \bar{\Sigma}^N}{1 + N_\psi \psi^\top \bar{\Sigma}^N \psi}.$$

Consequently,

$$\begin{aligned}\psi^\top \Sigma^N \psi &= \psi^\top \bar{\Sigma}^N \psi - \frac{N_\psi \left(\psi^\top \bar{\Sigma}^N \psi\right)^2}{1 + N_\psi \psi^\top \bar{\Sigma}^N \psi} \\ &= \frac{\psi^\top \bar{\Sigma}^N \psi}{1 + N_\psi \psi^\top \bar{\Sigma}^N \psi},\end{aligned}$$

which vanishes to zero as $N_\psi \to \infty$. By the Cauchy-Schwarz inequality,

$$\left(\varphi^\top \Sigma^N \psi\right)^2 \le \left(\varphi^\top \Sigma^N \varphi\right)\left(\psi^\top \Sigma^N \psi\right) \le \left(\varphi^\top \Sigma^0 \varphi\right)\left(\psi^\top \Sigma^N \psi\right),$$

implying that $\varphi^\top \Sigma^N \psi \to 0$ for all $\varphi \in \Phi$. Proposition 4 then implies that $v_\psi^{KG,n} \to 0$ a.s.

EC.1.6. Proof of Corollary 1

By Proposition 5 we have $\varphi^\top \Sigma^n \psi_i \to 0$ almost surely for all $\varphi \in \Phi$ and all $1 \le i \le k$. This implies $\varphi^\top \Sigma^n \psi \to 0$ almost surely for all $\varphi \in \Phi$. By Proposition 4, we then have $v_\psi^{KG,n} \to 0$ almost surely.

EC.1.7. Proof of Proposition 6

We prove this statement by contradiction. Fix ω and let $A_\omega \subseteq \Phi$ be the set of all $\psi \in \Phi$ for which $v_\psi^{KG,n}(\omega)$ does not converge to zero. Suppose that A_ω is non-empty. Then, Proposition 5 implies that $A_\omega \neq \Phi$ and also that any $\psi \in A_\omega$ has only been measured finitely many times on the sample path ω.

Since $|A_\omega|$ is finite, we can find a large enough N_1 such that, if $n > N_1$, then $\varphi^n(\omega) \notin A_\omega$. Furthermore, there exists some ε such that, for any N, there exists $n > N$ satisfying $\min_{\psi \in A_\omega} v^{KG,n}(\omega) > \varepsilon$ (in other words, this happens for infinitely many n). At the same time, for this ε, there also exists N_2 such that, for all $n > N_2$, $\max_{\psi \notin A_\omega} v_\psi^{KG,n}(\omega) < \varepsilon$. Consequently, there exists $n > \max(N_1, N_2)$ for which

$$\min_{\psi \in A_\omega} v^{KG,n}(\omega) > \varepsilon > \max_{\psi \notin A_\omega} v_\psi^{KG,n}(\omega).$$

Thus, any alternative in A_ω is preferable to any alternative not in A_ω at this time. However, since $n > N_1$, the KGUP policy must select an alternative not in A_ω, contradicting the definition of the policy. We conclude that $\lim_{n\to\infty} v_\psi^{KG,n} = 0$ a.s. for all $\psi \in \Phi$.

EC.1.8. Proof of Theorem 1

Consider a fixed ω. Propositions 4 and 6 imply that, for fixed $\psi \in \Phi$, there exists $\ell(\omega)$ such that $\varphi^\top \Sigma^n(\omega)\psi \to \ell(\omega)$ for all $\varphi \in \Phi$. However, these results by themselves are not sufficient to conclude whether $\ell(\omega)$ depends on ψ; note that the factor other than $\varphi^\top \Sigma^n \psi$ in (11) contains ψ, but has a limit due to martingale convergence. We now show that $\ell(\omega)$ does not depend on ψ.

By Proposition 6, we have $v_{\psi_1}^{KG,n}(\omega) \to 0$ and $v_{\psi_2}^{KG,n}(\omega) \to 0$ for any $\psi_1, \psi_2 \in \Phi$. Suppose that $\psi_1 \neq \psi_2$. By Proposition 4, we have

$$\varphi^\top \Sigma^\infty(\omega)\psi_1 = \ell_1(\omega), \qquad \varphi^\top \Sigma^\infty(\omega)\psi_2 = \ell_2(\omega)$$

for all $\varphi \in \Phi$. Now, fix some φ. Proposition 6 implies that $v_\varphi^{KG,n}(\omega) \to 0$. It then follows from Proposition 4 that there exists some $\ell(\omega)$ such that $\psi^\top \Sigma^\infty(\omega)\varphi = \ell(\omega)$ for all ψ. Therefore, $\ell_1(\omega) = \ell_2(\omega) = \ell(\omega)$.

Furthermore, as $N \to \infty$, there is at least one alternative that is measured infinitely often on the sample path ω. Combining Proposition 5 with the above results, we obtain $\ell(\omega) = 0$. Thus, $\psi^\top \Sigma^n(\omega)\psi \to 0$ for all $\psi \in \Phi$, as required. It then follows from (14) that the conditional variance vanishes to zero as well.

EC.1.9. Proof of Corollary 2

From Theorem 1, we have $\psi^\top \Sigma^n(\omega)\psi \to 0$, for almost every ω, and for any ψ chosen from among the r basis vectors. It follows that the sequence $v^\top \Sigma^n(\omega) v \to 0$ for any arbitrary v (not necessarily in Φ) that can be written as a linear combination of the basis vectors. From this it follows that $\mathrm{tr}(\Sigma^n) \to 0$ almost surely, implying that the largest eigenvalue of Σ^n converges to zero. From (3)-(4), recall that θ^n is identical to the least-squares estimator. It is well-known (Eicker 1963, Drygas 1976) that $\lambda_{\max}(\Sigma^n) \to 0$ is necessary and sufficient for consistency.

EC.1.10. Proof of Proposition 7

Since φ is a binary vector and we control $(\varphi_1, \varphi_2, \ldots, \varphi_{r-1})$, the maximum of $\varphi^\top \theta^n$ is simply obtained by letting $\varphi_j = 1$ when $\theta_j^n \geq 0$ and $\varphi_j = 0$ otherwise. Then, (8) can be rewritten as

$$v_\psi^{KG,n} = \mathbb{E}^n \left[\sum_{j \geq 1} (\theta_j^{n+1})^+ | \varphi^n = \psi \right] - \sum_{j \geq 1} (\theta_j^n)^+. \tag{EC.3}$$

Using (9) and (EC.3), we obtain

$$\begin{aligned} v_\psi^{KG,n} &= \sum_j \mathbb{E}_\psi^n \left[(\theta_j^n + u_j^n T_{s^n})^+ \right] - (\theta_j^n)^+ \\ &= \sum_{j\geq 1, u_j^n>0} \int_{-\frac{\theta_j^n}{u_j^n}}^{\infty} (\theta_j^n + u_j^n t) g_{s^n}(t)dt + \sum_{j\geq 1, u_j^n<0} \int_{-\infty}^{-\frac{\theta_j^n}{u_j^n}} (\theta_j^n + u_j^n t) g_{s^n}(t)dt - \sum_{j\geq 1, u_j^n\neq 0} (\theta_j^n)^+. \end{aligned} \tag{EC.4}$$

The conclusion follows after simple rearrangements of the terms in (EC.4).

EC.1.11. Proof of Proposition 8

Proof. We evaluate

$$\begin{aligned} \mathrm{tr}(C_j Z_j) &= \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} 0 & \theta^\top & 0^\top \\ \theta & 0 & t_j\Sigma \\ 0 & t_j\Sigma & 0 \end{bmatrix} \begin{bmatrix} 1 & (\varphi^j)^\top & (d_\psi)^\top \\ \varphi^j & \varphi^j(\varphi^j)^\top & \varphi^j d_\psi^\top \\ d_\psi & d_\psi(\varphi^j)^\top & d_\psi d_\psi^\top \end{bmatrix} \right) \\ &= (\varphi^j)^\top\theta + \mathrm{tr}(t_j\Sigma d_\psi(\varphi^j)^\top) \\ &= (\varphi^j)^\top\theta + t_j(\varphi^j)^\top\Sigma d_\psi. \end{aligned} \tag{EC.5}$$

The conclusion follows from comparing (19) and (EC.5) with (18).

EC.1.12. Proof of Proposition 9

The constraints on u^j and ψ' ensure that $u_{kl}^j = \varphi_k\psi_l$ and $\psi' = z\cdot\psi$ as long as $0\leq z\leq M$. Thus, only the constraint $\|P^{\frac{1}{2}}\psi'\|_2 \leq s$ remains to be justified. By the properties of ψ', it is equivalent to $z\|P^{\frac{1}{2}}\psi\|_2\leq s$, or

$$z \leq \frac{s}{\|P^{\frac{1}{2}}\psi\|_2} = \frac{\sum_{j=1}^J w_j t_j \sum_{k=1}^r \sum_{l=1}^r u_{kl}^j \Sigma_{kl}}{\|P^{\frac{1}{2}}\psi\|_2} = \frac{\sum_{j=1}^J w_j t_j \left(\varphi^j\right)^\top \Sigma\psi}{\|P^{\frac{1}{2}}\psi\|_2}.$$

Since z is maximized, the inequality becomes binding at optimality.

EC.2. Mixed-integer second-order cone reformulation of A-optimal policy

Given the belief parameter Σ, the A-optimal policy computes the optimal solution of the problem

$$v^* = \min_\varphi \mathrm{tr}\left(\Sigma - \frac{\Sigma\varphi\varphi^\top\Sigma}{1+\varphi^\top\Sigma\varphi}\right) \tag{EC.6}$$

subject to $A\varphi = h$ and $\varphi\in\{0,1\}^r$. Define $Q = \frac{A^\top A}{h^\top h} + \Sigma$, and consider the following MISOCO problem, where we have introduced the matrix $V\in\mathbb{R}^{r\times r}$ with additional constraints to linearize the definition $V_{kl} = \varphi_k\varphi_l$:

$$v_0^* = \min_{\varphi,s,z,V} z \tag{EC.7}$$

subject to

$$
\begin{aligned}
\left\| \begin{matrix} 2Q^{1/2}\varphi \\ s-z \end{matrix} \right\|_2 &\leq s+z \\
s &= \mathrm{tr}\left(\Sigma^2 V\right) \\
V_{kl} &\leq \varphi_k \\
V_{kl} &\leq \varphi_l \\
V_{kl} &\geq \varphi_k + \varphi_l - 1 \\
V_{kl} &\geq 0 \\
A\varphi &= h \\
\varphi &\in \{0,1\}^r .
\end{aligned}
$$

Proposition EC.1. *The problem in (EC.7) is a MISOCO reformulation of (EC.6), and the optimal values of these problems satisfy*

$$v^* = tr(\Sigma) - \frac{1}{v_0^*}.$$

Proof. The objective function of (EC.6) can be rewritten as

$$
\begin{aligned}
\mathrm{tr}\left(\Sigma - \frac{\Sigma\varphi\varphi^\top\Sigma}{1+\varphi^\top\Sigma\varphi}\right) &= \mathrm{tr}(\Sigma) - \frac{\mathrm{tr}\left(\Sigma\varphi\varphi^\top\Sigma\right)}{1+\varphi^\top\Sigma\varphi} \\
&= \mathrm{tr}(\Sigma) - \frac{\varphi^\top\Sigma\Sigma\varphi}{\varphi^\top Q\varphi},
\end{aligned}
$$

using the properties of the trace and the definition of Q. Since $\mathrm{tr}(\Sigma)$ is a constant term, we drop it for the purpose of optimization and reformulate (EC.6) as

$$
\begin{aligned}
v^* = \max_\varphi \ & \tfrac{\varphi^\top\Sigma^2\varphi}{\varphi^\top Q\varphi} \\
\text{s.t.} \quad & A\varphi = h \\
& \varphi \in \{0,1\}^r .
\end{aligned}
\tag{EC.8}
$$

Observing that the numerator and denominator in the objective function are positive, we can equivalently minimize the inverse of the objective function. Introducing $V_{kl} = \varphi_k\varphi_l$ and

$$s = \varphi^\top\Sigma^2\varphi = \mathrm{tr}\left(\Sigma^2\varphi\varphi^\top\right) = \mathrm{tr}\left(\Sigma^2 V\right),$$

allows us to rewrite the objective in (EC.8) as $\min_{\varphi,s} \frac{\varphi^\top Q\varphi}{s}$ subject to the old and new constraints. This is an example of a quadratic-over-linear objective, which are convex provided

s	t_1	t_2	t_3	t_4	t_5
3	-5.6124	-1.5520	0.0000	1.5520	5.6124
4	-3.4130	-1.1977	0.0000	1.1977	3.4130
5	-2.7943	-1.0636	0.0000	1.0636	2.7943
6	-2.5065	-0.9929	0.0000	0.9929	2.5065
7	-2.3406	-0.9493	0.0000	0.9493	2.3406
8	-2.2327	-0.9197	0.0000	0.9197	2.2327
9	-2.1569	-0.8982	0.0000	0.8982	2.1569
10	-2.1008	-0.8820	0.0000	0.8820	2.1008
11	-2.0576	-0.8693	0.0000	0.8693	2.0576
12	-2.0233	-0.8591	0.0000	0.8591	2.0233
13	-1.9954	-0.8507	0.0000	0.8507	1.9954
14	-1.9722	-0.8436	0.0000	0.8436	1.9722
15	-1.9527	-0.8377	0.0000	0.8377	1.9527
16	-1.9361	-0.8325	0.0000	0.8325	1.9361
17	-1.9217	-0.8281	0.0000	0.8281	1.9217
18	-1.9091	-0.8242	0.0000	0.8242	1.9091
19	-1.8980	-0.8207	0.0000	0.8207	1.8980
20	-1.8882	-0.8176	0.0000	0.8176	1.8882
∞	-1.7241	-0.7646	0.0000	0.7646	1.7241

Table EC.1 Optimal quantizations with $J = 5$.

that the quadratic form is positive semidefinite and the linear form is positive. Since these requirements are satisfied in our case, it remains to put the problem in epigraph form, leading to the objective $\min z$ and the additional constraint $z \geq \frac{\varphi^\top Q \varphi}{s}$. It is straightforward to verify that, for $s, z \geq 0$, this constraint is equivalent to the second-order cone constraint

$$\left\| \begin{matrix} 2Q^{1/2}\phi \\ s - z \end{matrix} \right\|_2 \leq s + z,$$

completing the reformulation. □

EC.3. Tables of Voronoi quantizations for the Student's t-distribution

In this appendix, we present tables of two Voronoi quantizations t^J of the standard Student's t-distribution with varying degrees of freedom s. The quantization does not exist when $s = 1$, as the mean of the distribution is undefined in this case, or when $s = 2$, since this corresponds to infinite variance. As s increases, the quantization approaches that of the standard normal distribution, which we also include.

References

Çinlar E (2011) *Probability and Stochastics*. Springer.

DeGroot MH (1970) *Optimal Statistical Decisions*. John Wiley and Sons.

Drygas H (1976) Weak and strong consistency of the least squares estimators in regression models. *Probability Theory and Related Fields* 34(2):119–127.

s	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
3	-22.3881	-7.3823	-3.2540	-1.4892	-0.4392	0.4392	1.4892	3.2540	7.3823	22.3881
4	-7.8620	-3.7060	-2.0375	-1.0601	-0.3315	0.3315	1.0601	2.0375	3.7060	7.8620
5	-5.3116	-2.8550	-1.6854	-0.9116	-0.2900	0.2900	0.9116	1.6854	2.8550	5.3116
6	-4.3417	-2.4886	-1.5190	-0.8364	-0.2682	0.2682	0.8364	1.5190	2.4886	4.3417
7	-3.8426	-2.2862	-1.4222	-0.7912	-0.2549	0.2549	0.7912	1.4222	2.2862	3.8426
8	-3.5409	-2.1581	-1.3590	-0.7610	-0.2459	0.2459	0.7610	1.3590	2.1581	3.5409
9	-3.3396	-2.0697	-1.3144	-0.7394	-0.2394	0.2394	0.7394	1.3144	2.0697	3.3396
10	-3.1959	-2.0052	-1.2813	-0.7231	-0.2345	0.2345	0.7231	1.2813	2.0052	3.1959
11	-3.0884	-1.9560	-1.2558	-0.7105	-0.2306	0.2306	0.7105	1.2558	1.9560	3.0884
12	-3.0049	-1.9173	-1.2355	-0.7005	-0.2276	0.2276	0.7005	1.2355	1.9173	3.0049
13	-2.9382	-1.8860	-1.2190	-0.6922	-0.2250	0.2250	0.6922	1.2190	1.8860	2.9382
14	-2.8837	-1.8601	-1.2053	-0.6853	-0.2229	0.2229	0.6853	1.2053	1.8601	2.8837
15	-2.8384	-1.8385	-1.1937	-0.6795	-0.2212	0.2212	0.6795	1.1937	1.8385	2.8384
16	-2.8001	-1.8200	-1.1839	-0.6746	-0.2196	0.2196	0.6746	1.1839	1.8200	2.8001
17	-2.7673	-1.8042	-1.1753	-0.6702	-0.2183	0.2183	0.6702	1.1753	1.8042	2.7673
18	-2.7389	-1.7904	-1.1679	-0.6665	-0.2172	0.2172	0.6665	1.1679	1.7904	2.7389
19	-2.7141	-1.7782	-1.1613	-0.6631	-0.2161	0.2161	0.6631	1.1613	1.7782	2.7141
20	-2.6922	-1.7675	-1.1555	-0.6602	-0.2152	0.2152	0.6602	1.1555	1.7675	2.6922
∞	-2.3451	-1.5913	-1.0578	-0.6099	-0.1996	0.1996	0.6099	1.0578	1.5913	2.3451

Table EC.2 Optimal quantizations with $J = 10$.

Eicker F (1963) Asymptotic normality and consistency of the least squares estimators for families of linear regressions. *The Annals of Mathematical Statistics* 34(2):447–456.

Frazier PI, Powell WB, Dayanik S (2009) The knowledge-gradient policy for correlated normal rewards. *INFORMS Journal on Computing* 21(4):599–613.