Online Supplement for "Spare Parts Inventory Management with Substitution-Dependent Reliability"

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1 Proofs

Proof of Proposition 1: Substituting formulations (1) and (3) into formulation (5) results in

$$\begin{aligned}
v^{\pi^n}(s) &= \min_{(o,a)\in A(s)} \Bigg\{ \sum_{i\in\mathcal{I}} c_i^o o_i + \sum_{i\in\mathcal{I}} c_i^h (x_i - \sum_{m\in\mathcal{M}} a_i^m) + \sum_{m\in\bar{Y}} c^m \big(1 - \sum_{i\in\mathcal{I}} a_i^m\big) + \\
&\lambda\mathbb{E}\bigg(\alpha_0 + \sum_{i\in\mathcal{I}} \alpha_i |x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}}| + \sum_{m\in\mathcal{M}} \beta_m \mathbf{1}_{\{y_m'=0\}} + \sum_{m\in\mathcal{M}} \zeta_m \sum_{i\in F(m)} p_i^m \mathbf{1}_{\{y_m'=i\}} \bigg| s\bigg)\Bigg\} \\
&= f_1(s,\alpha) + \min_{(o,a)\in A(s)} \Bigg\{ \sum_{i\in\mathcal{I}} c_i^o o_i - \sum_{i\in\mathcal{I}}\sum_{m\in\mathcal{M}} c_i^h a_i^m - \sum_{i\in\mathcal{I}}\sum_{m\in\bar{Y}} c^m a_i^m + \sum_{i\in\mathcal{I}} \lambda\alpha_i \mathbb{E}(|x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}}| \ \big| s) + \\
&\sum_{m\in\mathcal{M}} \lambda\beta_m \mathbb{E}(\mathbf{1}_{\{y_m'=0\}} \big| s) + \sum_{m\in\mathcal{M}} \lambda\zeta_m \sum_{i\in F(m)} p_i^m \mathbb{E}(\mathbf{1}_{\{y_m'=i\}} | s)\Bigg\} \\
&= f_1(s,\alpha) + \min_{(o,a)\in A(s)} \Bigg\{ \sum_{i\in\mathcal{I}} c_i^o o_i - \sum_{i\in\mathcal{I}}\sum_{m\in\mathcal{M}} c_i^h a_i^m - \sum_{i\in\mathcal{I}}\sum_{m\in\bar{Y}} c^m a_i^m + \sum_{i\in\mathcal{I}} \lambda\alpha_i \mathbb{E}(|x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}}| \ \big| s) + \\
&\sum_{m\in\mathcal{M}} \lambda\beta_m P(y_m' = 0\big| s) + \sum_{m\in\mathcal{M}} \lambda\zeta_m \sum_{i\in F(m)} p_i^m P(y_m' = i|s)\Bigg\}
\end{aligned}$$

where $f_1(s,\alpha)$ is a term independent of action, and s' denotes the state in the next period. Note that expectations are conditioned on the current state s. Since calculating the expectation $\mathbb{E}(|x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}}|)$ is intractable in closed form, we settle for relaxing $v^{\pi^n}(s)$. Because the absolute value function $|\cdot|$ is convex, by Jensen's inequality

$$\mathbb{E}(|x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}}|) \geq |\mathbb{E}(x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}})|.$$

Moreover,

$$\begin{aligned}
&P(y_m' = 0|s) = (1 - \sum_{i\in\mathcal{I}} a_i^m) + \sum_{i\in\mathcal{I}} a_i^m p_i^m, \ \ \forall m \in \bar{Y}, \\
&P(y_m' = 0|s) = p_{y_m}^m, \ \ \forall m \in \mathcal{M}\backslash\bar{Y}, \\
&P(y_m' = i|s) = (1 - p_i^m)\mathbf{1}_{\{y_m=i\}} + (1 - p_i^m) a_i^m, \ \ \forall m \in \mathcal{M}, i \in F(m),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\big(x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}} | s\big) = x_i + o_i - \sum_{m\in\bar{F}(i)} a_i^m - w_i \sum_{m\in\bar{F}(i)} P(y_m' = 0|s) \\
&= x_i + o_i - \sum_{m\in\bar{F}(i)} a_i^m - w_i \sum_{m\in\bar{F}(i)\cap\bar{Y}} \big((1 - \sum_{j\in\mathcal{I}} a_j^m) + \sum_{j\in\mathcal{I}} a_j^m p_j^m\big) - w_i \sum_{m\in\bar{F}(i)\backslash\bar{Y}} p_{y_m}^m.
\end{aligned}$$

Therefore, by rearranging terms and considering $A(s)$ one can construct the following mathematical program

$$
\begin{aligned}
& v^R(s) = f(s,\alpha,\beta,\zeta) + \min \sum_{i\in\mathcal{I}} c_i^o o_i - \sum_{i\in\mathcal{I}}\sum_{m\in\bar{Y}} (c_i^h + c^m + \lambda\beta_m(1-p_i^m))a_i^m + \sum_{i\in\mathcal{I}}\sum_{m\in\mathcal{M}\setminus\bar{Y}} c_i^h a_i^m + \\
& \sum_{i\in\mathcal{I}}\sum_{m\in\mathcal{M}} (\lambda\zeta_m p_i^m(1-p_i^m))a_i^m + \sum_{i\in\mathcal{I}} \lambda\alpha_i |\mathbb{E}(x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}})| \\
& \text{s.t.} \\
& \sum_{m\in\bar{F}(i)\cap\bar{Y}} a_i^m \le x_i, \ \ \forall i, \\
& \sum_{i\in F(m)} a_i^m \le 1, \ \ \forall m \in \bar{Y}, \\
& a_i^m = 0, \ \ \forall i, m \in \mathcal{M}\setminus\bar{Y}, \\
& o_i \ge 0, \ \ o_i \in \mathbb{Z}_+, \ \ a_i^m \in \{0,1\} \ \ \forall i, m.
\end{aligned}
$$

Let $z_i = |\mathbb{E}(x_i' - w_i \sum_{m\in\bar{F}(i)} \mathbf{1}_{\{y_m'=0\}})|$. Since $\alpha \ge 0$, the above nonlinear program is equivalent to the following mixed integer linear program

$$
\begin{aligned}
& v^R(s) = f(s,\alpha,\beta,\zeta) + \min \sum_{i\in\mathcal{I}} (c_i^o o_i + \lambda\alpha_i z_i) - \sum_{i\in\mathcal{I}}\sum_{m\in\bar{Y}} (c_i^h + c^m + \lambda\beta_m(1-p_i^m) - \lambda\zeta_m p_i^m(1-p_i^m))a_i^m \\
& \text{s.t.} \\
& \sum_{m\in\bar{F}(i)\cap\bar{Y}} a_i^m \le x_i, \ \ \forall i, \\
& \sum_{i\in F(m)} a_i^m \le 1, \ \ \forall m \in \bar{Y}, \\
& z_i \ge x_i + o_i - \sum_{m\in\bar{F}(i)} a_i^m - w_i \sum_{m\in\bar{F}(i)\cap\bar{Y}} \big((1 - \sum_{j\in\mathcal{I}} a_j^m) + \sum_{j\in\mathcal{I}} a_j^m p_j^m\big) - w_i \sum_{m\in\bar{F}(i)\setminus\bar{Y}} p_{y_m}^m, \ \ \forall i, \\
& -z_i \le x_i + o_i - \sum_{m\in\bar{F}(i)} a_i^m - w_i \sum_{m\in\bar{F}(i)\cap\bar{Y}} \big((1 - \sum_{j\in\mathcal{I}} a_j^m) + \sum_{j\in\mathcal{I}} a_j^m p_j^m\big) - w_i \sum_{m\in\bar{F}(i)\setminus\bar{Y}} p_{y_m}^m, \ \ \forall i, \\
& a_i^m = 0, \ \ \forall i, m \in \mathcal{M}\setminus\bar{Y}, \\
& o_i, z_i \ge 0, \ \ o_i \in \mathbb{Z}_+, \ \ a_i^m \in \{0,1\} \ \ \forall i, m.
\end{aligned}
$$

Rearranging the terms constructs formulation (6).

Proof of Proposition 2: Let $n_i = |\bar{F}(i)|$. For each i we have

$$
\hat{p}_i^{OP} = \min_{m\in\bar{F}(i)} p_{y_m}^m \le \hat{p}_i^{EX} = \frac{\sum_{m\in\bar{F}(i)} p_{y_m}^m}{n_i} \le \hat{p}_i^{RO} = \max_{m\in\bar{F}(i)} p_{y_m}^m. \tag{3}
$$

Let $\mathcal{X}^{OP} \sim binomial(n_i, \hat{p}_i^{OP})$, $\mathcal{X}^{EX} \sim binomial(n_i, \hat{p}_i^{EX})$, and $\mathcal{X}^{RO} \sim binomial(n_i, \hat{p}_i^{RO})$. Formulation (3) leads to

$$
\mathcal{X}^{OP} \le_{st} \mathcal{X}^{EX} \le_{st} \mathcal{X}^{RO}, \tag{4}
$$

where $\le_{st}$ denotes stochastic ordering. We know that $G_i(\cdot) = \frac{\hat{c}_i - c_i^o}{\hat{c}_i + c_i^h}$ is increasing in $\hat{c}_i$ for $\hat{c}_i > 0$ and

$$
0 < \hat{c}_i^{OP} = \min_{m\in\bar{F}(i)} c^m \le \hat{c}_i^{EX} = \frac{\sum_{m\in\bar{F}(i)} c^m}{n_i} \le \hat{c}_i^{RO} = \max_{m\in\bar{F}(i)} c^m.
$$

Therefore,

$$
G_i(x_i^H - y_i^H + o_i^{OP}) \le G_i(x_i^H - y_i^H + o_i^{EX}) \le G_i(x_i^H - y_i^H + o_i^{RO}). \tag{5}
$$

Finally, formulations (4) and (5) result in

$$
o_i^{OP} \le o_i^{EX} \le o_i^{RO}.
$$

2 Sampling States

As noted by de Farias and Roy (2004), φ regulates the quality of the approximation across S, and can therefore be used to target certain regions of the state space where one aims to obtain better approximations. In that regard, we would like to obtain better approximations in the states that are most likely to be visited in the near future when the "optimal" policy is used. For a policy π, define the distribution φ_π by

$$\varphi_\pi(s) := (1-\lambda)\sum_{t=0}^{\infty}\lambda^t \mathbb{P}_\pi\{s^t = s|s^0\}, \quad s \in S.$$

We would like to use φ_{π^*} in the objective function of formulation (7), and also in sampling $\hat{S}$ through ψ, as prescribed in de Farias and Roy (2004, Theorem 3.1., p.469). Unfortunately, one does not have prior access to π^*. Let π^0 denote an initial policy. Finding φ_{π^0} is computationally intractable, thus, we settle for approximating it using its empirical counterpart, $\hat{\varphi}_{\pi^0,T'}$, where

$$\hat{\varphi}_{\pi,T'}(S) := (1-\lambda)\sum_{t=0}^{T'}\lambda^t \mathbf{1}_{\{s^t(\omega)=s\}}, \quad s \in S,\, \pi \in \mathcal{P},$$

where $\{s^t(\omega) : t \geq 0\}$ represents the outcome of a Monte Carlo simulation and T' denotes the simulation budget. (In our numerical study we take the average over many replications.) We use this simulation run to select $\hat{S}$ as well: Computing ψ is infeasible, thus we approximate it as $\hat{\varphi}_{\pi,T'}$. The underlying motivation is that if on most simulation runs, states are visited at most once, thus we approximate $\mathbb{P}_\pi\{s^t = s|s^0 = s'\} \approx \sum_{r=1}^{R}\mathbf{1}_{\{s^t(\omega_r)=s\}}/R$, where R is the number of replications. This results in $\varphi \approx \psi$.

References

de Farias, D. P. and Roy, B. V. (2004), 'On constraint sampling in the linear programming approach to approximate dynamic programming', *Mathematics of Operations Research* **29**(3), 462–478.