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Online Supplement for “Solution of Monotone Complementarity and General Convex Programming Problems Using a Modified Potential Reduction Interior Point Method”*

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1. Definitions

We consider a monotone complementarity problem (MCP) of the following form

$$s = f(x) \quad (1)$$

$$0 \leq s \perp x \geq 0, \quad (2)$$

where $x, s \in R^n$, $f(x)$ is a continuously differentiable monotone mapping from $R_+^n := \{x \in R^n \mid x \geq 0\}$ to R^n , and the notation $0 \leq s \perp x \geq 0$ means that $(x, s) \geq 0$ and that $x^T s = 0$. This requirement is called the complementarity condition. Since f is a monotone mapping, for every $x^1, x^2 \in R_+^n$, we have

$$(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq 0. \quad (3)$$

Let $\nabla f(x)$ denote the Jacobian matrix of $f(x)$. If $\nabla f(x)$ is positive semidefinite for all $x > 0$, that is,

$$h^T \nabla f(x) h \geq 0, \forall x > 0, h \in R^n,$$

then $f(x)$ is a continuous monotone mapping.

Given an initial point $(x^0, s^0) \in R_{++}^{2n} := \{x, s \in R^n \mid x, s > 0\}$, we would like to compute a bounded sequence $\langle x^k, s^k \rangle \in R_{++}^{2n}, k = 0, 1, \dots$, such that

$$\lim_{k \rightarrow \infty} s^k - f(x^k) = 0. \quad (4)$$

If a limit point (x^*, s^*) of the sequence $\langle x^k, s^k \rangle$ satisfies (4), then (x^*, s^*) is called an *asymptotically feasible* solution of MCP. If there exists an *asymptotically feasible* solution (x^*, s^*) such that $(x^*)^T s^* = 0$, then MCP is said to be asymptotically solvable. In this case (x^*, s^*) is called an *asymptotically optimal or complementary* solution. If MCP has an asymptotically complementary solution, then it has an *asymptotically maximal complementary solution* (x', s') where the number of positive components of (x', s') is maximal. On the other hand, if there is no sequence $\langle x^k, s^k \rangle \in R_{++}^{2n}, k = 0, 1, \dots$, that satisfies (4), then MCP is said to be *strongly infeasible*. We note that if there is no solution $(x, s) \in R_+^{2n}$ that satisfies (1), then MCP is said to be *infeasible*.

Let $X := \text{diag}(x_1, \dots, x_n)$. We say that f satisfies a scaled Lipschitz condition (SLC) if there exists a monotone increasing function

$$\nu_f(\cdot) : (0, 1) \mapsto (1, \infty)$$

such that

$$\|X(f(x+h) - f(x) - \nabla f(x)h)\|_1 \leq \nu_f(\beta)h^T \nabla f(x)h, \quad (5)$$

whenever

$$h \in R^n, x \in R_{++}^n, \|X^{-1}h\|_\infty \leq \beta < 1.$$

2. Homogeneous formulation for the monotone complementarity problem

We consider an augmented homogeneous model related to MCP (HMCP):

$$s = \tau f(x/\tau), \quad (6)$$

$$\kappa = -x^T f(x/\tau), \quad (7)$$

$$0 \leq (s, \kappa) \perp (x, \tau) \geq 0. \quad (8)$$

Andersen and Ye (1999) showed statements 1–6 in the following theorem. Yoshise (2006) showed statement 7 in the more general setting of MCP over symmetric cones.

THEOREM 1. *Consider the MCP and the associated HMCP.*

1. *If ∇f is positive semidefinite in R_+^n , then ∇F is also positive semidefinite in R_{++}^{n+1} .*
2. *F is a continuous homogeneous function in R_{++}^{n+1} with degree 1 and for any $(x, \tau) \in R_{++}^{n+1}$, we have $(x, \tau)^T F(x, \tau) = 0$, and $(x, \tau)^T \nabla F(x, \tau) = -F(x, \tau)^T$.*
3. *If f is a continuous monotone mapping from R_+^n to R^n , then F is a continuous monotone mapping from R_{++}^{n+1} to R^{n+1} .*
4. *If f is scaled Lipschitz with ν_f , then F is also scaled Lipschitz and it satisfies condition (5) with*

$$\nu_F(\beta) = \left(1 + \frac{2\nu_f(2\beta/(1-\beta))}{1-\beta}\right) \left(\frac{1}{1-\beta}\right),$$

whenever

$$(h_x, h_\tau) \in R^{n+1}, (x, \tau) \in R_{++}^{n+1}, \|X^{-1}h_x, \tau^{-1}h_\tau\|_\infty \leq \beta < 1/3.$$

5. *HMCP is (asymptotically) feasible, and every (asymptotically) feasible solution is an (asymptotically) complementary solution.*
6. *If $\tau^* > 0$, then $(x^*/\tau^*, s^*/\tau^*)$ is an asymptotically complementary solution for MCP.*
7. *If MCP is strongly infeasible, then HMCP has an asymptotic solution with $\kappa^* > 0$. Conversely, if HMCP has an asymptotic solution $(x^*, \tau^*, s^*, \kappa^*)$ with $\kappa^* > 0$, then MCP is infeasible, and $(x^*/\kappa^*, s^*/\kappa^*)$ is a certificate to prove the infeasibility of MCP.*

For simplicity we let $\bar{n} := n + 1, x := (x, \tau) \in R_+^{\bar{n}}$ and $s := (s, \kappa) \in R_+^{\bar{n}}$. Let

$$r^k := s^k - F(x^k),$$

and let $(x^0, s^0) = e$ be the starting point, where e is the vector with all components equal to 1. Let $\mathcal{C}(\hat{\mu})$ denote a continuous trajectory such that

$$\mathcal{C}(\hat{\mu}) := \{(x^k, s^k) \mid s^k - F(x^k) = \hat{\mu}r^0, X^k s^k = \hat{\mu}e, 0 < \hat{\mu} \leq 1\}.$$

At iteration k with iterate $(x^k, s^k) > 0$, define a Newton direction (d_x, d_s) , which is obtained by solving the following system of linear equations:

$$d_s - \nabla F(x^k)d_x = -\eta r^k, \quad (9)$$

$$X^k d_s + S^k d_x = \gamma \mu^k e - X^k s^k. \quad (10)$$

Here $\eta, \gamma \in [0, 1]$, and $\mu^k := \frac{(x^k)^T s^k}{\bar{n}}$. For a step size $\alpha > 0$, let the new iterate be

$$x^+ := x^k + \alpha d_x > 0, \quad \text{and}$$

$$s^+ := s^k + \alpha d_s + (F(x^+) - F(x^k) - \alpha \nabla F(x^k)d_x) = F(x^+) + (1 - \alpha\eta)r^k > 0. \quad (11)$$

Andersen and Ye (1999) showed the following lemmas.

LEMMA 1. *The direction (d_x, d_s) satisfies $d_x^T d_s = d_x^T \nabla F(x^k)d_x + \eta(1 - \gamma - \eta)\bar{n}\mu^k$.*

LEMMA 2. *Let $r^+ = s^+ - F(x^+)$, and consider the new iterate (x^+, s^+) given by (9)–(11).*

Then,

(i) $r^+ = (1 - \alpha\eta)r^k,$

(ii) $(x^+)^T s^+ = (x^k)^T s^k(1 - \alpha(1 - \gamma)) + \alpha^2\eta(1 - \eta - \gamma)\bar{n}\mu^k.$

3. A practical potential reduction homogeneous algorithm

We consider the following potential function for the HMCP.

$$\Phi_\rho(x, s) := (\rho/2) \log \{(x^T s)^2 + \theta \|r\|^2\} - \sum_{j=1}^{\bar{n}} \log x_j s_j, \quad (12)$$

where θ is a constant parameter with positive value and $\rho \geq \bar{n} + \sqrt{\bar{n}}$.

To develop a practical potential reduction IPM, we impose the following condition on the iterates.

CONDITION 1.

- (i) $\lambda \leq \frac{(\tilde{x}^+)^T \tilde{s}^+}{(x^+)^T s^+} \leq \Lambda$, and $\lambda \leq \frac{(x^0)^T s^0 |r_j^k|}{(x^k)^T s^k |r_j^0|} \leq \Lambda$, $j = 1, \dots, \bar{n}$,
 (ii) $\Phi_{\bar{n}}(x^k, s^k) \leq \iota$,

where λ , Λ , and ι are given positive constants. Condition 1 (i) requires that the complementarity at the iterate $(\tilde{x}^+, \tilde{s}^+)$ generated using a favored direction and the theoretical iterate (x^+, s^+) are of the same order, and that the overall decrease in the infeasibility to complementarity ratio is maintained in Algorithm 1. Condition 1 (ii) requires the potential function (12) with parameter $\rho = \bar{n}$ is upper bounded by a constant amount. Note that if $\eta = 1 - \gamma$, then from Lemma 2 we have $(x^+)^T s^+ = (1 - \alpha\eta)(x^k)^T s^k$. As a result for the Newton direction, Condition 1 (i) is satisfied with $\lambda, \Lambda = 1$. Andersen and Ye (1999) shows that Condition 1 (i) is also satisfied by the Newton direction with appropriate step size. Convergence properties for an algorithm that has the additional flexibility of choosing more aggressive directions and step sizes is ensured under Condition 1.

Starting from an initial solution $(x^0, s^0) > 0$, our basic potential reduction IPM is now given as Algorithm 1. Note that Step 4 of Algorithm 1 checks if: (i) the reduction of the potential function is satisfactory; (ii) Condition 1 is satisfied. We note that the first safeguard guarantees global linear and polynomial time convergence properties, whereas the other safeguards make sure that the algorithm generates a maximal complementary solution of desired precision. Note that parameter ζ specifies the minimum reduction in the potential function at each iteration, and parameter ι specifies an upper bound on $\phi_{\bar{n}}(\cdot, \cdot)$. We note that this algorithm has the freedom in choosing a search direction \tilde{p}_x and \tilde{p}_s , and it allows different step sizes in the primal and dual spaces.

4. Convergence of the practical potential reduction homogeneous algorithm

We need to analyze the worst case decrease in (12) at each iteration of our potential reduction interior point method (IPM). Let

$$\begin{aligned} S^k &:= \text{diag}(s_1^k, \dots, s_{\bar{n}}^k), \quad D := (X^k S^k)^{0.5}, \quad D_{\min} := \min_{j=1, \dots, \bar{n}} (x_j^k s_j^k)^{0.5}, \\ p &:= \frac{\bar{n}}{\rho} \mu^k e - D^2 e = \frac{x^k{}^T s^k}{\rho} e - D^2 e, \quad \text{and } \alpha := \beta \frac{D_{\min}}{\|D^{-1} p\|}, \end{aligned} \quad (13)$$

where $\beta \in (0, 1/3)$ is a positive constant. Suppose (d_x, d_s) is computed by solving the system of linear equations (9) and (10) with $\eta = 1 - \gamma$ and $\gamma = \frac{\bar{n}}{\rho}$. Let

$$p_x := \alpha d_x,$$

Algorithm 1 Practical potential reduction IPM

Input: An initial solution $(x^0, s^0) > 0$, constants $\zeta < 0$ and $\iota > 0$.Output: An (asymptotically) maximal complementary solution (x^k, s^k) .

- 1: Let $k = 0$.
 - 2: **while** The termination criteria are not met **do**
 - 3: Compute $(\tilde{x}^+, \tilde{s}^+)$ using \tilde{p}_x, \tilde{p}_s for x^k and s^k .
 - 4: **if** $\Phi_\rho(\tilde{x}^+, \tilde{s}^+) - \Phi_\rho(x^k, s^k) \leq \zeta$ and Condition 1 is satisfied, **then**
 - 5: Set $(x^{k+1}, s^{k+1}) := (\tilde{x}^+, \tilde{s}^+)$.
 - 6: **else**
 - 7: Compute (x^+, s^+) using the Newton direction (d_x, d_s) with an appropriate step size α .
 - 8: Set $(x^{k+1}, s^{k+1}) := (x^+, s^+)$.
 - 9: **end if**
 - 10: Set $k := k + 1$.
 - 11: **end while**
-

$$\begin{aligned}
p_s &:= \alpha d_s + (F(x^+) - F(x^k) - \alpha \nabla F(x^k) d_x), \\
z &:= X^k (F(x^+) - F(x^k) - \nabla F(x^k) p_x), \quad \text{and} \\
q &:= z / \alpha,
\end{aligned} \tag{14}$$

where

$$x^+ := x^k + p_x > 0, \quad \text{and} \quad s^+ := s^k + p_s > 0.$$

We have

$$p = \gamma \mu^k e - D^2 e = X^k d_s + S^k d_x, \tag{15}$$

and

$$X^k p_s + S^k p_x - \alpha p = z = \alpha q. \tag{16}$$

Now suppose \tilde{p}_x, \tilde{p}_s be any search direction associated with unequal step sizes in the primal and dual spaces. We let

$$\tilde{x}^+ := x^k + \tilde{p}_x > 0, \quad \text{and} \quad \tilde{s}^+ := s^k + \tilde{p}_s > 0.$$

The following analysis follows some of the steps in Kojima et al. (1991) and in (Potra and Ye 1996, Sections 2 and 3). We drop the index k for an iteration to simplify the presentation.

PROPOSITION 1 (**Kojima et al. (1991), Lemma 2.3**). *Let u, v , and $w \in R^{\bar{n}}$ satisfy $u + v = w$ and $u^T v \geq 0$. Then*

$$\|u\| \leq \|w\|, \|v\| \leq \|w\|, \text{ and } u^T v \leq \frac{1}{2}\|w\|^2.$$

LEMMA 3. *Let p_x be defined as in (14). Then, $\|X^{-1}p_x\|_{\infty} \leq \beta$.*

Proof. We use the technique in Kojima et al. (1991) for the proof. We first rewrite equation (15) as

$$\frac{D^{-1}Xd_s}{\|D^{-1}p\|} + \frac{D^{-1}Sd_x}{\|D^{-1}p\|} = \frac{D^{-1}p}{\|D^{-1}p\|}. \quad (17)$$

From Lemma 1, we have

$$\frac{(D^{-1}Xd_s)^T D^{-1}Sd_x}{\|D^{-1}p\|\|D^{-1}p\|} = \frac{d_x^T d_s}{\|D^{-1}p\|^2} \geq 0. \quad (18)$$

Now using Proposition 1 in (17) and (18), we have

$$\frac{\|D^{-1}Sd_x\|}{\|D^{-1}p\|} \leq \frac{\|D^{-1}p\|}{\|D^{-1}p\|} = 1. \quad (19)$$

Using (13) and (19), we deduce the following relation

$$\frac{\|X^{-1}d_x\|}{\|D^{-1}p\|} = \frac{\|D^{-2}Sd_x\|}{\|D^{-1}p\|} \leq D_{\min}^{-1}. \quad (20)$$

Finally, using (13) and (20), we have

$$\|X^{-1}p_x\|_{\infty} = \alpha\|X^{-1}d_x\|_{\infty} \leq \beta D_{\min} \frac{\|X^{-1}d_x\|}{\|D^{-1}p\|} \leq \beta.$$

□

Potra and Ye (1996) showed the following lemma.

LEMMA 4 (**Potra and Ye (1996), Theorem 3.2**). *If d_x and d_s are the solution of the linear system (16), and if SLC condition (5) and Lemma 3 are satisfied, then*

$$\|z\| \leq \xi\beta^2 D_{\min}^2,$$

where $\xi = 0.25\nu_F(\beta)$ and z is defined in (14).

From the above lemma and (13) we have

$$\|D^{-1}q\| \leq \|q\|D_{\min}^{-1} = \frac{\|z\|}{\alpha}D_{\min}^{-1} \leq \frac{\xi\beta^2}{\alpha}D_{\min} = \xi\beta\|D^{-1}p\|. \quad (21)$$

Kojima et al. showed the following lemma.

LEMMA 5 (**Kojima et al. (1991), Lemma 2.5**). Suppose the search direction (d_x, d_s) is computed by solving the system of linear equations (9) and (10) with $\eta = 1 - \gamma$ and $\gamma = \frac{\bar{n}}{\rho}$ ($\rho \geq \bar{n} + \sqrt{\bar{n}}$), then

$$\frac{\rho}{x^T s} \|D^{-1}p\| = \left\| D^{-1}e - \frac{\rho}{x^T s} De \right\| \geq \frac{\sqrt{3}}{2} D_{\min}^{-1}.$$

In the following we define two real valued functions g_1 and g_2 , and establish results on their upper bounds. These two functions and the related upper bounds will be used later in our analysis. Let

$$g_1 := -\alpha\rho\eta - e^T (X^{-1}p_x + S^{-1}p_s) \quad \text{and} \quad (22)$$

$$g_2 := \frac{\|X^{-1}p_x\|^2 + \|S^{-1}p_s\|^2}{2(1 - \beta(1 + \xi\beta))}. \quad (23)$$

LEMMA 6. Let g_1 be defined as in (22). Suppose the search direction (d_x, d_s) is computed by solving the system of linear equations (9) and (10) with $\eta = 1 - \gamma$ and $\gamma = \frac{\bar{n}}{\rho}$ ($\rho \geq \bar{n} + \sqrt{\bar{n}}$). Let the step size α be defined as in (13). If SLC condition (5) (i) is satisfied, with a suitably chosen parameter $\beta \in (0, 1/3)$, then we have

$$g_1 \leq -\frac{\sqrt{3}}{2}(1 - \xi\beta)\beta + \xi\beta^2 \frac{\rho}{\sqrt{\bar{n}}}.$$

Proof. From (10) and $\gamma = 1 - \eta$, we have

$$-\alpha\rho\eta = \alpha\rho \frac{x^T d_s + s^T d_x}{\bar{n}\mu}. \quad (24)$$

An upper bound of g_1 is obtained in the following set of inequalities.

$$\begin{aligned} g_1 &= \alpha\rho \left(\frac{x^T d_s + s^T d_x}{x^T s} \right) - e^T (X^{-1}p_x + S^{-1}p_s) \\ &= \alpha \frac{\rho}{x^T s} e^T (X d_s + S d_x) - e^T D^{-2} (X p_s + S p_x) \\ &= \alpha \frac{\rho}{x^T s} e^T \left(p - \frac{x^T s}{\rho} D^{-2} (p + q) \right) \quad (\text{using (15) and (16)}) \\ &= \alpha \frac{\rho}{x^T s} e^T \left(\left(D - \frac{x^T s}{\rho} D^{-1} \right) D^{-1} p - \frac{x^T s}{\rho} D^{-2} q \right) \\ &= -\alpha \frac{\rho}{x^T s} (p^T D^{-1} D^{-1} p) - \alpha \frac{\rho}{x^T s} (D^{-1} p + De)^T D^{-1} q \quad (\text{using (13)}) \\ &= -\alpha \frac{\rho}{x^T s} (p^T D^{-1} D^{-1} (p + q)) - \alpha \frac{\rho}{x^T s} e^T q \\ &= -\alpha \frac{\rho}{x^T s} (\|D^{-1}p\|^2 + p^T D^{-1} D^{-1} q) - \frac{\rho}{x^T s} e^T z \quad (\text{using (14)}) \end{aligned}$$

$$\begin{aligned}
 &\leq -\alpha \frac{\rho}{x^T s} (\|D^{-1}p\|^2 - \|D^{-1}p\| \|D^{-1}q\|) + \frac{\rho}{x^T s} \|z\|_1 \\
 &\leq -\alpha \frac{\rho}{x^T s} (\|D^{-1}p\|^2 - \xi\beta \|D^{-1}p\|^2) + \frac{\rho}{x^T s} \sqrt{\bar{n}} \|z\| \quad (\text{using (21)}) \\
 &\leq -\alpha \frac{\rho}{x^T s} (1 - \xi\beta) \|D^{-1}p\|^2 + \frac{\rho}{x^T s} \sqrt{\bar{n}} \xi \beta^2 D_{\min}^2 \quad (\text{using Lemma 4}) \\
 &\leq -\frac{\sqrt{3}}{2} (1 - \xi\beta) \alpha D_{\min}^{-1} \|D^{-1}p\| + \frac{\rho}{\sqrt{\bar{n}}} \xi \beta^2 \quad (\text{using Lemma 5}) \\
 &= -\frac{\sqrt{3}}{2} (1 - \xi\beta) \beta + \frac{\rho}{\sqrt{\bar{n}}} \xi \beta^2. \quad (\text{using (13)})
 \end{aligned}$$

□

LEMMA 7. Let g_2 be defined as in (23). Under the hypothesis of Lemma 6, we have

$$g_2 \leq \frac{\beta^2(1 + \xi\beta)^2}{2(1 - \beta(1 + \xi\beta))}.$$

Proof. The following proof is based on techniques in Potra and Ye (1996). We have

$$\begin{aligned}
 2(1 - \beta(1 + \xi\beta))g_2 &= \|X^{-1}p_x\|^2 + \|S^{-1}p_s\|^2 \tag{25} \\
 &= \|D^{-2}Xp_s\|^2 + \|D^{-2}Sp_x\|^2 \\
 &\leq D_{\min}^{-2} (\|D^{-1}Xp_s\|^2 + \|D^{-1}Sp_x\|^2) \\
 &= D_{\min}^{-2} (\|D^{-1}(Xp_s + Sp_x)\|^2 - 2p_x^T p_s) \\
 &= D_{\min}^{-2} (\alpha^2 \|D^{-1}(p + q)\|^2 - 2p_x^T p_s) \quad (\text{using (16)}) \\
 &\leq D_{\min}^{-2} (\alpha^2 (\|D^{-1}p\| + \|D^{-1}q\|)^2 - 2p_x^T p_s) \\
 &\leq D_{\min}^{-2} (\alpha^2 (1 + \xi\beta)^2 \|D^{-1}p\|^2 - 2p_x^T p_s) \quad (\text{using (21)}) \\
 &= (\alpha D_{\min}^{-1} \|D^{-1}p\|)^2 (1 + \xi\beta)^2 - 2D_{\min}^{-2} p_x^T p_s \\
 &= \beta^2(1 + \xi\beta)^2 - 2D_{\min}^{-2} p_x^T p_s. \quad (\text{using (13)})
 \end{aligned}$$

The lower bound on $p_x^T p_s$ is obtained as follows.

$$\begin{aligned}
 p_x^T p_s &= p_x^T (\alpha d_s + (F(x^+) - F(x) - \nabla F(x)p_x)) \quad (\text{using (14)}) \tag{26} \\
 &= (x^+ - x)^T (F(x^+) - F(x)) + \alpha^2 (d_x^T d_s - d_x^T \nabla F(x) d_x) \quad (\text{using (14)}) \\
 &= (x^+ - x)^T (F(x^+) - F(x)) + \alpha^2 \eta (1 - \eta - \gamma) \bar{n} \mu \quad (\text{using Lemma 1}) \\
 &\geq 0 \quad (\text{from the monotonicity of } F).
 \end{aligned}$$

The result follows from using (25) and (26). □

The following proposition is frequently used in the IPM analysis.

PROPOSITION 2 (Karmarkar (1984)).

1. If $1 - \delta > 0$, then $\log \{1 - \delta\} \leq -\delta$.
2. If $\delta \in R^{\bar{n}}$ satisfies $\|\delta\|_{\infty} \leq v < 1$, then $\sum_{j=1}^{\bar{n}} \log \{1 - \delta_j\} \geq -e^T \delta - \frac{\|\delta\|^2}{2(1-v)}$.

Now we are ready to show a bound on the difference between the potential function values $\Phi_{\rho}(x^+, s^+)$ and $\Phi_{\rho}(x, s)$. Theorem 2 bounds the improvement in the potential function by taking the theoretical direction with a controllable step size.

THEOREM 2. Let $\xi = 0.25\nu_F(\beta) \geq 0.25$. Suppose the search direction (d_x, d_s) is computed by solving the system of linear equations (9) and (10) with $\eta = 1 - \gamma$ and $\gamma = \frac{\bar{n}}{\rho} (\rho \geq \bar{n} + \sqrt{\bar{n}})$. Let the step size α be defined as in (13). If SLC condition (5) is satisfied, and $\beta \in (0, 1/3)$ is chosen such that

$$\xi\beta \leq \frac{\sqrt{3}}{4(\sqrt{3} + 2\rho/\sqrt{\bar{n}})}, \quad \text{and} \quad (27)$$

$$\beta\sqrt{\xi}(1 + \xi\beta) \leq \frac{\sqrt{3}}{2(\sqrt{3} + 2\rho/\sqrt{\bar{n}})^{1/2}}, \quad (28)$$

then, (x^+, s^+) is strictly positive. Moreover,

$$\Phi_{\rho}(x^+, s^+) - \Phi_{\rho}(x, s) \leq -0.2 \frac{\beta}{1 - \beta}.$$

Proof. We first show that (x^+, s^+) is strictly positive. From Lemma 7 and (28), we have

$$\max \{ \|X^{-1}p_x\|_{\infty}, \|S^{-1}p_s\|_{\infty} \} \leq \beta(1 + \xi\beta) \leq \frac{\sqrt{3}}{2(\xi(\sqrt{3} + 2\rho/\sqrt{\bar{n}}))^{1/2}} < 1.$$

Hence, (x^+, s^+) is valid. We note that from Lemma 3, we have

$$\|X^{-1}p_x\|_{\infty} \leq \beta \leq \frac{\sqrt{3}}{4\xi(\sqrt{3} + 2\rho/\sqrt{\bar{n}})} \leq \frac{\sqrt{3}}{2(\xi(\sqrt{3} + 2\rho/\sqrt{\bar{n}}))^{1/2}}.$$

Therefore, the upper bound from Lemma 3 is still valid (and tighter). Now,

$$\begin{aligned} & \Phi_{\rho}(x^+, s^+) - \Phi_{\rho}(x, s) \\ &= (\rho/2) \log \{ (1 - \alpha\eta)^2 ((\bar{n}\mu)^2 + \|r\|^2) \} - \sum_{j=1}^{\bar{n}} \log \{ (x_j + (p_x)_j)(s_j + (p_s)_j) \} \end{aligned}$$

$$\begin{aligned}
 & -(\rho/2) \log \{(\bar{n}\mu)^2 + \|r\|^2\} + \sum_{j=1}^{\bar{n}} \log \{x_j s_j\} \quad (\text{using Lemma 2}) \\
 = & \rho \log \{1 - \alpha\eta\} - \sum_{j=1}^{\bar{n}} \log \left\{ 1 + \frac{(p_x)_j}{x_j} \right\} \left\{ 1 + \frac{(p_s)_j}{s_j} \right\} \\
 \leq & -\alpha\rho\eta - e^T (X^{-1}p_x + S^{-1}p_s) + \frac{\|X^{-1}p_x\|^2 + \|S^{-1}p_s\|^2}{2(1 - \beta(1 + \xi\beta))} \quad (\text{using Proposition 2}) \\
 = & g_1 + g_2. \quad (\text{using (22) and (23)})
 \end{aligned}$$

From Lemmas 6 and 7, we write

$$\begin{aligned}
 \Phi_\rho(x^+, s^+) - \Phi_\rho(x, s) & \leq -\frac{\sqrt{3}}{2}(1 - \xi\beta)\beta + \xi\beta^2\rho/\sqrt{\bar{n}} + \frac{\beta^2(1 + \xi\beta)^2}{2(1 - \beta(1 + \xi\beta))} \\
 & = \frac{\beta^2(1 + \xi\beta)^2}{2(1 - \beta(1 + \xi\beta))} + \beta \left(-\frac{\sqrt{3}}{2} + \xi\beta \left(\frac{\sqrt{3}}{2} + \frac{\rho}{\sqrt{\bar{n}}} \right) \right) \\
 & = \frac{\beta^2(1 + \xi\beta)^2}{2(1 - \beta(1 + \xi\beta))} - \frac{\sqrt{3}}{2}\beta \left(1 - \xi\beta \left(\frac{\sqrt{3} + 2(\rho/\sqrt{\bar{n}})}{\sqrt{3}} \right) \right) =: \zeta.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 2\frac{(1 - \beta(1 + \xi\beta))}{\beta}\zeta & = \beta(1 + \xi\beta)^2 - \sqrt{3}(1 - \beta(1 + \xi\beta)) \left(1 - \xi\beta \left(\frac{\sqrt{3} + 2(\rho/\sqrt{\bar{n}})}{\sqrt{3}} \right) \right) \\
 & \leq \beta(1 + 0.076)^2 - \sqrt{3}(3/4)(1 - \beta(1 + \xi\beta)) \quad (\text{using (27) and } \rho/\sqrt{\bar{n}} \geq 2) \\
 & \leq -0.4.
 \end{aligned}$$

Hence, we have

$$\zeta \leq -0.2 \frac{\beta}{1 - 2\beta}. \quad (29)$$

□

Since $\nu_F(\beta) : (0, 1/3) \mapsto (1, \infty)$ is a monotone increasing function depending on β , we can always choose a sufficiently small β that satisfies (27) and (28). As a result, a reduction in the potential function ($\zeta \leq -0.2 \frac{\beta}{1 - 2\beta}$) is guaranteed. However, choosing β is dependent on $\nu_F(\beta)$, and finding a maximal β that satisfies (27) and (28) requires the knowledge of the scaled Lipschitz constant $\nu_F(\beta)$, which may not be available in practice for an arbitrary convex function. Problems satisfying the scaled Lipschitz condition are discussed in Monteiro and Adler (1990), Potra and Ye (1993), Zhu (1992). Nevertheless, the analysis suggests that for an appropriate step size, the potential function value will reduce as long as the scaled Lipschitz constant is bounded. Consequently, the potential function can be

used as a guide to solve monotone complementarity and convex optimization problems satisfying this property. If the condition is not satisfied, then lack of sufficient reduction in the potential function indicates that the algorithm proposed here is not appropriate for the problem being solved. Finally, we note that Theorem 2 is similar to the result by Potra and Ye (1996), but with the difference that it allows us to choose a larger ρ , when β is sufficiently small.

In the following we will show that if potential function (12) can be reduced by at least a constant at each iteration, then Algorithm 1 generates an ϵ -complementary solution in polynomial time. Our analysis follows some of the steps in Güler and Ye (1993), Kojima et al. (1991), and Potra and Ye (1996).

THEOREM 3. *Let $\epsilon > 0$. Starting from a solution $(x^0, s^0) > 0$, if the value of potential function (12) can be reduced by at least a constant amount and if Condition 1 is satisfied at each iteration of Algorithm 1 then after at most $\mathcal{O}(\Phi_\rho(x^0, s^0) - \bar{n} \log \bar{n} + (\rho - \bar{n}) |\log \epsilon|)$ iterations, we will have a solution such that*

$$x^T s \leq \epsilon \quad \text{and} \quad \frac{|r_j|}{|r_j^0|} \leq \Lambda \frac{x^T s}{(x^0)^T s^0}, \quad j = 1, \dots, \bar{n}. \quad (30)$$

Proof. Using Condition 1 (i), we have

$$(\rho/2) \log \left\{ (x^T s)^2 \left(1 + \left(\lambda \frac{\|r^0\|}{(x^0)^T s^0} \right)^2 \right) \right\} - \sum_{j=1}^{\bar{n}} \log x_j s_j \leq \Phi_\rho(x, s),$$

which implies

$$\begin{aligned} & (\rho - \bar{n}) \log x^T s - \sum_{j=1}^{\bar{n}} \log \left\{ \frac{\bar{n} x_j s_j}{x^T s} \right\} + (\rho/2) \log \left\{ 1 + \left(\lambda \frac{\|r^0\|}{(x^0)^T s^0} \right)^2 \right\} \\ & \leq \Phi_\rho(x, s) - \bar{n} \log \bar{n}. \end{aligned} \quad (31)$$

We note that the second term of (31) is nonnegative due to arithmetic and geometric mean inequality, and obviously the third term is positive. Hence, it follows that

$$(\rho - \bar{n}) \log x^T s \leq \Phi_\rho(x, s) - \bar{n} \log \bar{n}.$$

Therefore, after at most $\mathcal{O}(\Phi_\rho(x^0, s^0) - \bar{n} \log \bar{n} + (\rho - \bar{n}) |\log \epsilon|)$ iterations, we have $\Phi_\rho(x, s) \leq \bar{n} \log \bar{n} - (\rho - \bar{n}) |\log \epsilon|$, and hence

$$x^T s \leq \epsilon.$$

Moreover, Condition 1 (i) implies that

$$\frac{|r_j|}{|r_j^0|} \leq \Lambda \frac{x^T s}{(x^0)^T s^0}, \quad j = 1, \dots, \bar{n}.$$

□

We now show that any limit point of a sequence $\langle x^k, s^k \rangle$ generated by Algorithm 1 is an asymptotically maximal complementary solution. Our analysis here follows some of the steps in (Güler and Ye 1993, Sections 2 and 4) and (Potra and Ye 1996, Section 4).

In the following analysis, we suppose Algorithm 1 is used from an initial positive point (x^0, s^0) . Recall that at iterate (x^k, s^k) , $(\tilde{x}^+, \tilde{s}^+)$ is a tentative point computed using any favored search direction using different step sizes in the primal and dual spaces, and it satisfies Condition 1; whereas (x^+, s^+) is computed using the Newton direction (d_x, d_s) and a fixed step size α . The proof of Lemma 8 only uses the fact that $\Phi_\rho(\cdot, \cdot)$ is reduced by an amount ζ , and is independent of the source of this reduction.

LEMMA 8. *Let λ and Λ be defined in Condition 1,*

$$C_1 := \frac{\rho - \bar{n}}{2} \log \left\{ \frac{((x^0)^T s^0)^2 + \Lambda^2 \|r^0\|^2}{((x^0)^T s^0)^2 + \lambda^2 \|r^0\|^2} \right\} \geq 0, \quad \text{and}$$

$$C_2 := C_1 + \max \{0, -(\rho - \bar{n}) \log \lambda\} \geq 0.$$

Then,

$$\Phi_{\bar{n}}(x^+, s^+) - \Phi_{\bar{n}}(x^k, s^k) \leq \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{(x^k)^T s^k} \right\}, \quad \text{and} \quad (32)$$

$$\Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+) - \Phi_{\bar{n}}(x^k, s^k) \leq \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{(x^k)^T s^k} \right\} + C_2, \quad (33)$$

where $\Phi_{\bar{n}}(x^k, s^k)$ is defined as in (12) with $\rho = \bar{n}$, and ζ is the reduction on the potential function required in Step 4 of Algorithm 1.

Proof. To simplify the presentation we drop the index k . Let $\tilde{r}^+ = \tilde{s}^+ - F(\tilde{x}^+)$. Using Condition 1 (i), we have

$$\|\tilde{r}^+\|^2 \geq \left(\lambda \|r^0\| \frac{(\tilde{x}^+)^T \tilde{s}^+}{(x^0)^T s^0} \right)^2, \quad \|r\|^2 \leq \left(\Lambda \|r^0\| \frac{x^T s}{(x^0)^T s^0} \right)^2, \quad \text{and} \quad (34)$$

$$(\tilde{x}^+)^T \tilde{s}^+ \geq \lambda (x^+)^T s^+. \quad (35)$$

First we bound the difference between the values of $\Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+)$ and $\Phi_{\bar{n}}(x, s)$.

$$\begin{aligned}
& \Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+) - \Phi_{\bar{n}}(x, s) \\
&= \Phi_{\rho}(\tilde{x}^+, \tilde{s}^+) - \Phi_{\rho}(x, s) - \Phi_{\rho}(\tilde{x}^+, \tilde{s}^+) + \Phi_{\rho}(x, s) + \Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+) - \Phi_{\bar{n}}(x, s) \\
&\leq \zeta - \frac{\rho - \bar{n}}{2} \log \left\{ ((\tilde{x}^+)^T \tilde{s}^+)^2 + \|\tilde{r}^+\|^2 \right\} \\
&\quad + \frac{\rho - \bar{n}}{2} \log \left\{ (x^T s)^2 + \|r\|^2 \right\} \text{ (using sufficient reduction condition in Step 4 of Algorithm 1)} \\
&\leq \zeta - \frac{\rho - \bar{n}}{2} \log \left\{ ((\tilde{x}^+)^T \tilde{s}^+)^2 + \left(\lambda \|r^0\| \frac{(\tilde{x}^+)^T \tilde{s}^+}{(x^0)^T s^0} \right)^2 \right\} \\
&\quad + \frac{\rho - \bar{n}}{2} \log \left\{ (x^T s)^2 + \left(\Lambda \|r^0\| \frac{x^T s}{(x^0)^T s^0} \right)^2 \right\} \text{ (using (34))} \\
&= \zeta - (\rho - \bar{n}) \log \left\{ \frac{(\tilde{x}^+)^T \tilde{s}^+}{x^T s} \right\} + C_1 \\
&\leq \zeta - (\rho - \bar{n}) \log \left\{ \frac{(x^+)^T s^+}{x^T s} \right\} - (\rho - \bar{n}) \log \lambda + C_1 \text{ (using (35))} \\
&\leq \zeta - (\rho - \bar{n}) \log \left\{ \frac{(x^+)^T s^+}{x^T s} \right\} + C_2 \\
&= \zeta - (\rho - \bar{n}) \log \{1 - \alpha \eta\} + C_2 \text{ (using Lemma 2)} \\
&= \zeta - (\rho - \bar{n}) \log \left\{ 1 + \alpha \frac{x^T d_s + s^T d_s}{x^T s} \right\} + C_2 \text{ (using (24))} \\
&= \zeta - (\rho - \bar{n}) \log \left\{ 1 + \alpha \frac{p^T e}{x^T s} \right\} + C_2 \text{ (using (15))}.
\end{aligned}$$

Now,

$$\begin{aligned}
\alpha \frac{p^T e}{x^T s} &= \frac{\alpha ((x^T s / \rho) e^T e - e^T D^2 e)}{x^T s} \text{ (using (13))} \\
&= \alpha \left(\frac{\bar{n}}{\rho} - 1 \right) \\
&= -\beta \frac{D_{\min}}{\|D^{-1} p\|} \left(\frac{\rho - \bar{n}}{\rho} \right) \text{ (using (13))} \\
&\geq -\beta D_{\min}^2 \frac{2(\rho - \bar{n})}{\sqrt{3} x^T s} \text{ (using Lemma 5)} \\
&\geq -D_{\min}^2 \frac{\sqrt{\bar{n}}(\rho - \bar{n})}{2x^T s \xi (\sqrt{3\bar{n}} + 2\rho)} \text{ (using (27))} \\
&\geq -D_{\min}^2 \frac{2\rho\sqrt{\bar{n}}}{x^T s (\sqrt{3\bar{n}} + 2\rho)} \text{ (because } \xi = 0.25\nu_F\beta \geq 0.25) \\
&\geq -D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s}.
\end{aligned}$$

Therefore,

$$\Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+) - \Phi_{\bar{n}}(x, s) \leq \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} + C_2.$$

The difference between the values of $\Phi_{\bar{n}}(x^+, s^+)$ and $\Phi_{\bar{n}}(x, s)$ can be constructed in a similar way. \square

The following lemma shows that $\Phi_{\bar{n}}(x^k, s^k)$ remains bounded at all iterates of Algorithm 1.

LEMMA 9. *There is a constant M_1 independent of k such that*

$$\Phi_{\bar{n}}(x^k, s^k) \leq M_1 \text{ for all } k,$$

where

$$M_1 = \max \left\{ \iota \bar{n} \log \bar{n} - (\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\sqrt{\bar{n}}} \right\} + C_2 + C_3, \Phi_{\bar{n}}(x^0, s^0) \right\}.$$

Proof. The index k is dropped for simplification. Let $C_3 := (\bar{n}/2) \log \left\{ 1 + \left(\Lambda \frac{\|r^0\|}{(x^0)^T s^0} \right)^2 \right\} > 0$. Using (34), we have

$$\begin{aligned} \Phi_{\bar{n}}(x, s) &= (\bar{n}/2) \log \left\{ (x^T s)^2 + \|r\|^2 \right\} - \sum_{j=1}^{\bar{n}} \log x_j s_j \\ &\leq (\bar{n}/2) \log \left\{ (x^T s)^2 + \left(\Lambda \|r^0\| \frac{x^T s}{(x^0)^T s^0} \right)^2 \right\} - \sum_{j=1}^{\bar{n}} \log x_j s_j \\ &= \bar{n} \log x^T s - \sum_{j=1}^{\bar{n}} \log x_j s_j + C_3. \end{aligned} \tag{36}$$

Recall that $\phi_{\bar{n}}(x, s) = \bar{n} \log x^T s - \sum_{j=1}^{\bar{n}} \log x_j s_j$ corresponds to the feasible potential function. In the following we analyze the behavior of $\phi_{\bar{n}}(x, s)$. Let $\iota > 0$ be sufficiently large (e.g., as given in Table 1) so that

$$\zeta - (\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\bar{n}^{\iota-0.5}} \right\} < 0, \tag{37}$$

where ζ is defined as in (29). We now consider the following two cases:

1. $\phi_{\bar{n}}(x, s) \geq \iota \bar{n} \log \bar{n}$;
2. $\phi_{\bar{n}}(x, s) \leq \iota \bar{n} \log \bar{n}$.

We will show that in the first case, the value of $\Phi_{\bar{n}}(x, s)$ is decreased if the algorithm chooses the Newton direction with a fixed step size; hence, $\Phi_{\bar{n}}(x^+, s^+) \leq M_1$, if $\Phi_{\bar{n}}(x, s) \leq M_1$. In

the second case, the value of $\Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+)$ is upper bounded by M_1 independently of k . This implies the algorithm is allowed to choose any favored search direction with different step sizes in the primal and dual spaces, as long as the safeguards are ensured.

Case 1:

It follows that

$$\begin{aligned}
& \bar{n} \log x^T s + \bar{n} \log D_{\min}^{-2} \geq \iota \bar{n} \log \bar{n}, \\
& \Rightarrow \log \{x^T s D_{\min}^{-2}\} \geq \iota \log \bar{n}, \\
& \Rightarrow \log \left\{ D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} \leq \log \left\{ \frac{1}{\bar{n}^{\iota-0.5}} \right\}, \\
& \Rightarrow -(\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} \leq -(\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\bar{n}^{\iota-0.5}} \right\}. \tag{38}
\end{aligned}$$

Now using (32), we have,

$$\begin{aligned}
\Phi_{\bar{n}}(x^+, s^+) & \leq \Phi_{\bar{n}}(x, s) + \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} \\
& \leq \Phi_{\bar{n}}(x, s) + \zeta - (\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\bar{n}^{\iota-0.5}} \right\} \quad (\text{using (38)}) \\
& < \Phi_{\bar{n}}(x, s). \quad (\text{using (37)})
\end{aligned}$$

Case 2:

Using (33), we have

$$\begin{aligned}
\Phi_{\bar{n}}(\tilde{x}^+, \tilde{s}^+) & \leq \Phi_{\bar{n}}(x, s) + \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} + C_2 \\
& \leq \bar{n} \log x^T s - \sum_{j=1}^{\bar{n}} \log x_j s_j + \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} \\
& \quad + C_2 + C_3 \quad (\text{using (36)}) \\
& \leq \iota \bar{n} \log \bar{n} + \zeta - (\rho - \bar{n}) \log \left\{ 1 - D_{\min}^2 \frac{\sqrt{\bar{n}}}{x^T s} \right\} + C_2 + C_3 \\
& < \iota \bar{n} \log \bar{n} - (\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\sqrt{\bar{n}}} \right\} + C_2 + C_3.
\end{aligned}$$

The result holds for

$$M_1 := \max \left\{ \iota \bar{n} \log \bar{n} - (\rho - \bar{n}) \log \left\{ 1 - \frac{1}{\sqrt{\bar{n}}} \right\} + C_2 + C_3, \Phi_{\bar{n}}(x^0, s^0) \right\}.$$

□

LEMMA 10. Let M_1 be the constant given in Lemma 9. There exists a constant M_2 (independent of k) such that

$$\frac{D_{\min}^2}{(x^k)^T s^k} = \frac{\min_{j=1, \dots, \bar{n}} \{x_j^k s_j^k\}}{(x^k)^T s^k} \geq M_2,$$

where

$$M_2 = \exp \{(\bar{n} - 1) \log \{\bar{n} - 1\} - M_1\}.$$

Proof. We drop the index k for simplification. Using the result in Lemma 9, we have

$$M_1 \geq \Phi_{\bar{n}}(x, s) = (\bar{n}/2) \log \{(x^T s)^2 + \|r\|^2\} - \sum_{j=1}^{\bar{n}} \log x_j s_j \geq \bar{n} \log x^T s - \sum_{j=1}^{\bar{n}} \log x_j s_j.$$

Hence, it follows that

$$\begin{aligned} \log x^T s - \log x_i s_i &\leq M_1 - (\bar{n} - 1) \log x^T s + \sum_{j \neq i} \log x_j s_j \\ &\leq M_1 - (\bar{n} - 1) \log \{x^T s - x_i s_i\} + \sum_{j \neq i} \log x_j s_j \\ &\leq M_1 - (\bar{n} - 1) \log \{\bar{n} - 1\} =: -\log M_2. \end{aligned}$$

Hence, the result holds for

$$M_2 = \exp \{(\bar{n} - 1) \log \{\bar{n} - 1\} - M_1\}.$$

□

Now let

$$\begin{aligned} \sigma(x) &:= \{j \in \{1, 2, \dots, \bar{n}\} : x_j \geq s_j\} \quad \text{and} \\ \sigma(s) &:= \{j \in \{1, 2, \dots, \bar{n}\} : s_j \geq x_j\}. \end{aligned} \tag{39}$$

THEOREM 4. Assume that $\Phi_{\rho}(x^k, s^k)$ is reduced by at least a constant amount at each iteration, and iterates are maintained while choosing search directions and step sizes satisfying Condition 1. Then, any limit point of the sequence $\langle x^k, s^k \rangle$ is an asymptotically maximal complementary solution for HMCP.

Proof. Let (x', s') be a maximal complementary solution for HMCP, and let (x^*, s^*) be a limit point of the sequence $\langle x^k, s^k \rangle$. In the following we show that (x^*, s^*) is an asymptotically maximal complementary solution, that is, $\sigma(x^*) = \sigma(x')$ and $\sigma(s^*) = \sigma(s')$.

Recall that $r^k = s^k - F(x^k)$. We use the notation $|r^k|^T := (|r_1^k|, \dots, |r_n^k|)$ and let $C_4 := \Lambda \frac{|r^0|^T x'}{(x^0)^T s^0} \geq 0$. Using Condition 1 (i), we have

$$(r^k)^T x' \leq |r^k|^T x' \leq \Lambda ((x^k)^T s^k) \frac{|r^0|^T x'}{(x^0)^T s^0} = C_4 (x^k)^T s^k. \quad (40)$$

From the monotonicity of $F(x)$, we have

$$\begin{aligned} & (x^k - x')^T (F(x^k) - F(x')) \geq 0, \\ \Rightarrow & -(x^k)^T s' - (s^k - r^k)^T x' \geq 0, \quad (\text{using Theorem 1 (statement 2)}) \\ \Rightarrow & (x^k)^T s' + (s^k)^T x' \leq (r^k)^T x' \leq C_4 (x^k)^T s^k, \quad (\text{using (40)}) \\ \Rightarrow & \sum_{j \in \sigma(x')} x'_j s_j^k + \sum_{j \in \sigma(s')} s'_j x_j^k \leq C_4 (x^k)^T s^k. \end{aligned}$$

As a result, if $j \in \sigma(x')$, then

$$C_4 (x^k)^T s^k \geq s_j^k x'_j \geq (s_j^k x_j^k) \frac{x'_j}{x_j^k} \geq \min_{j=1, \dots, \bar{n}} \{x_j^k s_j^k\} \frac{x'_j}{x_j^k}.$$

Using Lemma 10, the above relation implies

$$x_j^k \geq \frac{\min_{j=1, \dots, \bar{n}} \{x_j^k s_j^k\}}{C_4 (x^k)^T s^k} x'_j \geq \frac{M_2}{C_4} x'_j.$$

Hence, we have $x_j^* \geq (M_2/C_4)x'_j > 0$, consequently $\sigma(x^*) = \sigma(x')$. The proof of $\sigma(s^*) = \sigma(s')$ can be constructed in a similar way. \square

REMARK 1. Note that by performing simple block elimination on the system of linear equations (9) and (10), we obtain the following alternative formulation:

$$X(\alpha \nabla F(x) d_x - \alpha \eta r) + \alpha S d_x - \alpha p = 0. \quad (41)$$

Hence, d_x is a direction obtained by applying a single Newton step to the equation

$$z(h) := X(F(x + \alpha h) - F(x) - \alpha \eta r) + \alpha S h - \alpha p = 0, \quad (42)$$

with starting point $h = 0$. If F is linear (for example, LP formulated as a HMCP model), then the linear system (42) can be solved exactly. In this case Theorem 2 with $\xi = 0$ reduces to the result of Mehrotra and Huang (2012).

Suppose F is nonlinear and does not satisfy the SLC. Given positive constants ξ and β that satisfy conditions (27) and (28), if one can compute an approximate solution δ_x such that

$$\|z(\delta_x)\| = \|X(F(x + \alpha\delta_x) - F(x) - \alpha\eta r) + \alpha S\delta_x - \alpha p\| \leq \xi\beta^2 \min_{j=1,\dots,\bar{n}} (x_j s_j), \quad (43)$$

then the result of Theorem 2 is still satisfied. However, finding a direction δ_x may require many Newton steps, and an explicit bound on the number of Newton steps is unknown.

REMARK 2. Note that from Lemma 2 we have $(x^+)^T s^+ = (1 - \alpha\eta)(x^k)^T s^k$. As a result, if the Newton direction is chosen in the algorithm with suitable chosen parameter $\eta = 1 - \gamma$, then Condition 1 (i) are satisfied as the infeasibility residual and the complementarity are reduced at the same rate.

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Problem	Before Presolving Rows	Before Presolving Cols	After Presolving Rows	After Presolving Cols	Primal Objective	Dual Objective	Presolve Time	Solver Time	Iters	Avg Diis	Prop Φ (%)
aug2d	10000	20200	9604	19800	1.68741175E+06	1.68741175E+06	0.03	3.9	12	1.33	4.39
aug2dc	10000	20200	10000	20200	1.81836807E+06	1.81836807E+06	0.02	4.07	12	1.33	4.48
aug2dcqp	10000	20200	10000	20200	6.49813474E+06	6.49813473E+06	0.02	2.4	13	2	6.81
aug2dcqp	10000	20200	10000	20200	6.23701204E+06	6.23701200E+06	0.01	2.64	14	2	7.5
aug3d	1000	3873	512	2673	5.54067726E+02	5.54067726E+02	0.01	0.27	8	1.5	3.89
aug3dc	1000	3873	1000	3873	7.71262439E+02	7.71262439E+02	0.01	0.38	8	1.5	4.38
aug3dcqp	1000	3873	1000	3873	9.93362148E+02	9.93362145E+02	0.01	0.36	11	2	4.84
aug3dcqp	1000	3873	1000	3873	6.75237672E+02	6.75237670E+02	0.01	0.39	13	2	5.59
cont-050	2401	2597	2401	2597	-4.56385090E+00	-4.56385090E+00	0.01	0.65	11	2	3.49
cont-100	9801	10197	9801	10197	-4.64439787E+00	-4.64439787E+00	0.01	4.55	11	2	2.11
cont-101	10098	10197	10098	10197	1.95527325E-01	1.95527325E-01	0.01	4.06	9	2	1.95
cont-200	39601	40397	39601	40397	-4.68487592E+00	-4.68487592E+00	0.04	43.34	11	2	0.99
cont-201	40198	40397	40198	40397	1.92483373E-01	1.92483373E-01	0.04	42.09	10	1.9	1
cont-300	90298	90597	90298	90597	1.91512286E-01	1.91512286E-01	0.08	247.59	13	1.92	0.47
cvxqp1_m	500	1000	500	1000	1.08751157E+06	1.08751157E+06	0	0.41	10	2	2.25
cvxqp1_s	50	100	50	100	1.15907181E+04	1.15907181E+04	0	0.08	8	2	1.32
cvxqp2_m	250	1000	250	1000	8.20155431E+05	8.20155431E+05	0	0.23	10	2	2.76
cvxqp2_s	25	100	25	100	8.12094048E+03	8.12094048E+03	0	0.1	10	1.9	1.08
cvxqp3_m	750	1000	750	1000	1.36282874E+06	1.36282874E+06	0	0.65	12	2	1.41
cvxqp3_s	75	100	75	100	1.19434322E+04	1.19434322E+04	0	0.09	10	2	0
dpklo1	77	133	77	133	3.70096217E-01	3.70096217E-01	0	0.04	3	2	0
dtoc3	9998	14999	9997	14996	2.35262481E+02	2.35262481E+02	0.02	0.93	4	2	8.75
dual1	1	85	1	85	3.50129664E-02	3.50129664E-02	0	0.1	12	2	2.06
dual2	1	96	1	96	3.37336761E-02	3.37336761E-02	0	0.08	9	2	0
dual3	1	111	1	111	1.35755838E-01	1.35755838E-01	0	0.09	9	2	1.25
dual4	1	75	1	75	7.46090842E-01	7.46090840E-01	0	0.08	10	2	1.32
dualc1	215	223	13	21	6.15525083E+03	6.15525083E+03	0	0.26	25	1.48	0
dualc2	229	235	9	15	3.55130769E+03	3.55130769E+03	0	0.21	20	1.55	0
dualc5	278	285	1	8	4.27232327E+02	4.27232327E+02	0.01	0.12	13	1.62	0
dualc8	503	510	15	22	1.83093588E+04	1.83093588E+04	0.01	0.2	19	1.63	0.5
genhs28	8	10	8	10	9.27173692E-01	9.27173694E-01	0.01	0.03	3	2	0
gouldqp2	349	699	349	699	1.84275300E-04	1.84273273E-04	0	0.16	13	2	3.25
gouldqp3	349	699	349	699	2.06278397E+00	2.06278397E+00	0.01	0.14	11	2	4.58
hs118	17	32	17	32	6.64820450E+02	6.64820450E+02	0.01	0.09	10	2	0
hs21	1	3	1	3	-9.99599996E+01	-9.99600001E+01	0.01	0.08	9	2	0
hs268	5	10	5	10	2.58580312E-05	-5.06859105E-05	0	0.1	12	2	0
hs35	1	4	1	4	1.11111111E-01	1.11111111E-01	0	0.05	5	2	0
hs35mod	1	4	1	3	2.50000007E-01	2.49999989E-01	0	0.09	11	2	0
hs51	3	5	3	5	0.00000000E+00	9.97371075E-11	0.01	0.03	3	2	0
hs52	3	5	3	5	5.32664756E+00	5.32664757E+00	0	0.04	3	2	0
hs53	3	5	3	5	4.09302326E+00	4.09302302E+00	0.01	0.04	4	2	0
hs76	3	7	3	7	-4.68181818E+00	-4.68181818E+00	0	0.05	5	2	0
ksip	1001	1021	1001	1021	5.75797941E-01	5.75797941E-01	0.01	0.38	16	2	3.45

Problem	Before Presolving		After Presolving		Primal Objective	Dual Objective	Presolve Time	Solver Time	Hers	Avg Dirs	Prop Φ (%)
	Rows	Cols	Rows	Cols							
laser	1000	2002	1000	2002	2.40960135E+06	2.40960134E+06	0.01	0.21	10	2	4.04
liswet1	10000	20002	10000	20002	3.61224021E+01	3.61224021E+01	0.03	3.21	17	1.89	7.53
liswet2	10000	20002	10000	20002	2.49980788E+01	2.49980759E+01	0.02	2.68	17	2	7.32
liswet3	10000	20002	10000	20002	2.50012181E+01	2.50012181E+01	0.02	2.55	16	2	7.39
liswet4	10000	20002	10000	20002	2.50001164E+01	2.50001107E+01	0.02	2.76	18	2	8.06
liswet5	10000	20002	10000	20002	2.50343102E+01	2.50341783E+01	0.02	2.54	16	2	7.91
liswet6	10000	20002	10000	20002	2.49957529E+01	2.49957445E+01	0.02	3	18	2	7.82
liswet7	10000	20002	10000	20002	4.98840891E+02	4.98840891E+02	0.02	3.71	19	1.89	7.31
liswet8	10000	20002	10000	20002	7.14470067E+02	7.14470059E+02	0.02	5.47	37	1.78	7.76
liswet9	10000	20002	10000	20002	1.96325126E+03	1.96325126E+03	0.02	5.12	31	1.65	7.18
liswet10	10000	20002	10000	20002	4.94858035E+01	4.94857840E+01	0.02	3.18	21	1.95	7.6
liswet11	10000	20002	10000	20002	4.95239569E+01	4.95239569E+01	0.02	3.23	22	1.95	7.82
liswet12	10000	20002	10000	20002	1.73692743E+03	1.73692743E+03	0.02	4.89	32	1.75	7.6
lotschd	7	12	7	12	2.39841590E+03	2.39841589E+03	0	0.05	9	2	0
mosarqp1	700	3200	700	3200	-9.52875441E+02	-9.52875445E+02	0	0.21	9	2	5.5
mosarqp2	600	1500	600	1500	-1.59748211E+03	-1.59748212E+03	0	0.14	9	2	7.35
powell20	10000	20000	10000	20000	5.20895828E+10	5.20895828E+10	0.02	1.46	8	1.88	7.39
primal1	85	410	85	410	-3.50129646E-02	-3.50129670E-02	0	0.11	9	2	2.97
primal2	96	745	96	745	-3.37336714E-02	-3.37336763E-02	0	0.1	7	2	1.04
primal3	111	856	111	856	-1.35755836E-01	-1.35755837E-01	0	0.21	9	2	4.71
primal4	75	1564	75	1564	-7.46090837E-01	-7.46090843E-01	0	0.2	9	2	5.08
primalc1	9	239	9	239	-6.15525083E+03	-6.15525083E+03	0	0.14	18	1.83	1.49
primalc2	7	238	7	237	-3.55130769E+03	-3.55130769E+03	0	0.2	25	1.68	3.96
primalc5	8	295	8	206	-4.27232327E+02	-4.27232327E+02	0	0.07	10	1.9	1.43
primalc8	8	528	8	528	-1.83094298E+04	-1.83094298E+04	0	0.15	12	1.83	3.36
q25fv47	820	1876	790	1846	1.37444479E+07	1.37444478E+07	0.02	1.55	27	2	4.48
qadlittle	56	138	55	137	4.80318859E+05	4.80318859E+05	0	0.07	11	2	0
qafiro	27	51	27	51	-1.59078179E+00	-1.59078179E+00	0	0.08	14	1.86	0
qbandm	305	472	252	419	1.63523420E+04	1.63523420E+04	0	0.17	20	2	0.6
qbeaconf	173	295	107	219	1.64712060E+05	1.64712060E+05	0	0.1	14	2	1.96
qbore3d	233	334	123	225	3.10020080E+03	3.10020080E+03	0	0.2	19	1.79	2.45
qbrandy	220	303	136	246	2.83751149E+04	2.83751149E+04	0	0.13	17	2	0
qcapri	271	482	243	438	6.67932932E+07	6.67932932E+07	0	0.37	36	1.72	2.72
qe226	223	472	214	463	2.12653433E+02	2.12653433E+02	0	0.17	16	2	2.48
qetamacr	400	816	334	669	8.67603696E+04	8.67603693E+04	0	0.49	32	2	2.29
qffff80	524	1028	322	826	8.73147464E+05	8.73147456E+05	0	0.39	25	1.92	3.37
qforplan	161	492	122	450	7.45663147E+09	7.45663144E+09	0	0.39	31	1.71	2.86
qftrdxpn	616	1160	590	1134	1.00790585E+11	1.00790585E+11	0	0.39	30	1.9	3.39
qgrow15	300	645	300	645	-1.01693640E+08	-1.01693640E+08	0	0.28	22	2	2.21
qgrow22	440	946	440	946	-1.49628953E+08	-1.49628953E+08	0	0.43	26	2	3.77
qgrow7	140	301	140	301	-4.27987139E+07	-4.27987139E+07	0	0.18	21	2	3.91
qisrael	174	316	174	316	2.53478378E+07	2.53478376E+07	0	0.28	28	1.75	1.83
qpcblend	74	114	74	114	-7.84254164E-03	-7.84254440E-03	0	0.11	17	2	1.94

Problem	Before Presolving Rows	Before Presolving Cols	After Presolving Rows	After Presolving Cols	Primal Objective	Dual Objective	Presolve Time	Solver Time	Iters	Avg Dtrs	Prop Φ (%)
qpcboe1	351	726	343	718	1.15039140E+07	1.15039140E+07	0	0.27	22	2	2.26
qpcboe2	166	305	126	265	8.17196226E+06	8.17196221E+06	0	0.22	26	1.88	1.83
qpcstair	356	614	356	532	6.20438749E+06	6.20438744E+06	0	0.32	25	1.76	1.92
qpilotno	975	2446	866	2151	4.72858690E+06	4.72858690E+06	0.01	1.84	42	1.86	2.8
qpreipe	91	204	76	141	-2.66615999E+02	-2.66616001E+02	0	0.12	19	2	0.88
qptest	2	4	2	4	4.37187500E+00	4.37187500E+00	0	0.03	6	2	0
qsc205	205	317	203	315	-5.81395349E-03	-5.81395349E-03	0	0.12	17	2	1.69
qscagr25	471	671	469	669	2.01737938E+08	2.01737938E+08	0	0.22	22	1.95	4.13
qscagr7	129	185	127	183	2.68659486E+07	2.68659485E+07	0	0.14	22	1.82	2.13
qscfxm1	330	600	311	581	1.68826916E+07	1.68826916E+07	0	0.29	25	2	2.79
qscfxm2	660	1200	622	1162	2.77761616E+07	2.77761615E+07	0	0.49	33	2	3.56
qscfxm3	990	1800	933	1743	3.08163545E+07	3.08163545E+07	0	0.73	35	1.94	3.37
qscorpio	388	466	355	436	1.88050955E+03	1.88050955E+03	0	0.15	14	1.93	2.8
qscrs8	490	1275	445	1230	9.04560015E+02	9.04560013E+02	0.01	0.41	24	1.96	2.7
qscsd1	77	760	77	760	8.66666669E+00	8.66666667E+00	0	0.15	9	2	2.05
qscsd6	147	1350	147	1350	5.08082139E+01	5.08082139E+01	0	0.29	14	2	2.2
qscsd8	397	2750	397	2750	9.40763578E+02	9.40763570E+02	0.01	0.38	11	2	3.76
qscstap1	300	660	284	644	1.41586111E+03	1.41586111E+03	0.01	0.33	17	2	1.88
qscstap2	1090	2500	1033	2443	1.73502651E+03	1.73502649E+03	0.01	0.51	13	2	2.45
qscstap3	1480	3340	1408	3268	1.43875468E+03	1.43875468E+03	0.01	0.64	15	2	3.95
qseba	515	1036	514	1033	8.14818004E+07	8.14818004E+07	0.01	0.64	26	1.92	1.75
qshare1b	117	253	112	248	7.20078318E+05	7.20078318E+05	0	0.47	29	1.76	1.96
qshare2b	96	162	96	162	1.17036917E+04	1.17036917E+04	0	0.25	17	2	0.41
qshell	536	1777	487	1475	1.57263685E+12	1.57263684E+12	0.01	0.98	38	1.79	1.76
qship04l	402	2166	323	2104	2.42001553E+06	2.42001553E+06	0.01	0.38	14	2	2.93
qship04s	402	1506	235	1356	2.42499367E+06	2.42499367E+06	0.01	0.31	14	2	4.26
qship08l	778	4363	638	4247	2.37604062E+06	2.37604062E+06	0.01	1.12	14	2	4.74
qship08s	778	2467	366	2079	2.38572885E+06	2.38572885E+06	0.01	0.41	14	1.93	3.31
qship12l	1151	5533	784	5226	3.01887658E+06	3.01887658E+06	0.02	2.06	16	2	4.04
qship12s	1151	2869	412	2190	3.05696225E+06	3.05696225E+06	0.01	0.73	17	2	3.13
qsierra	1227	2735	1212	2705	2.37504582E+07	2.37504582E+07	0	0.54	20	2	5.53
qstair	356	614	356	532	7.98545276E+06	7.98545275E+06	0	0.36	23	1.87	2.29
qstandat	359	1274	346	855	6.41183839E+03	6.41183839E+03	0	0.21	13	2	1.96
s268	5	10	5	10	2.58580312E-05	-5.06859105E-05	0	0.06	12	2	0
stadat1	3999	6000	3999	6000	-2.85268639E+07	-2.85268640E+07	0.01	1.03	12	2	4.5
stadat2	3999	6000	3999	6000	-3.26266649E+01	-3.26266649E+01	0.01	1.32	18	2	5.61
stadat3	7999	12000	7999	12000	-3.57794529E+01	-3.57794530E+01	0.01	2.5	19	2	7.01
tame	1	2	1	2	0.00000000E+00	-4.19289224E-10	0	0.04	3	2	0
ubh1	12000	18009	12000	17997	1.11600082E+00	1.11600082E+00	0.03	1.19	3	2	10.74
yao	2000	4002	2000	4000	1.97704256E+02	1.97704256E+02	0.01	1.11	23	1.57	4.52
zezevic2	2	4	2	4	-4.12499999E+00	-4.12500002E+00	0	0.04	5	2	0
Avg							0.01	3.44	15.92	1.92	3.03

Table 2: iOptimize performance on Maros and Mészáros's QP test problems

Problem	Before Presolving		After Presolving		Primal Objective	Dual Objective	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
	Rows	Cols	Rows	Cols							
QQ-aug2d	10001	20201	10001	20201	1.68741182E+06	1.68741175E+06	0.01	3.75	8	1.75	4.14
QQ-aug2dc	10001	20201	10001	20201	1.81836815E+06	1.81836807E+06	0.02	3.47	8	1.75	4.7
QQ-aug2dcqp	10001	20201	10001	20201	6.49813474E+06	6.49813474E+06	0.75	6.79	19	1.68	5.16
QQ-aug2dqp	10001	20201	10001	20201	6.23701203E+06	6.23701202E+06	0.73	6.7	21	1.71	5.38
QQ-aug3d	1001	3874	1001	3874	5.54067753E+02	5.54067816E+02	0.01	0.56	5	2	2.03
QQ-aug3dc	1001	3874	1001	3874	7.71262488E+02	7.71262604E+02	0.01	0.21	5	2	6.38
QQ-aug3dcqp	1001	3874	1001	3874	9.93362146E+02	9.93362144E+02	0.04	0.44	11	2	5.74
QQ-aug3dqp	1001	3874	1001	3874	6.75237669E+02	6.75237664E+02	0.03	0.43	12	2	6.31
QQ-cont-050	2402	2598	2402	2598	-4.56385143E+00	-4.56385137E+00	0.01	0.75	11	2	3.51
QQ-cont-100	9802	10198	9802	10198	-4.64439991E+00	-4.64439983E+00	0.01	5.59	12	2	2.2
QQ-cont-101	10099	10198	10099	10198	1.95527204E-01	1.95527225E-01	0.01	5.21	11	2	1.97
QQ-cont-200	39602	40398	39602	40398	-4.68487611E+00	-4.68487609E+00	0.05	49.43	13	2	1.32
QQ-cont-201	40199	40398	40199	40398	1.92483103E-01	1.92483155E-01	0.04	57.03	12	2	0.84
QQ-cont-300	90299	90598	90299	90598	1.91512063E-01	1.91512092E-01	0.1	261.3	12	2	0.43
QQ-cvxqp1-m	501	1001	501	1001	1.08751109E+06	1.08751098E+06	0.01	0.38	10	2	2.16
QQ-cvxqp1-s	51	101	51	101	1.15907177E+04	1.15907177E+04	0	0.05	8	2	4.17
QQ-cvxqp2-m	251	1001	251	1001	8.20155390E+05	8.20155375E+05	0	0.22	10	2	2.86
QQ-cvxqp2-s	26	101	26	101	8.12093991E+03	8.12093970E+03	0	0.05	9	2	0
QQ-cvxqp3-m	751	1001	751	1001	1.36282857E+06	1.36282854E+06	0.01	0.78	13	2	1.57
QQ-cvxqp3-s	76	101	76	101	1.19434322E+04	1.19434317E+04	0	0.06	9	2	0
QQ-dpklol	78	134	78	134	3.70096112E-01	3.70096109E-01	0	0.04	5	2	0
QQ-dtoc3	9999	15000	9999	14998	2.35262505E+02	2.35262481E+02	0.31	2.22	7	1.71	5.97
QQ-dual1	2	86	2	86	3.50125784E-02	3.50123964E-02	0	0.07	13	2	1.47
QQ-dual2	2	97	2	97	3.373345240E-02	3.37334725E-02	0	0.07	12	2	1.45
QQ-dual3	2	112	2	112	1.35755782E-01	1.35755743E-01	0	0.08	12	2	1.35
QQ-dual4	2	76	2	76	7.46090320E-01	7.46089988E-01	0	0.06	10	2	1.92
QQ-dualc1	216	224	216	224	6.15524425E+03	6.15524190E+03	0	0.14	21	2	3.62
QQ-dualc2	230	236	230	236	3.55130765E+03	3.55130761E+03	0	0.12	17	2	2.68
QQ-dualc5	279	286	279	286	4.27232103E+02	4.27231914E+02	0	0.07	9	2	4.76
QQ-dualc8	504	511	504	511	1.83093532E+04	1.83093473E+04	0	0.13	13	2	6.56
QQ-gehs28	9	11	9	11	9.27173702E-01	9.27173694E-01	0	0.03	5	2	0
QQ-gouldqp2	350	700	350	700	1.84274493E-04	1.84273733E-04	0	0.15	15	2	4.76
QQ-gouldqp3	350	700	350	700	2.062783287E+00	2.06278325E+00	0	0.12	10	2	4.24
QQ-hs118	18	33	18	33	6.64820445E+02	6.64820439E+02	0	0.05	11	2	0
QQ-hs21	2	4	2	4	-9.99600032E+01	-9.99600019E+01	0	0.05	10	2	0
QQ-hs268	6	11	6	11	-2.12146135E-07	-7.69541657E-07	0	0.07	15	2	0
QQ-hs35	2	5	2	5	1.11111110E-01	1.11111107E-01	0	0.03	6	2	0
QQ-hs35mod	2	5	2	5	2.49999954E-01	2.49999959E-01	0	0.06	12	2	0
QQ-hs51	4	6	4	6	4.24138339E-08	3.60237236E-08	0	0.02	5	2	4.76
QQ-hs52	4	6	4	6	5.32664757E+00	5.32664757E+00	0	0.02	5	2	0
QQ-hs53	4	6	4	6	4.09302193E+00	4.09302286E+00	0	0.03	6	2	0
QQ-hs76	4	8	4	8	-4.68181818E+00	-4.68181818E+00	0	0.03	6	2	0
QQ-ksip	1002	1022	1002	1022	5.75797941E-01	5.75797940E-01	0.01	0.29	13	2	3.42

Problem	Before Presolving Rows	Before Presolving Cols	After Presolving Rows	After Presolving Cols	Primal Objective	Dual Objective	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
QQ-laser	1001	2003	1001	2003	2.40960146E+06	2.40960145E+06	0	0.6	19	1.53	4.79
QQ-liswet1	10001	20003	10001	20003	3.61224031E+01	3.61224021E+01	0.02	4.87	21	1.95	5.63
QQ-liswet2	10001	20003	10001	20003	2.49980889E+01	2.49980663E+01	0.02	4.23	17	2	5.29
QQ-liswet3	10001	20003	10001	20003	2.50012248E+01	2.50012132E+01	0.02	3.94	17	2	6.13
QQ-liswet4	10001	20003	10001	20003	2.50001177E+01	2.50001054E+01	0.02	4.26	18	2	5.67
QQ-liswet5	10001	20003	10001	20003	2.50342522E+01	2.50342269E+01	0.02	4.44	20	2	5.95
QQ-liswet6	10001	20003	10001	20003	2.49957486E+01	2.49957422E+01	0.02	4.22	19	2	6.16
QQ-liswet7	10001	20003	10001	20003	4.98841004E+02	4.98840891E+02	0.02	5.85	22	1.73	5.4
QQ-liswet8	10001	20003	10001	20003	7.14470246E+02	7.14470012E+02	0.02	7.68	30	1.7	5.48
QQ-liswet9	10001	20003	10001	20003	1.96325241E+03	1.96325122E+03	0.02	10.43	36	1.67	5
QQ-liswet10	10001	20003	10001	20003	4.94858015E+01	4.94857829E+01	0.02	6.31	29	2	5.84
QQ-liswet11	10001	20003	10001	20003	4.95239698E+01	4.95239560E+01	0.02	5.3	23	1.96	6.63
QQ-liswet12	10001	20003	10001	20003	1.73692791E+03	1.73692742E+03	0.02	9.62	38	1.68	5.43
QQ-lotschd	8	13	8	13	2.39841591E+03	2.39841578E+03	0	0.03	9	2	0
QQ-mosarqp1	701	3201	701	3201	-9.52875445E+02	-9.52875470E+02	0	0.27	10	2	6.3
QQ-mosarqp2	601	1501	601	1501	-1.59748212E+03	-1.59748217E+03	0	0.15	9	2	5.56
QQ-powell20	10001	20001	10001	20001	5.20895810E+10	5.20895828E+10	0.02	7.8	19	1.37	3.72
QQ-primal1	86	411	86	411	-3.50129655E-02	-3.50129677E-02	0	0.1	10	2	1.04
QQ-primal2	97	746	97	746	-3.37336759E-02	-3.37336761E-02	0	0.13	9	2	3.42
QQ-primal3	112	857	112	857	-1.35755836E-01	-1.35755837E-01	0	0.26	11	2	4.96
QQ-primal4	76	1565	76	1565	-7.46090838E-01	-7.46090843E-01	0	0.22	9	2	4.46
QQ-primalc1	10	240	10	240	-6.15525075E+03	-6.15525083E+03	0	0.11	19	2	0.97
QQ-primalc2	8	239	8	239	-3.55130754E+03	-3.55130761E+03	0	0.09	17	2	0
QQ-primalc5	9	296	9	296	-4.27232136E+02	-4.27232268E+02	0	0.07	10	2	3.13
QQ-primalc8	9	529	9	529	-1.83094193E+04	-1.83094258E+04	0	0.11	12	2	1.83
QQ-q25fv47	821	1877	800	1856	1.37444358E+07	1.37444267E+07	0.02	1.74	28	1.93	3.16
QQ-qadlittle	57	139	56	138	4.80318851E+05	4.80318850E+05	0	0.05	12	2	0
QQ-qafiro	28	52	28	52	-1.59078178E+00	-1.59078188E+00	0	0.04	14	2	4.65
QQ-qbandm	306	473	278	445	1.63523409E+04	1.63523397E+04	0	0.18	21	2	2.27
QQ-qbeaconf	174	296	141	257	1.64711933E+05	1.6471173E+05	0	0.09	13	2	3.53
QQ-qbore3d	234	335	200	303	3.10020162E+03	3.10020194E+03	0	0.18	19	2	3.39
QQ-qbrandy	221	304	141	251	2.83751074E+04	2.83751029E+04	0	0.16	17	2	2.61
QQ-qcapri	272	483	262	457	6.67932279E+07	6.67932452E+07	0.01	0.3	32	1.69	3.36
QQ-qe226	224	473	223	472	2.12653431E+02	2.12653417E+02	0	0.15	17	2	1.39
QQ-qetamacr	401	817	401	814	8.67601160E+04	8.67601476E+04	0	0.59	33	2	3.97
QQ-qffiff80	525	1029	491	995	8.73147438E+05	8.73146951E+05	0	0.49	26	2	4.02
QQ-qforplan	162	493	136	467	7.456661267E+09	7.456660721E+09	0.01	0.27	23	2	3.04
QQ-qgfrdxpn	617	1161	591	1135	1.00790508E+11	1.00790507E+11	0.01	0.3	24	2	4.36
QQ-qgrow15	301	646	301	646	-1.01693609E+08	-1.01693652E+08	0	0.26	21	2	2.76
QQ-qgrow22	441	947	441	947	-1.49628886E+08	-1.49628968E+08	0	0.41	24	2	3.28
QQ-qgrow7	141	302	141	302	-4.27987148E+07	-4.27987170E+07	0	0.15	21	2	4.7
QQ-qisrael	175	317	175	317	2.53478001E+07	2.53477766E+07	0	0.28	29	1.83	2.89
QQ-qpcblend	75	115	75	115	-7.84259484E-03	-7.84259090E-03	0	0.08	16	2	5.41

Problem	Before Presolving		After Presolving		Primal		Dual		Presolve Time	Solver Time	Iters	Avg Divs	Prop Φ (%)
	Rows	Cols	Rows	Cols	Objective	Objective	Objective	Objective					
QQ-qpboe1	352	727	349	724	1.15039007E+07	1.15039019E+07	0.01	0.34	24	1.96	5.18		
QQ-qpboe2	167	306	141	280	8.17195892E+06	8.17195821E+06	0	0.25	29	1.97	4.18		
QQ-qpboe3	357	615	357	615	6.20438137E+06	6.20437660E+06	0.01	0.31	21	2	3.99		
QQ-qpilotno	976	2447	955	2248	4.72823223E+06	4.72827122E+06	0.04	1.23	30	2	3.07		
QQ-qrecipe	92	205	84	149	-2.66616053E+02	-2.66616081E+02	0	0.09	18	2	2.27		
QQ-qpctest	3	5	3	5	4.37187497E+03	4.37187483E+03	0	0.02	7	2	0		
QQ-qsc205	206	318	204	316	-5.81401694E-03	-5.81421806E-03	0	0.1	17	2	2.13		
QQ-qscagr25	472	672	472	672	2.01737737E+08	2.01737657E+08	0	0.36	29	1.66	3.13		
QQ-qscagr7	130	186	129	185	2.68659112E+07	2.68658869E+07	0	0.09	19	2	3.33		
QQ-qscfxm1	331	601	323	593	1.68826874E+07	1.68826848E+07	0	0.35	30	1.77	3.8		
QQ-qscfxm2	661	1201	646	1186	2.77761554E+07	2.77761498E+07	0	0.69	36	1.81	4.15		
QQ-qscfxm3	991	1801	969	1779	3.08163509E+07	3.08163491E+07	0	0.97	37	1.84	4.3		
QQ-qscfxpmo	389	467	373	456	1.88050938E+03	1.88050937E+03	0	0.16	16	2	2.65		
QQ-qscsc8	491	1276	491	1276	9.04559788E+02	9.04559689E+02	0	0.38	24	2	2.96		
QQ-qscscd1	78	761	78	761	8.66666691E+00	8.66666672E+00	0	0.12	11	2	3.6		
QQ-qscscd6	148	1351	148	1351	5.08082149E+01	5.08082137E+01	0	0.2	15	2	6.91		
QQ-qscscd8	398	2751	398	2751	9.40763587E+02	9.40763560E+02	0	0.33	12	2	4.03		
QQ-qscscap1	301	661	301	661	1.41586111E+03	1.41586111E+03	0	0.19	20	2	3.17		
QQ-qscscap2	1091	2501	1091	2501	1.73502650E+03	1.73502649E+03	0	0.38	13	2	4.79		
QQ-qscscap3	1481	3341	1481	3341	1.43875468E+03	1.43875468E+03	0	0.52	14	2	4.59		
QQ-qscscapb	516	1037	516	1037	8.14817931E+07	8.14817942E+07	0.02	0.43	28	1.86	4.47		
QQ-qshare1b	118	254	113	249	7.20078322E+05	7.20078214E+05	0	0.15	26	2	1.4		
QQ-qshare2b	97	163	97	163	1.17036812E+04	1.17036812E+04	0	0.09	19	2	1.15		
QQ-qshell	537	1778	537	1775	1.57263633E+12	1.57263551E+12	0.02	1.01	30	2	4.25		
QQ-qship041	403	2167	357	2163	2.42001501E+06	2.42001490E+06	0	0.32	15	2	3.27		
QQ-qship04s	403	1507	269	1415	2.42499308E+06	2.42499280E+06	0	0.22	14	2	2.9		
QQ-qship081	779	4364	695	4346	2.37604021E+06	2.37604015E+06	0.02	1.53	15	2	3.9		
QQ-qship08s	779	2468	487	2242	2.38572832E+06	2.38572819E+06	0.01	0.59	15	2	5.05		
QQ-qship121	1152	5534	921	5412	3.01887662E+06	3.01887624E+06	0.03	3.32	19	2	2.88		
QQ-qship12s	1152	2870	688	2515	3.05696198E+06	3.05696007E+06	0.01	0.84	19	2	4.73		
QQ-qstair	1228	2736	1228	2721	2.37504456E+07	2.37504474E+07	0	0.64	20	2	5.06		
QQ-qstair	357	615	357	546	7.98544934E+06	7.98544810E+06	0.01	0.33	24	1.92	2.51		
QQ-qstandat	360	1275	359	1192	6.41183858E+03	6.41183826E+03	0.01	0.29	17	2	4.2		
QQ-s268	6	11	6	11	-2.12146135E-07	-7.69541657E-07	0	0.04	15	2	0		
QQ-stadat1	4000	6001	4000	6001	-2.85268640E+07	-2.85268640E+07	0.01	1.7	19	1.53	5.02		
QQ-stadat2	4000	6001	4000	6001	-3.26266629E+01	-3.26266666E+01	0.01	1.01	18	2	7.01		
QQ-stadat3	8000	12001	8000	12001	-3.57794533E+01	-3.57794533E+01	0.01	2.18	18	2	6.65		
QQ-tame	2	3	2	3	-3.27685749E-10	-6.53153282E-10	0	0.02	6	2	0		
QQ-ubh1	12001	18010	12001	17998	1.11600080E+00	1.11600080E+00	0.47	1.46	5	2	11.09		
QQ-yao	2001	4003	2001	4003	1.97704256E+02	1.97704256E+02	0.02	1.01	23	1.87	5.14		
QQ-zecevic2	3	5	3	5	-4.12500000E+00	-4.12500003E+00	0	0.02	7	2	0		
Avg							0.03	4.08	16.49	1.95	3.37		

Table 3: iOptimize performance on Mittelmann's QCQP test problems

Problem	Before Presolving		After Presolving		Primal Objective		Dual Objective		Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
	Rows	Cols	Rows	Cols	Objective	Objective	Time	Time					
airport	42	84	42	84	4.79527017E+04	4.79526915E+04	0	0	0.17	28	1.29	1.86	
polak4	3	3	3	3	2.43646932E-08	1.65838221E-07	0	0	0.04	13	2	0	
makela1	2	4	2	4	-1.41421356E+00	-1.41421356E+00	0	0	0.03	8	2	4.35	
makela2	3	3	3	3	7.20000270E+00	7.20000353E+00	0	0	0.02	7	2	5.26	
makela3	20	21	20	21	2.31888348E-11	0.00000000E+00	0	0	0.03	8	2	0	
gigomez1	3	5	3	5	-2.99999998E+00	-2.99999992E+00	0	0	0.03	9	1.67	0	
hanging	180	288	180	288	-6.20176047E+02	-6.20176047E+02	0	0	0.16	10	2	3.45	
madsschj	158	81	158	81	-7.97283702E+02	-7.97283702E+02	0	0	0.14	11	1.64	2.27	
rosemnx	4	5	4	5	-4.39999956E+01	-4.39999945E+01	0	0	0.03	8	2	0	
optward	2	8	2	8	-1.09999913E+00	-1.09999913E+00	0	0	0.03	10	2	0	
optprloc	30	35	30	35	-1.64197690E+01	-1.64197857E+01	0	0	0.06	16	1.81	0	
Avg							0.00	0.07	11.64	1.86	1.56		

Table 4: iOptimize performance on Cute QCQP test problems

Problem	Before Presolving Rows	After Presolving Cols	Primal Objective	Dual Objective	Presolve Time	Solver Time	Iters	Avg Dns	Prop Φ (%)		
bigbank	814	1773	814	1773	-4.20569613E+06	-4.20566623E+06	0.01	5.89	21	1.67	6.53
gridnet6	3844	7565	3844	7565	2.06554692E+02	2.06554667E+02	0.02	2.76	7	2	8.87
gridnetf	3844	7565	3844	7565	2.42109044E+02	2.421089909E+02	0.02	24.67	13	2	8.05
gridmeth	36	61	36	61	3.96262686E+01	3.96262678E+01	0	0.09	6	2	1.16
gridneti	36	61	36	61	4.02474662E+01	4.02474593E+01	0.01	0.08	8	2	0
dallas1	598	837	598	837	-2.02604132E+05	-2.02605958E+05	0	1.93	33	1.7	6.72
dallas2	119	164	119	164	-4.81981888E+04	-4.82005984E+04	0.01	0.59	31	1.68	2.41
dallas3	29	44	29	44	-3.23932257E+04	-3.23932491E+04	0.01	0.49	31	1.65	1.44
polak1	2	3	2	3	2.71828183E+00	2.71828183E+00	0.01	0.04	5	2	0
polak2	2	11	2	11	5.45981500E+01	5.45981501E+01	0	0.04	5	2	2.7
polak3	10	12	10	12	5.93300334E+00	5.93300332E+00	0	0.09	8	2	2.35
polak5	2	3	2	3	4.99999999E+01	5.00000001E+01	0	0.05	5	2	4.55
smbank	64	117	64	117	-7.12929192E+06	-7.12929437E+06	0	0.49	27	1.3	1.86
cantlivr	1	5	1	5	1.33995627E+00	1.33995688E+00	0	0.16	12	2	0.65
cb2	3	3	3	3	1.95222449E+00	1.95222448E+00	0	0.08	8	2	0
cb3	3	3	3	3	2.00000001E+00	2.00000001E+00	0.01	0.07	7	2	0
chacomm1	3	3	3	3	1.95222449E+00	1.95222448E+00	0	0.08	8	2	0
chacomm2	3	3	3	3	2.00000001E+00	2.00000001E+00	0	0.07	7	2	0
dipigri	4	7	4	7	6.80630123E+02	6.80629688E+02	0	0.1	10	2	0
gpp	498	499	498	499	1.44009283E+04	1.44009270E+04	0.03	2.3	10	2	7.93
hong	1	4	1	4	1.34730668E+00	1.34730660E+00	0	0.15	15	2	0
loadbal	31	51	31	51	4.52851049E-01	4.52850913E-01	0	0.19	21	1.9	0.53
svanberg	5000	5000	5000	5000	8.36142276E+03	8.36142275E+03	0.1	99.69	14	2	11.59
antenna\antenna2	166	49	166	49	1.57986028E+00	1.57986028E+00	0.01	8.62	16	1.94	8.09
antenna\antenna_vareps	166	50	166	50	1.22309383E+00	1.22309383E+00	0.01	11.65	18	1.78	7.19
braess\trafequill	361	722	361	722	5.26883710E+01	5.2688376E+01	0.01	1.06	16	2	1.1
braess\trafequill2	437	798	437	798	5.26883638E+01	5.26883618E+01	0	1.66	13	2	7.98
braess\trafequillf	309	856	309	856	5.15910514E+01	5.15910257E+01	0	1.62	15	2	0.66
braess\trafequill2sf	385	932	385	932	5.15910455E+01	5.15910368E+01	0.01	2.1	12	2	7.69
elena\chemeq	12	38	12	38	-1.91087044E+03	-1.91087044E+03	0.01	0.28	20	1.5	0.73
elena\s383	1	14	1	14	7.28593654E+05	7.28593512E+05	0	0.24	21	1.33	0.85
firfilter\fir-convex	243	163	243	163	1.04648765E+00	1.04648766E+00	0	0.65	13	2	7.41
markowitz\growthopt	1	8	1	8	-1.17084820E-01	-1.17084818E-01	0	0.06	7	2	0
wbv\antenna2	1197	1180	1197	1180	1.09359976E+00	1.09359979E+00	0	80.82	13	2	9.25
wbv\lowpass2	200	219	200	219	1.05908595E+00	1.05908552E+00	0	7.42	22	2	6.96
batch	73	106	73	106	2.59180316E+05	2.59134654E+05	0	0.11	12	2	0.97
stockcycle	97	481	97	481	1.17916252E+05	1.17915998E+05	0	0.17	10	2	3.66
synthes1	6	10	6	10	7.59284810E-01	7.59283839E-01	0	0.04	6	2	0
synthes2	14	21	14	21	-5.54416888E-01	-5.54416891E-01	0	0.05	8	2	2.04
synthes3	23	34	23	34	1.50821846E+01	1.50821840E+01	0	0.05	8	2	3.85
trimloss2	24	53	24	53	7.18307944E-01	7.18309570E-01	0	0.07	10	2	4.11
trimloss4	64	145	64	145	1.70933112E+00	1.70933131E+00	0	0.12	12	2	1.77
trimloss5	90	216	90	216	1.17886860E+00	1.17886902E+00	0	0.21	13	2	7.39
trimloss6	120	287	120	287	1.30565472E+00	1.30565507E+00	0	0.26	12	2	7.39
trimloss7	154	436	154	436	5.93496892E-01	5.93497829E-01	0	0.57	13	2	8.08
trimloss12	384	1028	384	1028	2.31187177E+00	2.31187184E+00	0	6.4	17	2	9
Avg							0.01	5.75	13.46	1.92	3.77

Table 5: iOptimize performance on CUTE CP test problems

Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize
aug2d	6	12	laser	12	10	qpcboei1	20	22
aug2dc	6	12	liswet1	26	19	qpcboei2	21	26
aug2dcqp	13	13	liswet2	21	17	qpcstair	22	25
aug2dqp	15	14	liswet3	21	16	qpilotho	35§	42
aug3d	6	8	liswet4	26§	18	qrecipe	16	19
aug3dc	6	8	liswet5	25	16	qptest	6	6
aug3dcqp	11	11	liswet6	24	18	qsc205	16	17
aug3dqp	12	13	liswet7	31	19	qscagr25	20	22
cont-050	11	11	liswet8	38	37	qscagr7	19	22
cont-100	12	11	liswet9	37	31	qscfxm1	25	25
cont-101	13	9	liswet10	36	21	qscfxm2	30	33
cont-200	12	11	liswet11	31	22	qscfxm3	30	35
cont-201	15	10	liswet12	30	32	qscorp10	13	14
cont-300	16	13	lotschd	10	9	qscrs8	21	24
cvxqp1_m	10	10	mosarqp1	9	9	qscsd1	10	9
cvxqp1_s	8	8	mosarqp2	9	9	qscsd6	13	14
cvxqp2_m	10	10	powell20	10§	8	qscsd8	12	11
cvxqp2_s	9	10	primal1	10	9	qscstap1	18	17
cvxqp3_m	12	12	primal2	8	7	qscstap2	13	13
cvxqp3_s	10	10	primal3	9	9	qscstap3	14	15
dpk101	5	3	primal4	10	9	qseba	16	26
dtoc3	5	4	primalc1	18	18	qshare1b	23	29
dual1	14	12	primalc2	15	25	qshare2b	17	17
dual2	12	9	primalc5	8	10	qshell	33	38
dual3	12	9	primalc8	10	12	qship041	14	14
dual4	12	10	q25fv47	28	27	qship04s	13	14
dualc1	10	25	qadlittle	11	11	qship081	13	14
dualc2	10	20	qafro	12	14	qship08s	13	14
dualc5	7	13	qbandm	19	20	qship121	17	16
dualc8	7	19	qbeaconf	14	14	qship12s	17	17
genhs28	4	3	qbre3d	13	19	qsierra	18	20
gouldqp2	14	13	qbrandy	14	17	qstair	22	23
gouldqp3	10	11	qcapri	26	36	qstandat	15	13
hs118	10	10	qe226	14	16	s268	22	12
hs21	9	9	qetamacr	27	32	stadat1	25	12
hs268	22	12	qffff80	22	25	stadat2	41	18
hs35	5	5	qforplan	23	31	stadat3	46	19
hs35mod	11	11	qgfrdxpn	23	30	tame	3	3
hs51	5	3	qgrow15	22	22	ubh1	6	3
hs52	4	3	qgrow22	25	26	yao	23	23
hs53	6	4	qgrow7	23	21	zezevic2	5	5
hs76	5	5	qisrael	21	28			
ksip	21	16	qpcblend	19	17	Avg	16.22	15.92

¹ §: Near optimal solution found.

Table 6: Comparison with MOSEK on Maros and Mészáros's QP test problems

Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize
QQ-aug2d1	11		QQ-laser	135		QQ-qpchoeil	24	24
QQ-aug2dc	9		QQ-liswet1	41?		QQ-qpchoe2	29	29
QQ-aug2dcp	18		QQ-liswet2	21		QQ-qpchoe3	24	21
QQ-aug2dqp	†	21	QQ-liswet3	17		QQ-qpchoe4	39	30
QQ-aug3d	6		QQ-liswet4	17		QQ-qpchoe5	16	18
QQ-aug3dc	7		QQ-liswet5	15		QQ-qptest	7	7
QQ-aug3dcp	17		QQ-liswet6	17		QQ-qpsc205	17	17
QQ-aug3dqp	22		QQ-liswet7	34?		QQ-qscagr25	20	29
QQ-aug3dqp	14		QQ-liswet8	49?		QQ-qscagr7	18	19
QQ-cont-050	14		QQ-liswet9	26?		QQ-qscfxm1	43	30
QQ-cont-100	15		QQ-liswet10	18?		QQ-qscfxm2	39	36
QQ-cont-101	12		QQ-liswet11	27?		QQ-qscfxm3	26	37
QQ-cont-200	19		QQ-liswet12	24?		QQ-qscorp10	17	16
QQ-cont-201	14		QQ-lotschd	15		QQ-qscrs8	24	24
QQ-cont-300	15		QQ-mosarqp1	15		QQ-qscsd1	13	11
QQ-cvxqpl_m	15		QQ-mosarqp2	14		QQ-qscsd6	17	15
QQ-cvxqpl_s	15		QQ-powell20	†		QQ-qscsd8	16	12
QQ-cvxqpl2_m	18		QQ-primal1	11		QQ-qscsd1	17	15
QQ-cvxqpl2_s	12		QQ-primal2	9		QQ-qscsd6	16	12
QQ-cvxqpl3_m	18		QQ-primal3	13		QQ-qscsd8	16	12
QQ-cvxqpl3_s	14		QQ-primal4	9		QQ-qscsd1	18	13
QQ-dpk101	6		QQ-primac1	11		QQ-qscsd6	18	14
QQ-dtocc3	11		QQ-primac2	20		QQ-qscsd1	62	28
QQ-dual1	14		QQ-primac5	18		QQ-qshare1b	27	26
QQ-dual2	13		QQ-primac8	11		QQ-qshare2b	19	19
QQ-dual3	14		QQ-q25fv47	15		QQ-qshell	26	30
QQ-dual4	13		QQ-qadlittle	39		QQ-qship041	21	15
QQ-dualc1	17		QQ-qafiro	19		QQ-qship04s	18	14
QQ-dualc2	14		QQ-qbandm	16		QQ-qship081	20	15
QQ-dualc5	11		QQ-qbeaconf	21		QQ-qship121	21	15
QQ-dualc8	18		QQ-qchore3d	12		QQ-qship12s	24	19
QQ-genhs28	5		QQ-qcappri	14		QQ-qsierra	24	20
QQ-gouldqp2	13		QQ-qetamacr	17		QQ-qstair	31	24
QQ-gouldqp3	13		QQ-qfifff80	26		QQ-qstandat	16	17
QQ-hs118	11		QQ-qforplan	17		QQ-qstadar2	13	15
QQ-hs21	13		QQ-qgfrdxpn	24		QQ-qstadar3	18	18
QQ-hs268	13		QQ-qgrow15	30		QQ-qstadar1	53	19
QQ-hs35	6		QQ-qgrow22	1808		QQ-qstadar2	18	18
QQ-hs35mod	13		QQ-qgrow7	25		QQ-tame	5	6
QQ-hs51	5		QQ-qisrael	21		QQ-ubh1	20	5
QQ-hs52	6		QQ-qpblend	28		QQ-yao	278	23
QQ-hs53	10			21		QQ-zecevic2	6	7
QQ-hs76	8			28		Avg1	20.91	16.43
QQ-ksip	15			20		Avg2	18.34	16.26

†: Max number of iterations reached.
 ‡: Near optimal solution found.
 †?: Terminated with unknown status.

Table 7: Comparison with MOSEK on Mittelmann's QCQP test problems

Problem	MOSEK	iOptimize
airport	26	28
polak4	10	13
makela1	8	8
makela2	7	7
makela3	6	8
gigomez1	8	9
hanging	13	10
madsschj	10	11
rosenmx	9	8
optreward	10	10
optprloc	18	16
Avg	11.36	11.64

Table 8 Comparison with MOSEK on Cute QCQP test problems

Problem	MOSEK	iOptimize	Ipopt	Knitro
bigbank	20	21	24	19
gridnetf	9	7	5	4
gridnetf	16	13	27	17
gridneth	8	6	6	3
gridneti	10	8	7	6
dallasl	74	33	249	245
dallasm	128	31	‡	‡
dallass	†	31	†	135
polak1	6	5	4	2
polak2	6	5	4	1
polak3	10	8	‡	‡
polak5	6	5	24	12
sembank	14	27	12	11
cantilvr	12	12	5	3
cb2	8	8	4	3
cb3	7	7	4	4
chaconn1	8	8	4	3
chaconn2	7	7	4	4
dipigri	14	10	6	2
gpp	12	10	22	11
hong	13	15	11	7
loadbal	22	21	13	7
svanberg	14	14	30	16
<hr/>				
antenna\antenna2	21§	16	178	58
antenna\antenna_vareps	19	18	106	61
braess\trafequil	15	16	18	22
braess\trafequil2	11	13	26	13
braess\trafequilsf	14	15	18	24
braess\trafequil2sf	12	12	25	12
elena\chemeq	23	20	37	16
elena\s383	85	21	18	8
firfilter\fir_convex	33§	13	27	11
markowitz\growthopt	8	7	8	7
wbv\antenna2	15	13	31	16
wbv\lowpass2	30§	22	31	66
<hr/>				
batch	13	12	13	16
stockcycle	11	10	23	12
synthes1	6	6	6	6
synthes2	8	8	10	7
synthes3	8	8	9	8
trimloss2	10	10	13	7
trimloss4	11	12	16	8
trimloss5	13	13	20	10
trimloss6	12	12	23	11
trimloss7	13	13	35	14
trimloss12	19	17	41	22
<hr/>				
Avg1	18.53	13.07	–	–
Avg2	13.02	11.98	–	–

¹ †: Max number of iterations reached.

² §: Near optimal solution found.

³ ‡: Function evaluate error from AMPL.

Table 9 Comparison with MOSEK, Ipopt, and Knitro on CP test problems

Problem	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
iQQ-aug2d	0.01	7.14	16	1.69	4.78
iQQ-aug2dc	0.01	6.8	16	1.69	4.34
iQQ-aug2dcqp	0.78	13.44	29	1.34	4.5
iQQ-aug2dqp	0.73	13.4	29	1.34	4.44
iQQ-aug3d	0	4.72	35	1.09	2.43
iQQ-aug3dc	0	1.06	16	1.5	5.13
iQQ-aug3dcqp	0.04	2.59	33	1.09	4.1
iQQ-aug3dqp	0.03	2.68	37	1.05	4.4
iQQ-cont-050	0	0.76	9	2	4.72
iQQ-cont-100	0.01	6.7	14	1.93	2.17
iQQ-cont-101	0	3.33	9	2	2.35
iQQ-cont-200	0.03	47.74	14	2	1.71
iQQ-cont-201	0.03	26.08	9	2	2.09
iQQ-cont-300	0.05	104.79	9	2	1.57
iQQ-cvxqp1_m	0	2.19	26	1.38	1.38
iQQ-cvxqp1_s	0	2.24	35	1.2	0.45
iQQ-cvxqp2_m	0	2.14	29	1.34	0.9
iQQ-cvxqp2_s	0	2.41	37	1.19	0.92
iQQ-cvxqp3_m	0	2.85	30	1.23	1.8
iQQ-cvxqp3_s	0	1.22	35	1.17	0.98
iQQ-dpklo1	0	0.36	13	2	0.83
iQQ-dtoc3	0.32	4.21	15	1.87	5.9
iQQ-dual1	0	1.14	31	1.32	0.88
iQQ-dual2	0	0.52	16	1.69	0
iQQ-dual3	0	0.63	18	1.67	1.61
iQQ-dual4	0	1.23	33	1.33	0
iQQ-dualc1	0	1.06	35	1.66	0
iQQ-dualc2	0	1.19	35	1.57	0.59
iQQ-dualc5	0	1.65	44	1.32	1.04
iQQ-dualc8	0	1.06	34	1.68	1.9
iQQ-genhs28	0	0.37	13	2	0
iQQ-gouldqp2	0	0.27	10	2	0
iQQ-gouldqp3	0	0.56	16	1.63	1.82
iQQ-hs118	0	0.27	11	2	0
iQQ-hs21	0.02	1.2	35	1.11	0.08
iQQ-hs268	0	1.52	45	1.11	0
iQQ-hs35	0	0.52	16	1.75	0
iQQ-hs35mod	0	0.54	16	1.75	0.19
iQQ-hs51	0	1.07	27	1.19	0
iQQ-hs52	0	0.73	26	1.31	0
iQQ-hs53	0	1.41	39	1.13	0
iQQ-hs76	0	0.36	13	2	0
iQQ-ksip	0	1.73	42	1.05	3.3

Problem	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
iQQ-laser	0	0.4	13	2	5.22
iQQ-liswet1	0.01	16.43	42	1.14	4.19
iQQ-liswet2	0.02	17.23	43	1.12	4.9
iQQ-liswet3	0.01	15	38	1.16	5.15
iQQ-liswet4	0.01	15.72	40	1.13	5.16
iQQ-liswet5	0.01	18.09	47	1.17	4.49
iQQ-liswet6	0.02	16.79	43	1.14	4.64
iQQ-liswet7	0.01	15.81	40	1.13	5.12
iQQ-liswet8	0.01	14.03	33	1.15	4.37
iQQ-liswet9	0.01	15.69	37	1.14	4.69
iQQ-liswet10	0.02	16.5	42	1.14	4.97
iQQ-liswet11	0.01	16.33	42	1.14	4.9
iQQ-liswet12	0.01	15.73	38	1.13	4.62
iQQ-lotschd	0	0.3	12	2	0
iQQ-mosarqp1	0	2.04	33	1.09	4.55
iQQ-mosarqp2	0	1.41	38	1.08	3.57
iQQ-powell20	0.02	6.5	16	1.63	3.5
iQQ-primal1	0.01	1.28	43	1.09	1.41
iQQ-primal2	0	1.31	37	1.14	2.47
iQQ-primal3	0	1.58	41	1.1	2.24
iQQ-primal4	0.01	1.56	36	1.17	4.44
iQQ-primalc1	0	1.37	42	1.26	1.03
iQQ-primalc2	0	1.32	36	1.33	1.52
iQQ-primalc5	0	0.97	23	1.39	0.93
iQQ-primalc8	0	1.1	32	1.28	1.01
iQQ-q25fv47	0.01	1.4	15	2	6.42
iQQ-qadlittle	0	0.34	14	2	0.59
iQQ-qafiro	0	1.37	39	1.15	0.07
iQQ-qbandm	0	0.34	14	1.86	3.02
iQQ-qbeaconf	0	0.41	14	1.93	0.5
iQQ-qbore3d	0	0.51	18	1.89	0
iQQ-qbrandy	0	0.48	16	1.88	0.41
iQQ-qcapri	0	0.62	22	1.91	1.63
iQQ-qe226	0	1.41	36	1.22	1.85
iQQ-qetamacr	0	1.09	34	1.21	3.17
iQQ-qffff80	0	0.57	20	1.8	3.57
iQQ-qforplan	0.01	0.25	11	2	1.63
iQQ-qgfrdxpn	0.01	0.47	19	2	2.42
iQQ-qgrow15	0	0.44	16	1.94	1.15
iQQ-qgrow22	0.01	0.54	18	1.89	0
iQQ-qgrow7	0	0.56	18	1.89	0.54
iQQ-qisrael	0	0.47	19	2	1.3
iQQ-qpcblend	0	1.5	38	1.16	0.27

Problem	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
iQQ-qpcboei1	0.01	0.31	12	2	6.06
iQQ-qpcboei2	0	0.44	17	1.94	4.65
iQQ-qpcstair	0.01	0.64	20	1.75	1.27
iQQ-qpilotno	0.04	0.65	16	2	3.84
iQQ-qrecipe	0	0.75	20	1.6	0.13
iQQ-qpctest	0	0.56	16	1.81	0.18
iQQ-qsc205	0	1.51	41	1.24	1.35
iQQ-qscagr25	0	0.32	14	2	0.32
iQQ-qscagr7	0	0.33	14	2	1.22
iQQ-qscfxm1	0	0.4	15	1.93	1.79
iQQ-qscfxm2	0	0.42	16	1.94	2.44
iQQ-qscfxm3	0	0.54	16	1.94	4.71
iQQ-qscorpio	0	0.95	25	1.24	1.91
iQQ-qscrs8	0	0.63	22	1.73	6.45
iQQ-qscsd1	0	0.84	20	1.6	1.31
iQQ-qscsd6	0	0.89	22	1.64	2.07
iQQ-qscsd8	0	2.02	37	1.11	5.11
iQQ-qscstap1	0	1.07	36	1.19	1.14
iQQ-qscstap2	0	2.26	40	1.13	3.58
iQQ-qscstap3	0	2.26	32	1.22	3.65
iQQ-qseba	0.02	0.57	20	1.85	5.36
iQQ-qshare1b	0	0.44	17	1.94	2.33
iQQ-qshare2b	0	0.47	13	1.69	2.13
iQQ-qshell	0.02	0.82	17	2	6.4
iQQ-qship04l	0	0.31	11	2	0
iQQ-qship04s	0.01	0.25	11	2	6.88
iQQ-qship08l	0.01	1.63	16	1.94	4.56
iQQ-qship08s	0	0.72	17	1.88	3.73
iQQ-qship12l	0.01	2.96	17	1.71	4.74
iQQ-qship12s	0.01	0.99	18	1.83	2.98
iQQ-qsierra	0	0.64	17	2	5
iQQ-qstair	0	0.39	17	2	1.05
iQQ-qstandat	0.01	1.37	39	1.18	2.05
iQQ-s268	0	1.56	45	1.11	0.06
iQQ-stadat1	0.01	3.13	26	1.23	4.19
iQQ-stadat2	0	4.39	35	1.14	4.42
iQQ-stadat3	0.01	6.18	25	1.2	4.25
iQQ-tame	0	0.23	10	2	0
iQQ-ubh1	0.46	5.73	26	1.96	6.21
iQQ-yao	0.01	3.17	38	1.05	4.85
iQQ-zecevic2	0	0.23	9	2	0
Avg	0.02	4.40	25.30	1.56	2.48

Table 10 iOptimize performance on Maros and Mészáros’s QCQP infeasible test problems

Problem	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
iairport	0.01	0.6	19	1.63	0.67
ipolak4	0	0.81	25	1.28	0.12
imakela1	0	1.11	32	1.13	0
imakela2	0	1.13	32	1.13	0
imakela3	0	0.32	14	2	0
igigomez1	0	1.02	26	1.19	0
ihanging	0	1.17	30	1.13	4.27
imadsschj	0.06	0.53	17	1.59	0.76
irosemmx	0	1	35	1.14	0
ioptreward	0	0.64	20	1.55	0
ioptprloc	0	0.92	21	1.52	0.11
Avg	0.01	0.84	24.64	1.39	0.54

Table 11 iOptimize performance on CUTE QCQP infeasible test problems

Problem	Presolve Time	Solver Time	Iters	Avg Dirs	Prop Φ (%)
ibigbank	0	17.12	27	1.33	6.06
igrinete	0	81.51	14	1.71	6.79
igrinetf	0	377.58	36	1.08	5.67
igrineth	0.02	0.78	23	1.39	0.26
igrineti	0	1.23	40	1.07	1.15
idallas [‡]					
idallasm	0.01	1.96	47	1.4	2.26
idallass	0	1.58	45	1.22	0.51
ipolak1	0	0.41	16	1.81	0
ipolak2	0	0.32	13	2	0
ipolak3	0	0.77	27	1.3	0.52
ipolak5	0	0.41	13	2	0.25
ismbank	0	1.13	31	1.13	1.7
icantilvr	0	0.29	12	2	0.35
icb2	0	0.94	32	1.19	0
icb3	0.01	1.25	31	1.16	0.24
ichaconn1	0	0.95	32	1.19	0
ichaconn2	0	1.26	31	1.16	0.08
idipigri	0	1.29	32	1.19	0.08
igpp	0	34.24	38	1.11	5.34
ihong	0	0.92	22	1.45	0.11
iloadbal	0	0.37	14	2	0.27
isvanberg	0.08	632.99	27	1.22	6.65
<hr/>					
antenna-iantenna2	0	68.79	186	1.65	13.31
antenna-iantenna_vareps	0	48.02	47	1.13	5.83
braess-itrafеquїl	0	4.77	38	1.37	3.1
braess-itrafеquїl2	0	1.87	9	2	7.34
braess-itrafеquїl2sf	0	9.23	46	1.26	2.71
braess-itrafеquїl2sf	0	2.56	9	2	7.16
elena-ichemeq	0	0.06	14	2	4.92
elena-is383	0	0.06	14	1.93	1.64
fırfilter-ifır_convex	0	0.75	13	1.92	6.03
markowitz-ıgrowthopt	0	0.08	16	1.69	0
wbv-ıantenna2	0.06	119.88	17	2	7.97
wbv-ılowpass2	0	4.54	13	1.92	7.35
<hr/>					
ibatch	0	0.15	22	1.36	3.47
ıstockcycle	0	0.88	35	1.14	6.29
ısynthes1	0	0.09	25	1.36	0
ısynthes2	0	0.09	23	1.35	1.15
ısynthes3	0	0.18	40	1.1	3.91
ıtrimloss2	0	0.1	15	1.93	0
ıtrimloss4	0	0.15	16	1.94	7.43
ıtrimloss5	0	0.22	15	2	6.07
ıtrimloss6	0	0.37	15	2	7.32
ıtrimloss7	0	0.84	16	2	6.58
ıtrimloss12	0.01	7.79	16	2	7.41
<hr/>					
Avg	0.00	31.79	28.07	1.56	3.45

[‡] ‡: Function evaluate error from AMPL.

Table 12 iOptimize performance on Cute CP infeasible test problems

Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize	Problem	MOSEK	iOptimize
iQQ-aug2d	†	16	iQQ-laser	29	13	iQQ-qpcboei1	†	12
iQQ-aug2dc	†	16	iQQ-liswet1	31	42	iQQ-qpcboei2	†	17
iQQ-aug2dcqp	†	29	iQQ-liswet2	30	43	iQQ-qpcstair	†	20
iQQ-aug2dqp	24	29	iQQ-liswet3	30	38	iQQ-qpilotno	†	16
iQQ-aug3d	21	35	iQQ-liswet4	30	40	iQQ-qrecipe	12	20
iQQ-aug3dc	16	16	iQQ-liswet5	21	47	iQQ-qptest	15	16
iQQ-aug3dcqp	121	33	iQQ-liswet6	30	43	iQQ-qsc205	16	41
iQQ-aug3dqp	95	37	iQQ-liswet7	24	40	iQQ-qscagr25	383?	14
iQQ-cont-050	†	9	iQQ-liswet8	26	33	iQQ-qscagr7	38	14
iQQ-cont-100	†	14	iQQ-liswet9	24	37	iQQ-qscfxm1	†	15
iQQ-cont-101	†	9	iQQ-liswet10	30	42	iQQ-qscfxm2	†	16
iQQ-cont-200	†	14	iQQ-liswet11	30	42	iQQ-qscfxm3	†	16
iQQ-cont-201	70	9	iQQ-liswet12	24	38	iQQ-qscorpio	31	25
iQQ-cont-300	†	9	iQQ-lotschd	26	12	iQQ-qscrs8	52	22
iQQ-cvxqp1_m	†	26	iQQ-mosarqp1	100	33	iQQ-qscsd1	28	20
iQQ-cvxqp1_s	21	35	iQQ-mosarqp2	66	38	iQQ-qscsd6	25	22
iQQ-cvxqp2_m	97	29	iQQ-powell20	†	16	iQQ-qscsd8	30	37
iQQ-cvxqp2_s	18	37	iQQ-primal1	31	43	iQQ-qscstap1	38	36
iQQ-cvxqp3_m	†	30	iQQ-primal2	32	37	iQQ-qscstap2	†	40
iQQ-cvxqp3_s	22	35	iQQ-primal3	33	41	iQQ-qscstap3	87	32
iQQ-dpktol	†	13	iQQ-primal4	34	36	iQQ-qshare1b	28	20
iQQ-dtocc3	24	15	iQQ-primalc1	36	42	iQQ-qshare2b	51	17
iQQ-dual1	10	31	iQQ-primalc2	19	36	iQQ-qshell	175	13
iQQ-dual2	9	16	iQQ-primalc5	22	23	iQQ-qship	20	17
iQQ-dual3	9	18	iQQ-primalc8	24	32	iQQ-qship04l	15	11
iQQ-dual4	9	33	iQQ-q25fv47	†	15	iQQ-qship04s	15	11
iQQ-dualc1	18	35	iQQ-qadlittle	†	14	iQQ-qship08l	†	16
iQQ-dualc2	15	35	iQQ-qafiro	70	39	iQQ-qship08s	†	17
iQQ-dualc5	10	44	iQQ-qbandm	37	14	iQQ-qship12l	†	17
iQQ-dualc8	30	34	iQQ-qbeaconf	26	14	iQQ-qship12s	†	18
iQQ-genhs28	†	13	iQQ-qbore3d	10	18	iQQ-qstair	†	17
iQQ-gouldqp2	10	10	iQQ-qbrandy	†	16	iQQ-qstandat	353?	17
iQQ-gouldqp3	14	16	iQQ-qcapri	†	22	iQQ-s268	340	39
iQQ-hs118	232	11	iQQ-qe226	281	36	iQQ-stadat1	26	45
iQQ-hs21	40	35	iQQ-qetamacr	†	34	iQQ-stadat2	16	26
iQQ-hs268	26	45	iQQ-qfffff80	†	20	iQQ-stadat3	†	35
iQQ-hs35	72	16	iQQ-qforplan	21	11	iQQ-stadat3	†	25
iQQ-hs35mod	25	16	iQQ-qgfrdxpn	21	19	iQQ-tame	6	10
iQQ-hs51	29	27	iQQ-qgrow15	22	16	iQQ-ubh1	†	26
iQQ-hs52	22	26	iQQ-qgrow22	25	18	iQQ-yao	27	38
iQQ-hs53	27	39	iQQ-qgrow7	22	18	iQQ-zecevic2	8	9
iQQ-hs76	69	13	iQQ-qisrael	†	19	Avg1	42.20	28.15
iQQ-ksip	30	42	iQQ-qpcblend	105	38	Avg2	27.39	28.15

¹ †: Max number of iterations reached.

² ?: Terminated with unknown status.

Table 13: Comparison with MOSEK on Maros and Mészáros's QCQP infeasible test problems

Problem	MOSEK	iOptimize
iairport	10	19
ipolak4	†	25
imakela1	28	32
imakela2	29	32
imakela3	8	14
igomez1	26	26
ihanging	32	30
imadsschj	24	17
irosemmx	27	35
ioptreward	11	20
ioptprloc	8	21
Avg	20.30	24.60

¹ †: Max number of iterations reached.

Table 14 Comparison with MOSEK on Cute QCQP infeasible test problems

Problem	MOSEK	iOptimize	Ipopt	Knitro
ibigbank	196	27	48	42
igrinet	134	14	21	†
igrinetf	†	36	42	†
igrineth	202	23	25	18
igrinethi	†	40	40	133 §
idallasl	‡	‡	50§	‡
idallasm	‡	47	‡	0
idallasm	‡	45	‡	0
ipolak1	50	16	‡	‡
ipolak2	7	13	‡	‡
ipolak3	178	27	◇	†
ipolak5	7	13	37	†
ismbank	165	31	34	37
icantilvr	0	12	25	0
icb2	196	32	◇	18
icb3	196	31	†	8
ichaconn1	196	32	◇	6
ichaconn2	196	31	10	17
idipigri	157	32	188	†
igpp	†	38	65	†
ihong	11	22	16	†
iloadbal	123	14	22	†
isvanberg	†	27	48	53
<hr/>				
antenna\iantenna2	†	186	†	156
antenna\iantenna_vareps	†	47	†	†
braess\itrafequil	43	38	36	126
braess\itrafequil2	7	9	82	50
braess\itrafequilsf	47	46	34	103
braess\itrafequil2sf	†	9	54	91
elena\ichemeq	14	14	102	16
elena\is383	7	14	67	†
firfilter\ifir_convex	0	13	99	0
markowitz\igrowthopt	15	16	22	†
wbv\iantenna2	0	17	32	0
wbv\ilowpass2	0	13	32	0
<hr/>				
ibatch	0	22	32	158
istockcycle	117	35	72	†
isynthes1	9	25	35	35
isynthes2	62	23	48	46
isynthes3	53	40	94	†
itrimloss2	0	15	52	0
itrimloss4	0	16	40	0
itrimloss5	0	15	60	0
itrimloss6	0	15	55	0
itrimloss7	0	16	61	0
itrimloss12	0	16	73	0
<hr/>				
Avg1	66.33	19.44	–	–
Avg2	11.74	16.04	–	–

¹ †: Max number of iterations reached.

² ‡: Function evaluate error from AMPL.

³ §: Solver terminates with a near optimal status.

⁴ ◇: Ipopt fails in restoration phase.

Table 15 Comparison with MOSEK, Ipopt, and Knitro on CP infeasible test problems