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Electronic Companion to Technical Note: The $MAP_t/Ph_t/\infty$ Queueing System and Multiclass $[MAP_t/Ph_t/\infty]^K$ Queueing Network

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In this paper we demonstrate how a key adjustment to known numerically exact methods for evaluating time-dependent moments of the number of entities in the $Ph_t/Ph_t/\infty$ queueing system and $[Ph_t/Ph_t/\infty]^K$ queueing network may be implemented to capture the effect of autocorrelation that may be present in arrivals to the more general $MAP_t/Ph_t/\infty$ queueing system and multiclass $[MAP_t/Ph_t/\infty]^K$ queueing network. The MAP_t is more general than the Ph_t arrival process in that it allows for stationary *non-renewal* point processes, as well as the time-dependent generalization of non-renewal point processes. Modeling real-world systems with bursty arrival processes such as those in telecommunications and transportation, for example, necessitate the use of non-renewal processes. Finally, we show that the covariance of the number of entities at different nodes and times may be described by a single closed differential equation.

Key words: queues; queueing; algorithms; phase-type distribution; nonstationary processes; infinite server; MAPs; queueing networks; time-dependent; transient

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Translating to and from Lucantoni MAP Notation

We can construct the Lucantoni representation directly from our representation $(\mathbf{A},\boldsymbol{\lambda})$, namely

$$\mathbf{D}_0 = \boldsymbol{\Lambda}(\mathbf{A}_1 - \mathbf{I}),$$

where $\mathbf{\Lambda}$ is a diagonal matrix with nonzero elements λ_j, for $j=1,2,\ldots,m_A$, and $\mathbf{I}$ is the identity matrix, while

$$(\mathbf{D}_1)_{jh} = \lambda_j \left(\sum_{k=1}^{v_A} a_{j,m_A+k}\alpha_{kh} \right),$$

for $j,h=1,2,\ldots,m_A$.

Notice that the reverse construction (to ours from Lucantoni) may be performed by assigning to each transient phase a unique absorbing phase to which to transition in the event of an arrival (i.e., setting $v_A = m_A$). Given Lucantoni matrices $(\mathbf{D}_0, \mathbf{D}_1)$ we can specify our representation $(\mathbf{A},\boldsymbol{\lambda})$ by setting

$$\lambda_j = -(\mathbf{D}_0)_{jj}$$

$$a_{jh} = \begin{cases} 0, & \text{if } h=j, \\ ((\mathbf{D}_0)_{jh})/\lambda_j & \text{otherwise,} \end{cases}$$

$$a_{j,m_A+h} = \begin{cases} 1-\sum_{r=1}^{m_A} a_{jr}, & \text{if } h=j, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{jh} = ((\mathbf{D}_1)_{jh})/(\lambda_j a_{j,m_A+j}).$$

for $j,h=1,2,\ldots,m_A$. Notice that this defines $\mathbf{A}_2$ as a diagonal matrix.

The Closed System of MDEs for the $MAP_t/Ph_t/\infty$ Queueing System

THEOREM 2. *For the $MAP_t/Ph_t/\infty$ with $t\geq s$, $s\geq 0$, and i and $j=1,2,\ldots m_B$,*

$$C_{i,j}(s,t)' \equiv -\mu_j(t)C_{i,j}(s,t) + \sum_{r=1}^{m_B} \mu_r(t)b_{r,j}(t)C_{i,j}(s,t) \tag{1}$$

where $b_{r,j}(t)$ is the Markov routing probability for entities proceeding to node j after having finished their service at node r at time t, for r and $j=1,2,\ldots m_B$, and $t\geq s$.

Proof:

$$C_{i,j}(s,t)' \equiv \mathrm{E}_{ij}(s,t)' - (\mathrm{E}_i(s)\mathrm{E}_j(t))',$$

where $\mathrm{E}_{ij}(s,t)' \equiv \mathrm{E}'\left[N_i(s)N_j(t)\right]$. Now using the partial marginal Kolmogorov equations defined in the earlier text we have

$$\begin{aligned}
\mathrm{E}_{ij}(s,t)' = & \sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \frac{\partial}{\partial t}\mathrm{P}(N_i(s)=n_i, N_j(t)=n_j) \\
= & -\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A} a_{h,m_A+r}(t)\mathrm{P}(N_i(s)=n_i, N_j(t)=n_j, A(t)=h) \\
& -\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \left[1-b_{jj}(t)\right]\mu_j(t) n_j \mathrm{P}(N_i(s)=n_i, N_j(t)=n_j) \\
& -\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \sum_{\substack{r=1\\ r\neq j}}^{m_B}\mu_r(t)b_{rj}(t)\sum_{n_r=0}^{\infty} n_r \mathrm{P}(N_i(s)=n_i, N_j(t)=n_j, N_r(t)=n_r) \\
& +\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A} a_{h,m_A+r}(t)\mathrm{P}(N_i(s)=n_i, N_j(t)=n_j-1, A(t)=h) \\
& +\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \mu_j(t)\left[1-b_{jj}(t)\right](n_j+1)\,\mathrm{P}(N_i(s)=n_i, N_j(t)=n_j+1) \\
& +\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j \sum_{\substack{r=1\\ r\neq j}}^{m_B}\mu_r(t)b_{rj}(t)\sum_{n_r=0}^{\infty} n_r \mathrm{P}(N_i(s)=n_i, N_j(t)=n_j-1, N_r(t)=n_r) \\
= & -\beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A} a_{h,m_A+r}(t)\mathrm{E}_{ij,h}(s,t) - \mu_j(t)\left[1-b_{jj}(t)\right]\mathrm{E}_{ijj}(s,t,t) \\
& -\sum_{\substack{r=1\\ r\neq j}}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_{irj}(s,t,t) + \beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A} a_{h,m_A+r}(t)\mathrm{E}_{i(j+1),h}(s,t) \\
& +\mu_j(t)\left[1-b_{jj}(t)\right]\mathrm{E}_{ij(j-1)}(s,t,t) + \sum_{\substack{r=1\\ r\neq j}}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_{ir(j+1)}(s,t,t) \\
= & \ \beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A} a_{h,m_A+r}(t)\mathrm{E}_{i,h}(s,t) + \sum_{\substack{r=1\\ r\neq j}}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_{ir}(s,t) \\
& +\mu_j(t)\left[1-b_{jj}(t)\right]\left(\mathrm{E}_{ijj}(s,t,t)-\mathrm{E}_{ij}(s,t)\right) - \mu_j(t)\left[1-b_{jj}(t)\right]\mathrm{E}_{ijj}(s,t,t)
\end{aligned}$$

$$= \quad \beta_j(t)\sum_{h=1}^{m_A}\lambda_h(t)\sum_{r=1}^{v_A}a_{h,m_A+r}(t)\mathrm{E}_{i,h}(s,t)+\sum_{r=1}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_{ir}(s,t)\mu_j(t)\mathrm{E}_{ij}(s,t)$$

And using the fact that $\mathrm{E}_{i,h}(s,t)=\mathrm{P}(t;\cdot,\ell)\mathrm{E}_i(s)$ and the result for $\mathrm{E}_j(t)'$ from Nelson and Taaffe (2004), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\mathrm{E}_i(s)\mathrm{E}_j(t)\right) =&\ \mathrm{E}_i(s)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_j(t)\right)\\ =&\ \mathrm{E}_i(s)\beta_j(t)\sum_{\ell=1}^{m_A}\sum_{r=1}^{v_A}a_{h,m_A+r}(t)\lambda_\ell(t)\mathrm{P}(t;\cdot,\ell)+\mathrm{E}_i(s)\left(\sum_{\ell=1}^{m_B}b_{\ell j}(t)\mu_\ell(t)\mathrm{E}_\ell(t)-\mu_\ell(t)\mathrm{E}_j(t)\right)\end{aligned}$$

So combining these terms

$$\begin{aligned}C_{i,j}(s,t)' =&-\mu_j(t)\mathrm{E}_{ij}(s,t)+\sum_{r=1}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_{ir}(s,t)-\mathrm{E}_i(s)\sum_{r=1}^{m_B}\mu_r(t)b_{rj}(t)\mathrm{E}_r(t)+\mu_j(t)\mathrm{E}_i(s)\mathrm{E}_j(t)\\ =&-\mu_j(t)C_{i,j}(s,t)+\sum_{r=1}^{m_B}\mu_r(t)b_{rj}(t)C_{i,j}(s,t)\end{aligned}$$

□

Notice that the functional form of this result is 1) the same as the Massey and Whitt Whitt and Massey (1993) result for the $[M_t/M_t/\infty]^K$ and $M_t/G_t/\infty$ system, and 2) *independent* of the arrival process. The latter point is intuitive since entities arriving to the system after time s do not interact/interfere with entities already present in the system by time s.

THEOREM 3. *The $MAP_t/Ph_t/\infty$ arrival-process state-probability differential equations (ADEs) are:*

$$\mathrm{P}(t;\cdot,k)'=\sum_{\ell=1}^{m_A}\lambda_\ell(t)\mathrm{P}(t;\cdot,\ell)\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t;\cdot,\ell)-\lambda_k(t)\mathrm{P}(t;\cdot,k)$$

for $k=1,2,\ldots,m_A$.

Proof: While we can show this result by straightforward and tedious algebraic manipulation of the expression

$$\mathrm{P}(t;\cdot,k)' \equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty}\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k)',$$

we also note that the arrival process is independent of the service process and therefore we can formulate the KFEs associated with this simple time-dependent Markov arrival process directly. $\square$

The *marginal first-moment differential equations* (MMDEs) are shown below and we observe that they are *not* closed. Let $\mathrm{E}_j(t) \equiv \mathrm{E}[N_j(t)]$.

THEOREM 4. *The $MAP_t/Ph_t/\infty$ marginal first-moment differential equations (MMDEs) are:*

$$\mathrm{E}_j(t)' \equiv \frac{d}{dt}\mathrm{E}_j(t) = \beta_j(t)\sum_{\ell=1}^{m_A}\lambda_\ell(t)\left(\sum_{k=1}^{v_A}a_{\ell,m_A+k}(t)\right)\mathrm{P}(t;\cdot,\ell) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell j}(t)\mu_\ell(t)\mathrm{E}_\ell(t) - [1-b_{jj}(t)]\mathrm{E}_j(t)$$

for $j = 1,2,\ldots,m_B$.

THEOREM 5. *The $MAP_t/Ph_t/\infty$ p^{th}-partial-moment differential equations (p^{th} PMDEs):*

$$\begin{aligned}\frac{d}{dt}\mathrm{E}^p_{j,k}(t) = &-\lambda_k(\mathrm{t})\,\mathrm{E}^p_{j,k}(t) + [1-b_{jj}(t)]\,\mu_j(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)\,(-1)^{p-q}\\ &+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\,\mathrm{E}^p_{j,\ell}(t)\\ &+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\\ &+\sum_{\ell=1}^{m_A}\lambda_\ell(t)\,a_{\ell k}(t)\,\mathrm{E}^p_{j,\ell}(t) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)\,b_{\ell j}(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j\ell,k}(t)\end{aligned}$$

for $i = 1,2,\ldots,m_A$, $j = 1,2,\ldots,m_B$, and $p = 0,1,2,\ldots$.

Proof:

$$
\begin{aligned}
\frac{d}{dt}\mathrm{E}^p_{j,k}(t) &\equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_j^p \mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k)' \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_j^p \times \\
&\quad \Bigg\{ -[1-a_{kk}(t)]\lambda_k(t)\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k) \\
&\quad - \sum_{\ell=1}^{m_B} n_\ell\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k) \\
&\quad + \sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{h=1}^{m_B}\mathrm{I}_{[n_h>0]}\beta_h(t)\mathrm{P}(t;n_1,\ldots,n_h-1,\ldots,n_{m_B},\ell) \\
&\quad + \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},\ell) \\
&\quad + \sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)[n_\ell+1]\mu_\ell(t)\mathrm{P}(t;n_1,\ldots,n_\ell+1,\ldots,n_{m_B},k) \\
&\quad + \sum_{\ell=1}^{m_B}\mathrm{I}_{[n_\ell>0]}\sum_{\substack{h=1\\ h\neq\ell}}^{m_B} b_{h\ell}(t)[n_h+1]\mu_h(t)\mathrm{P}(t;n_1,\ldots,n_\ell-1,\ldots,n_h+1,\ldots,n_{m_B},k) \Bigg\} \\
&= -[1-a_{kk}(t)]\lambda_k(t)\mathrm{E}^p_{j,k}(t) - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t) \\
&\quad + \sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\left\{\sum_{\substack{h=1\\ h\neq j}}^{m_B}\beta_h(t)\mathrm{E}^p_{j,\ell}(t) + \beta_j(t)\sum_{n_j=0}^{\infty}[n_j+1]^p\mathrm{P}(t;n_j,\ell)\right\} \\
&\quad + \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_\ell+1]\mathrm{P}(t;n_{\ell+1},n_j,k) \\
&\quad + b_{j,m_B+1}(t)\mu_j(t)\sum_{n_j=0}^{\infty} n_j^p[n_j+1]\mathrm{P}(t;n_j+1,k)
\end{aligned}
$$

$$+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\left\{\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=1}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_\ell-1,n_j,n_h+1,k)\right.$$

$$\left.+\mu_j(t)b_{j\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_j+1]\mathrm{P}(t;n_\ell-1,n_j+1,k)\right\}$$

$$+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_j=1}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_j-1,n_h+1,k)$$

$$=-\lambda_k(t)\mathrm{E}^p_{j,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t)$$

$$+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h(t)}\alpha_{h,k}(t)\right)\lambda_\ell(t)\ \left\{[1-\beta_j(t)]\mathrm{E}^p_{j,\ell}(t)+\beta_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\right\}$$

$$+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t)+\sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p n_\ell\mathrm{P}(t;n_\ell,n_j,k)$$

$$+b_{j,m_B+1}(t)\mu_j(t)\sum_{n_j=1}^{\infty}[n_j-1]^p\ n_j\mathrm{P}(t;n_j,k)$$

$$+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_\ell,n_j,n_h+1,k)$$

$$+\mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{n_j=1}^{\infty}[n_j-1]^p n_j\sum_{n_\ell=0}^{\infty}\mathrm{P}(t;n_\ell,n_j,k)$$

$$+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty}[n_j+1]^p[n_h+1]\mathrm{P}(t;n_j,n_h+1,k)$$

$$=-\lambda_k(t)\mathrm{E}^p_{j,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t)$$

$$
\begin{aligned}
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\\
&+(1-\beta_j(t))\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\,\mathrm{E}^p_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t)+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t)\\
&+b_{j,m_B+1}(t)\mu_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}\\
&+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p n_h \mathrm{P}(t;n_\ell,n_j,n_h,k)\\
&+\mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{n_j=1}^{\infty}[n_j-1]^p n_j \mathrm{P}(t;n_j,k)\\
&+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_j=0}^{\infty}[n_j+1]^p n_h \mathrm{P}(t;n_j,n_h,k)\\
=&-\lambda_k(t)\mathrm{E}^p_{j,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t)\\
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\,\mathrm{E}^p_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t)+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
&+b_{j,m_B+1}(t)\mu_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^{p}_{jh,k}(t)\\
&+\mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q}_{jh,k}(t)\\
=&-\lambda_k(t)\mathrm{E}^{p}_{j,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^{p}_{j\ell,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t)\\
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q}_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}^{p}_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^{p}_{j,\ell}(t)+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^{p}_{j\ell,k}(t)\\
&+\mu_j(t)[b_{j,m_B+1}(t)+1-b_{jj}(t)-b_{j,m_B+1}(t)]\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}\\
&+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^{p}_{jh,k}(t)+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q}_{jh,k}(t)\\
=&-\lambda_k(t)\mathrm{E}^{p}_{j,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^{p}_{j\ell,k}(t)\\
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q}_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}^{p}_{j,\ell}(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t)\\
&+\mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}\\
&+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^p_{jh,k}(t) + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t)\\
=& -\lambda_k(t)\mathrm{E}^p_{j,k}(t) - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)\\
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t)\\
&+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) + \mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}\\
&+\sum_{\substack{h=1\\ h\neq j}}^{m_B}[1-b_{hh}(t)-b_{h,m_B+1}(t)-b_{hj}(t)]\mu_h(t)\mathrm{E}^p_{jh,k}(t) + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t)\\
=& -\lambda_k(t)\mathrm{E}^p_{j,k}(t) - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)\\
&+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t)\\
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}_{j,k}^{q+1}(t)(-1)^{p-q} \\
&+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{j\ell,k}^{p}(t)-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)b_{\ell,m_B+1}(t)\mathrm{E}_{j\ell,k}^{p}(t) \\
&-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)b_{\ell j}(t)\mathrm{E}_{j\ell,k}^{p}(t)+\sum_{\substack{h=1\\ h\neq j}}^{m_B}b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}_{jh,k}^{q}(t) \\
=&-\lambda_k(t)\mathrm{E}_{j,k}^{p}(t)+\beta_j(t)\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}_{j,\ell}^{q}(t) \\
&+\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{j,\ell}^{p}(t)+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{j,\ell}^{p}(t) \\
&+\mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}_{j,k}^{q+1}(t)(-1)^{p-q}+\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}b_{\ell j}(t)\mu_l(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}_{jh,k}^{q}(t)
\end{aligned}
$$

□

THEOREM 6. *The $MAP_t/Ph_t/\infty$ cross-product partial-moment differential equations are*

$$
\begin{aligned}
\mathrm{E}_{ij,k}(t)' = &\sum_{\ell=1}^{m_B} b_{\ell i}(t)\mu_\ell \mathrm{E}_{\ell j,k}(t) + \sum_{\ell=1}^{m_B} b_{\ell j}(t)\mu_\ell \mathrm{E}_{\ell i,k}(t) \\
&- b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t) - b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t) \\
&- [\mu_i(t)+\mu_j(t)]\mathrm{E}_{ij,k}(t) - \lambda_k(t)\mathrm{E}_{ij,k}(t) \\
&+ \sum_{\ell=1}^{m_A} \lambda_\ell(t) a_{\ell k}(t)\mathrm{E}_{ij,\ell}(t) + \sum_{\ell=1}^{m_A} \lambda_\ell(t)\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\mathrm{E}_{ij,\ell}(t) \\
&+ \beta_i(t)\sum_{\ell=1}^{m_A} \lambda_\ell(t)\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\mathrm{E}_{j,\ell}(t) \\
&+ \beta_j(t)\sum_{\ell=1}^{m_A} \lambda_\ell(t)\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\mathrm{E}_{i,\ell}(t).
\end{aligned}
$$

$i = 1, 2, \ldots, m_B$ *and* $j = 1, 2, \ldots, m_B$, $i \neq j$, *and* $k = 1, 2, \ldots, m_A$.

Proof:

$$
\begin{aligned}
\mathrm{E}_{ij,k}(t)' &\equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_i n_j \mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k)' \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_i n_j \times
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg\{ - [1-a_{kk}(t)]\lambda_k(t)\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k) \\
&- \sum_{\ell=1}^{m_B} n_\ell \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k) \\
&+ \sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\sum_{h=1}^{m_B}\mathrm{I}_{[n_h>0]}\beta_h(t)\mathrm{P}(t;n_1,\ldots,n_h-1,\ldots,n_{m_B},\ell) \\
&+ \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},\ell) \\
&+ \sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)[n_\ell+1]\mu_\ell(t)\mathrm{P}(t;n_1,\ldots,n_\ell+1,\ldots,n_{m_B},k) \\
&+ \sum_{\ell=1}^{m_B}\mathrm{I}_{[n_\ell>0]}\sum_{\substack{h=1\\ h\neq\ell}}^{m_B} b_{h\ell}(t)[n_h+1]\mu_h(t)\mathrm{P}(t;n_1,\ldots,n_\ell-1,\ldots,n_h+1,\ldots,n_{m_B},k) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= &-[1-a_{kk}(t)]\lambda_k(t)\mathrm{E}_{ij,k}(t) - \sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t) \\
&-\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t) \\
&\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\Bigg\{\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B}\beta_h(t)\mathrm{E}_{ij,\ell}(t) \\
&\quad + \beta_i(t)\sum_{n_i=0}^{\infty}[n_i+1]n_j\mathrm{P}(t;n_i,n_j,\ell) + \beta_j(t)\sum_{n_j=0}^{\infty}n_i[n_j+1]\mathrm{P}(t;n_i,n_j,\ell)\Bigg\} \\
&+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_\ell+1]\mathrm{P}(t;n_\ell+1,n_i,n_j,k) \\
&\quad + b_{i,m_B+1}(t)\mu_i(t)\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_i+1]\mathrm{P}(t;n_i+1,n_j,k) \\
&\quad + b_{j,m_B+1}(t)\mu_j(t)\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_j+1]\mathrm{P}(t;n_i,n_j+1,k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\Bigg\{\sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=1}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j (n_h+1)\mathrm{P}(t;n_\ell-1,n_i,n_j,n_h+1,k)\\
&\qquad+\mu_i(t)b_{i\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j[n_i+1]\mathrm{P}(t;n_\ell-1,n_i+1,n_j,k)\\
&\qquad+\mu_j(t)b_{j\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j[n_j+1]\mathrm{P}(t;n_\ell-1,n_i,n_j+1,k)\Bigg\}\\
&+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty} n_i n_j[n_h+1]\mathrm{P}(t;n_i-1,n_j,n_h+1,k)\\
&+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty} n_i n_j[n_h+1]\mathrm{P}(t;n_i,n_j-1,n_h+1,k)\\
&+b_{ji}(t)\mu_j(t)\sum_{n_j=0}^{\infty}\sum_{n_i=1}^{\infty} n_i n_j[n_j+1]\mathrm{P}(t;n_i-1,n_j+1,k)\\
&+b_{ij}(t)\mu_i(t)\sum_{n_j=1}^{\infty}\sum_{n_i=0}^{\infty} n_i n_j[n_i+1]\mathrm{P}(t;n_j-1,n_i+1,k)\\
&+\sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
=&-[1-a_{kk}(t)]\lambda_k(t)\mathrm{E}_{ij,k}(t)\\
&-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t)\\
&-\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t)\\
&\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\Bigg\{[1-\beta_i(t)-\beta_j(t)]\mathrm{E}_{ij,\ell}(t)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \beta_i(t)[\mathrm{E}_{ij,\ell}(t) + \mathrm{E}_{j,\ell}(t)] + \beta_j(t)[\mathrm{E}_{ij,\ell}(t) + \mathrm{E}_{i,\ell}(t)]\Bigg\} \\
&+ \sum_{\substack{\ell=1 \\ \ell\neq i \\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t) \\
&+ b_{i,m_B+1}(t)\mu_i(t)\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty}[n_i-1]n_jn_i\mathrm{P}(t;n_i,n_j,k) \\
&+ b_{j,m_B+1}(t)\mu_j(t)\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty}n_i[n_j-1]n_j\mathrm{P}(t;n_i,n_j,k) \\
&+ \sum_{\substack{\ell=1 \\ \ell\neq i \\ \ell\neq j}}^{m_B}\Bigg\{ \sum_{\substack{h=1 \\ h\neq \ell \\ h\neq i \\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty} n_in_jn_h\mathrm{P}(t;n_\ell,n_i,n_j,n_h,k) \\
&\quad + \mu_i(t)b_{i\ell}(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty}[n_i-1]n_jn_i\mathrm{P}(t;n_\ell,n_i,n_j,k) \\
&\quad + \mu_j(t)b_{j\ell}(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty}n_i[n_j-1]n_j\mathrm{P}(t;n_\ell,n_i,n_j,k)\Bigg\} \\
&+ \sum_{\substack{h=1 \\ h\neq i \\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}[n_i+1]n_jn_h\mathrm{P}(t;n_i,n_j,n_h,k) \\
&+ \sum_{\substack{h=1 \\ h\neq i \\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_i[n_j+1]n_h\mathrm{P}(t;n_i,n_j,n_h,k) \\
&+ b_{ji}(t)\mu_j(t)\sum_{n_j=1}^{\infty}\sum_{n_i=0}^{\infty}[n_i+1][n_j-1]n_j\mathrm{P}(t;n_i,n_j,k) \\
&+ b_{ij}(t)\mu_i(t)\sum_{n_j=0}^{\infty}\sum_{n_i=1}^{\infty}[n_i-1][n_j+1]n_i\mathrm{P}(t;n_j,n_i,k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
=\;& -\lambda_k(t)\mathrm{E}_{ij,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t)\\
&-\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t)\\
&\sum_{\ell=1}^{m_A}\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\left\{\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\right\}\\
&+b_{i,m_B+1}(t)\mu_i(t)\left[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\right]+b_{j,m_B+1}(t)\mu_j(t)\left[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\right]\\
&+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\Bigg\{\sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)\\
&\qquad+\mu_i(t)b_{i\ell}(t)\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big]+\mu_j(t)b_{j\ell}(t)\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\Big]\Bigg\}\\
&+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\Big]+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\Big]\\
&+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
&+b_{ji}(t)\mu_j(t)\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{j,k}(t)-\mathrm{E}_{j,k}(t)\Big]\\
&+b_{ij}(t)\mu_i(t)\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{i,k}(t)-\mathrm{E}_{i,k}(t)\Big]\\
=\;& -\lambda_k(t)\mathrm{E}_{ij,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)b_{\ell\ell}(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t)\\
&-\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t)
\end{aligned}
$$

$$+\sum_{\ell=1}^{m_A}\left\{\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\right\}$$

$$+b_{i,m_B+1}(t)\mu_i(t)\left[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\right]+b_{j,m_B+1}(t)\mu_j(t)\left[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\right]$$

$$+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)$$

$$+\mu_i(t)\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}b_{i\ell}(t)\left[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\right]+\mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}b_{j\ell}(t)\left[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\right]$$

$$+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B}b_{hi}(t)\mu_h(t)\left[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\right]+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B}b_{hj}(t)\mu_h(t)\left[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\right]$$

$$+b_{ji}(t)\mu_j(t)\left[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{j,k}(t)-\mathrm{E}_{j,k}(t)\right]$$

$$+b_{ij}(t)\mu_i(t)\left[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{i,k}(t)-\mathrm{E}_{i,k}(t)\right]$$

$$+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)$$

$$=-\lambda_k(t)\mathrm{E}_{ij,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)b_{\ell\ell}(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\ell=1}^{m_B}\mu_\ell(t)b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t)$$

$$-\mu_i(t)\mathrm{E}^2_{ij,k}(t)-\mu_j(t)\mathrm{E}^2_{ji,k}(t)$$

$$+\sum_{\ell=1}^{m_A}\left\{\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\right\}$$

$$-b_{i,m_B+1}(t)\mu_i(t)\mathrm{E}_{ij,k}(t)-b_{j,m_B+1}(t)\mu_j(t)\mathrm{E}_{ij,k}(t)$$

$$+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)$$

$$
\begin{aligned}
&+\mu_i(t)\left\{\sum_{\substack{\ell=1\\ \ell\neq i}}^{m_B} b_{i\ell}(t)\right\}\left[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\right]+\mu_j(t)\left\{\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\right\}\left[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\right]\\
&+\sum_{\substack{h=1\\ h\neq i}}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\Big]-b_{ji}(t)\mu_j(t)\mathrm{E}^2_{ji,k}(t)\\
&+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\Big]-b_{ij}(t)\mu_i(t)\mathrm{E}^2_{ij,k}(t)\\
&-b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t)-b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t)+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
=&-\lambda_k(t)\mathrm{E}_{ij,k}(t)-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\ell=1}^{m_B}\mu_\ell(t)b_{\ell m_B+1}(t)\mathrm{E}_{ij\ell,k}(t)\\
&-\mu_i(t)\mathrm{E}^2_{ij,k}(t)-\mu_j(t)\mathrm{E}^2_{ji,k}(t)\\
&+\sum_{\ell=1}^{m_A}\Bigg\{\left(\sum_{h=1}^{v_A} a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\Bigg\}\\
&-b_{i,m_B+1}(t)\mu_i(t)\mathrm{E}_{ij,k}(t)-b_{j,m_B+1}(t)\mu_j(t)\mathrm{E}_{ij,k}(t)\\
&+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B}\mu_h(t)\mathrm{E}_{ijh,k}(t)-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)\\
&-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{h,m_B+1}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)\\
&+\mu_i(t)\Big(\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big)\Big(1-b_{i,m_B+1}(t)\Big)+\mu_j(t)\Big(\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\Big)\Big(1-b_{j,m_B+1}(t)\Big)\\
&+\sum_{h=1}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\Big]+\sum_{h=1}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\Big]\\
&-b_{ji}(t)\mu_j(t)\mathrm{E}^2_{ji,k}(t)-b_{ij}(t)\mu_i(t)\mathrm{E}^2_{ij,k}(t)-b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t)-b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
&-\mu_i(t)b_{ii}(t)\mathrm{E}^2_{ij,k}(t)-\mu_j(t)b_{jj}(t)\mathrm{E}^2_{ji,k}(t)+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
=&-\lambda_k(t)\mathrm{E}_{ij,k}(t)+\sum_{\ell=1}^{m_B}\mu_\ell(t)b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t)\\
&+\sum_{\ell=1}^{m_A}\Big\{\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\}\\
&-\mu_i(t)\mathrm{E}_{ij,k}(t)-\mu_i(t)b_{i,m_B+1}(t)\mathrm{E}^2_{ij,k}(t)-\mu_j(t)\mathrm{E}_{ij,k}(t)-\mu_j(t)b_{j,m_B+1}(t)\mathrm{E}^2_{ji,k}(t)\\
&-\sum_{\substack{h=1\\h\neq i\\h\neq j}}^{m_B}b_{h,m_B+1}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)+\sum_{h=1}^{m_B}b_{hi}(t)\mu_h(t)\mathrm{E}_{jh,k}(t)+\sum_{h=1}^{m_B}b_{hj}(t)\mu_h(t)\mathrm{E}_{ih,k}(t)\\
&-b_{ji}(t)\mu_j(t)\mathrm{E}_{ij,k}(t)-b_{ij}(t)\mu_i(t)\mathrm{E}_{ij,k}(t)+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)\\
=&\sum_{h=1}^{m_B}b_{hi}(t)\mu_h(t)\mathrm{E}_{jh,k}(t)+\sum_{h=1}^{m_B}b_{hj}(t)\mu_h(t)\mathrm{E}_{ih,k}(t)\\
&-b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t)-b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t)-\left[\mu_i(t)+\mu_j(t)\right]\mathrm{E}_{ij,k}(t)\\
&+\sum_{\ell=1}^{m_A}\Big\{\left(\sum_{h=1}^{v_A}a_{\ell,m_A+h}(t)\alpha_{h,k}(t)\right)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\}\\
&-\lambda_k(t)\mathrm{E}_{ij,k}(t)+\sum_{\ell=1}^{m_A}a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)
\end{aligned}
$$

□

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