Online Supplement to “A Joint Vehicle Routing and Speed Optimization Problem”

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This is an online supplement to the main paper [1].

1 Proof of Proposition 1

We prove the statement by contradiction. Suppose the speed optimization problem (3) has an optimal speed vector $\mathbf{v}$ with three consecutive vertices j, i, k such that customer i is non-active ($t_i \in (a_i, b_i)$) and $v_{ji} \neq v_{ik}$. We show that we can create a new feasible speed vector with a strictly lower cost than the optimal speed vector. The new speed vector differs from the optimal speed vector only by the speeds on arcs (j,i) and (i,k).

Let the departure time at vertex j be T_j and the arriving time at vertex k be T_k. We first assume that $v_{ji} < v_{ik} \leq u$. The cost over arcs (j,i) and (i,k) is $d_{ji}f(v_{ji}) + d_{ik}f(v_{ik})$. We create two new speeds $v'_{ji} = v_{ji} + d_{ik}\delta$ and $v'_{ik} = v_{ik} - d_{ji}\delta$ with

$$0 < \delta < \min\{\frac{v_{ik} - v_{ji}}{d_{ik}}, \frac{v_{ik} - v_{ji}}{d_{ji}}, \frac{v_{ik}^2 - v_{ji}^2}{v_{ji}d_{ik} + v_{ik}d_{ji}}\}.$$

We will show that with these two new speeds, the vehicle is able to depart vertex j at time T_j, serve customer i within its time window, arrive at node k at time T_k, and incur a lower cost over arcs (j,i) and (i,k).

According to the choice of δ, we have $v'_{ji}, v'_{ik} \in (v_{ji}, v_{ik})$, so v'_{ji} and v'_{ik} are feasible speeds. Since the function $f(v)$ is strictly convex in v, then

$$\frac{f(v'_{ji}) - f(v_{ji})}{d_{ik}\delta} = \frac{f(v_{ji} + d_{ik}\delta) - f(v_{ji})}{d_{ik}\delta} < \frac{f(v_{ik}) - f(v_{ik} - d_{ji}\delta)}{d_{ji}\delta} = \frac{f(v_{ik}) - f(v'_{ik})}{d_{ji}\delta}.$$

Thus $d_{ji}f(v'_{ji}) + d_{ik}f(v'_{ik}) < d_{ji}f(v_{ji}) + d_{ik}f(v_{ik})$, which implies the new speeds incur less cost on the route. Now we show that with the new speeds, the vehicle is able to leave vertex j at T_j and arrive at vertex k at time T_k. The original travel time over arcs (j,i) and (i,k) is

$$T = \frac{d_{ji}}{v_{ji}} + \frac{d_{ik}}{v_{ik}},$$

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and the travel time with the new speeds is

$$T' = \frac{d_{ji}}{v_{ji} + d_{ik}\delta} + \frac{d_{ik}}{v_{ik} - d_{ji}\delta}.$$

Then

$$\begin{aligned}
T' - T &= \frac{d_{ji}}{v_{ji} + d_{ik}\delta} + \frac{d_{ik}}{v_{ik} - d_{ji}\delta} - (\frac{d_{ji}}{v_{ji}} + \frac{d_{ik}}{v_{ik}}) \\
&= \frac{d_{ji}d_{ik}\delta[(v_{ji}d_{ik} + v_{ik}d_{ji})\delta - (v_{ik}^2 - v_{ji}^2)]}{v_{ik}v_{ji}(v_{ik} - d_{ji}\delta)(v_{ji} + d_{ik}\delta)} \\
&< 0.
\end{aligned}$$

The last inequality follows from the choice of δ. Since the total travel time between vertex j and vertex k decreases with the new speeds, with sufficiently small δ the vehicle is able to satisfy the time window constraints for all customers while incurring a lower cost. The case with $v_{ji} > v_{ik}$ can be proven in a similar way.

2 Proof of Proposition 2

Let $\bar{z}$ be a feasible solution to the LP relaxation of (2). For any route $r \in \Omega_1$ such that $\bar{z}_r > 0$, let $I(r)$ and $s(r)$ denote the set of active customers and their respective start times when route r is traversed in an optimal fashion according to optimization problem (3) (if multiple optima exist, one can be picked arbitrarily).

Then, we can define $z'_{r,I(r),s(r)} = \bar{z}_r$ for all $r \in \Omega_1$ such that $\bar{z}_r > 0$ and by definition, $c_{r,I(r),s(r)} = c_r$. By having all other components of z' equal to zero, we have that z' is a feasible solution to the LP relaxation of (5) with the same cost as $\bar{z}$.

Conversely, let z' be an optimal solution to the LP relaxation of (5). We first note that if $z'_{r,I,s} > 0$ and $z'_{r,J,t} > 0$, then it must be the case that $c_{r,I,s} = c_{r,J,t}$. To see that, observe that the coefficients α_{ir} of both variables in constraints (5b) are the same. Thus, if $c_{r,I,s} < c_{r,J,t}$ then we can increase $z'_{r,I,s}$ by ϵ and decrease $z'_{r,J,t}$ by ϵ for some $\epsilon > 0$ and get a feasible solution of strictly smaller cost. In addition, both costs above are the same as the optimal cost c_r of solving problem (3) for route r, since otherwise a similar argument can be applied.

Thus, we can define $\bar{z}_r = \sum\limits_{I,s} z'_{r,I,s}$. Note that, due to the above observation, $c_r\bar{z}_r = \sum\limits_{I,s} c_{r,I,s}z'_{r,I,s}$ and thus we get a feasible solution to the LP relaxation of (2) with the same cost as the optimal solution to (5).

3 Proof of Proposition 4

Suppose there exists L^1 satisfying conditions 2–4. Let $(P, v_P) \in E(L^2)$. For any $v_2 \in S_v^2$ such that $(\bar{v}^2, v_2, v_P)$ is a feasible speed vector over $P^2 \oplus P$, conditions 1, 2, 3, and (4-a) ensure that there exists $v_1 \in S_v^1$ such that $(\bar{v}^1, v_1, v_P)$ is a feasible speed vector over $P^1 \oplus P$. Therefore, $(P, v_P) \in E(L^1)$.

We are going to show that for any triple $(P^2 \oplus P, I^2, \mathbf{s}^2)$ induced by $v_2 \in S_v^2$ that leads to a feasible speed vector $(\bar{v}^2, v_2, v_P)$ on $P^2 \oplus P$, there exists a triple $(P^1 \oplus P, I^3, \mathbf{s}^3)$ with a smaller reduced cost. It is sufficient to consider the optimal speed vector $\mathbf{v}^*$ over $P^2 \oplus P$ that is consistent with label L^2, i.e., the components of $\mathbf{v}^*$ from the depot to vertex i_{w^2} are $\bar{v}^2$, and customers after i_{w^2} and before i^2 (including i^2) are all seamless. Let v_2^* be the component of $\mathbf{v}^*$ corresponding to the speed over the arc entering vertex i^2. Then the vehicle travels along $P^2 \oplus P$ in the following manner: start from the depot to vertex i_{w^2} with speed $\bar{v}^2$, leave from vertex i_{w^2} to vertex i^2 with speed v_2^*, and leave from vertex i^2 back to the depot with speeds $\mathbf{v}_P^*$, the components of $\mathbf{v}^*$ that correspond to P. Assume that the vehicle returns to the depot at time $t_{P^2\oplus P}$, and the optimal cost on P is F^*. Thus the total cost of route $P^2 \oplus P$ with speed vector $\mathbf{v}^*$ is $C^2(v_2) + F^*$.

Now we show how the route $P^1 \oplus P$ admits a triple $(P^1 \oplus P, I^3, \mathbf{s}^3)$ with a better reduced cost than $(P^2 \oplus P, I^2, \mathbf{s})$ corresponding to the speed vector $\mathbf{v}^*$. Since $v_2 \in S_v^2$, by (4-a) there exists $v_1 \in S_v^1$ such that $T^1(v_1) \le T^2(v_2)$. Therefore, it is feasible to travel along $P^1 \oplus P$ in the following manner: travel from the depot to vertex i_{w^1} along P^1 with speed vector $\bar{v}^1$, travel from vertex i_{w^1} to vertex i^1 with speed v_1, finish serving vertex i^1 at $T^1(v_1)$, leave vertex i^1 at time $T^2(v_2)$, travel along P using speeds given by the corresponding components in $\mathbf{v}^*$, and return to the depot at time $t_{P^2\oplus P}$. Then the total cost of traveling along route $P^1 \oplus P$ in the above manner is $C^1(v_1) + F^*$. Figure 1 illustrates the comparison between $P^1 \oplus P$ and $P^2 \oplus P$.

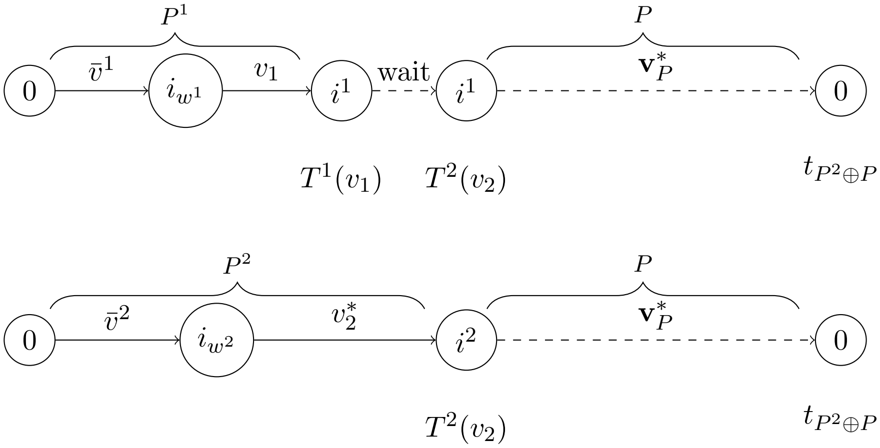

Figure 1: An illustration of comparison between $P^1 \oplus P$ and $P^2 \oplus P$.

Note that traveling along $P^1 \oplus P$ in the above manner does not necessarily respect the set of active customers in L^1. Nonetheless it is a feasible way to travel along $P^1 \oplus P$. Then there must exists a triple $(P^1 \oplus P, I^3, \mathbf{s}^3)$ such that $c_{P^1\oplus P, I^3, \mathbf{s}^3}$ is at most $C^1(v_1) + F^*$. Thus

$$\begin{aligned}
&\bar{c}_{P^1\oplus P, I^3, \mathbf{s}^3} - \bar{c}_{P^2\oplus P, I^2, \mathbf{s}} \le [C^1(v_1) + F^* - (\mu^1 + \mu + \nu)] \\
&-[C^2(v_2) + F^* - (\mu^2 + \mu + \nu)] \\
&=[C^1(v_1) - \mu^1] - [C^2(v_2) - \mu^2] \\
&<0.
\end{aligned}$$

The last inequality follows from condition (4-b).

4 Proof of Proposition 6

To check if Condition (4-b) holds is equivalent to solve a one-dimensional constrained optimization problem with decision variable v_2. To see this, fix $v_2 \in [v_2^{\min}, v_2^{\max}]$ and define

$$\phi(v_2) = \min\ C^1(v_1) \tag{2a}$$

$$\text{s.t. } T^1(v_1) = s^1 + \Gamma^1 + \frac{D^1}{v_1} \le s^2 + \Gamma^2 + \frac{D^2}{v_2} = T^2(v_2) \tag{2b}$$

$$v_1^{\min} \le v_1 \le v_1^{\max}. \tag{2c}$$

Constraint (2b) is equivalent to $v_1 \ge \beta(v_2)$. Based on the assumption that Condition (4-a) holds and Proposition 1 in the Appendix, we have $\beta(v_2^{\max}) \le v_1^{\max}$. Thus $\beta(v_2) \le v_1^{\max}$ for any $v_2 \in S_v^2$. Therefore, $\phi(v_2) = C^1(v_1^{\min})$ if $\beta(v_2) \le v_1^{\min}$ and $\phi(v_2) = C^1(\beta(v_2))$ otherwise. Define

$$\psi(v_2) = \phi(v_2) - C^2(v_2)$$
$$z^* = \max\{\psi(v_2) \mid v_2 \in [v_2^{\min}, v_2^{\max}]\}$$

Then (4-b) is satisfied if and only if

$$z^* - \mu^1 + \mu^2 < 0.$$

We consider the following three cases to compute z^*.

1. $T^1(v_1^{\min}) \le T^2(v_2^{\max})$. Then for any $v \in [v_2^{\min}, v_2^{\max}]$, $v_1^{\min} \ge \beta(v)$ and $\phi(v) = C^1(v_1^{\min})$. Thus $\psi(v) = C^1(v_1^{\min}) - C^2(v)$ and $z^* = C^1(v_1^{\min}) - C^2(v_2^{\min})$.

2. $T^2(v_2^{\max}) < T^1(v_1^{\min}) \le T^2(v_2^{\min})$. Then $\beta(v_2^{\min}) \le v_1^{\min} < \beta(v_2^{\max})$. Since β is monotonically increasing by Proposition 1 in the Appendix, there exists $\tilde{v} \in [v_2^{\min}, v_2^{\max}]$ such that $\beta(\tilde{v}) = v_1^{\min}$. In particular, $\tilde{v} = D^2 v_1^{\min}/(D^1 - \delta v_1^{\min})$. Then $\phi(v) = C^1(v_1^{\min})$ for $v \in [v_2^{\min}, \tilde{v}]$, and $\phi(v) = C^1(\beta(v))$ for $v \in [\tilde{v}, v_2^{\max}]$.

   When $v \in [v_2^{\min}, \tilde{v}]$, the function $\psi(v) = C^1(v_1^{\min}) - C^2(v)$. Its maximum is attained at $C^1(v_1^{\min}) - C^2(v_2^{\min})$. When $v \in [\tilde{v}, v_2^{\max}]$, the function $\psi(v) = C^1(\beta(v)) - C^2(v) = H(v)$, which attains its maximum at either $\tilde{v}, v_2^{\max}$, or v^* if it is well defined and $v^* \in [\tilde{v}, v_2^{\max}]$. Note that $\psi(\tilde{v}) = C^1(v_1^{\min}) - C^2(\tilde{v}) \le C^1(v_1^{\min}) - C^2(v_2^{\min})$. Therefore from Lemmas 1 and 2 in the Appendix, z^* equals to the maximum of $C^1(v_1^{\min}) - C^2(v_2^{\min})$, $C^1(\beta(v_2^{\max})) - C^2(v_2^{\max})$, or $C^1(v^*) - C^2(v^*)$ if v^* is well-defined and $v^* \in [\tilde{v}, v_2^{\max}]$.

3. $T^2(v_2^{\min}) < T^1(v_1^{\min})$. Then for any $v_2 \in [v_2^{\min}, v_2^{\max}]$, $\beta(v) > v_1^{\min}$ and $\phi(v) = C^1(\beta(v))$. Thus $\psi(v) = C^1(\beta(v)) - C^2(v) = H(v)$ for any $v \in [v_2^{\min}, v_2^{\max}]$. Therefore from Lemmas 1 and 2 in the Appendix, the maximum of $H(v)$ is attained at $v_2^{\min}$, $v_2^{\max}$, or v^* if v^* is well-defined and $v^* \in [v_2^{\min}, v_2^{\max}]$.

5 Properties of functions β and H

The following proposition summarizes some properties of function β, where are easy to verify.

Proposition 1.

- *For any $v \in [v_2^{\min}, v_2^{\max}]$ and $\beta(v) \in [v_1^{\min}, v_1^{\max}]$, $T^1(\beta(v)) = T^2(v)$.*

- *Function β is monotonically increasing.*

- *$T^1(v_1) \le T^2(v_2)$ if and only if $\beta(v_2) \le v_1$. In particular, $T^1(v_1^{\max}) \le T^2(v_2^{\max})$ if and only if $\beta(v_2^{\max}) \le v_1^{\max}$.*

Lemma 1. *When $v \in [l, u]$, the derivative of $H(v)$ is 0 if and only if $\beta(v) = v$. If v^* is well-defined, then the derivative of $H(v)$ equals to 0 at $v = v^*$.*

Proof. Since $H(v) = F^1_{\text{speed}} + D^1 f(\beta(v)) - F^2_{\text{speed}} - D^2 f(v)$, its derivative

$$\begin{aligned} H'(v) &= D^1 f'(\beta(v))\beta'(v) - D^2 f'(v) \\ &= D^1 f'(\beta(v)) \frac{D^1 D^2}{(D^2/v + \delta)^2 v^2} - D^2 f'(v) \\ &= D^2 f'(\beta(v)) (\frac{D^1}{D^2/v + \delta})^2 \frac{1}{v^2} - D^2 f'(v) \\ &= D^2 f'(\beta(v)) \frac{(\beta(v))^2}{v^2} - D^2 f'(v). \end{aligned}$$

Thus $H'(v) = 0$ if and only if $f'(\beta(v))(\beta(v))^2 - G'(v)v^2 = 0$. The function $f'(v)v^2$ is strictly increasing for $v \in [v_F, u]$, since $f'(v)$ is non-negative and strictly increasing in $[v_F, u]$ and v^2 is non-negative and strictly increasing in $[0, \infty)$. When $v \in [v_F, u]$, $f'(\beta(v))(\beta(v))^2 - f'(v)v^2 = 0$ if and only if $\beta(v) = v$. When v^* is well-defined, $\beta(v) = v$ if and only if $v = v^*$. ☐

Lemma 2. *Suppose $\delta = 0$. Then the maximum of $H(v)$ over an interval $[v^{\min}, v^{\max}]$ with $v^{\min} > 0$ is attained at $v^{\max}$ if $D^1 > D^2$, at $v^{\min}$ if $D^1 < D^2$, and at any point in $[v^{\min}, v^{\max}]$ if $D^1 = D^2$.*

Proof. When $D^1 = D^2$, $\beta(v) = v$ for any v and $H(v)$ is constant. When $D^1 > D^2$, $\beta(v) > v$ and $H'(v) > 0$ for any $v \in [v^{\min}, v^{\max}]$, based on the proof of Lemma 1. Thus $H(v)$ is strictly increasing and its maximum is attained at $v^{\max}$. The case of $D^1 < D^2$ can be proved in a similar way. ☐

6 Detailed computational results of the BCP algorithms with different route choices for the PRP instances

	Elementary				2-cycle-free				q-route			
	LP				LP				LP			
Instance	Value	Time	BUB	Time	Value	Time	BUB	Time	Value	Time	BUB	Time
UK10A-1	170.64	0.48	170.64	0.5	170.64	1.43	170.64	1.4	170.64	1.81	170.64	1.8
UK10A-2	204.88	0.26	204.88	0.3	204.88	1.00	204.88	1.0	204.88	1.18	204.88	1.2
UK10A-3	200.34	0.70	200.34	0.7	200.34	0.88	200.34	0.9	200.34	0.53	200.34	0.5
UK10A-4	189.88	0.22	189.88	0.2	189.88	1.35	189.88	1.4	189.88	1.40	189.88	3.5
UK10A-5	175.59	0.50	175.59	0.5	175.59	3.40	175.59	3.4	175.59	2.85	175.59	2.9
UK10A-6	214.48	0.24	214.48	0.2	214.48	0.53	214.48	0.5	214.48	0.78	214.48	0.8
UK10A-7	190.14	0.12	190.14	0.1	190.14	1.74	190.14	1.7	190.14	2.81	190.14	2.8
UK10A-8	222.17	0.03	222.17	0.0	222.17	0.15	222.17	0.1	222.17	0.14	222.17	0.1
UK10A-9	174.54	0.12	174.54	0.1	174.54	2.07	174.54	2.1	174.54	6.69	174.54	6.7
UK10A-10	189.82	0.39	189.82	0.4	189.82	0.42	189.82	0.4	189.82	0.59	189.82	0.6
UK10B-1	246.44	0.01	246.44	0.0	246.44	0.01	246.44	0.0	246.44	0.01	246.44	0.0
UK10B-2	303.73	0.01	303.73	0.0	303.73	0.01	303.73	0.0	303.73	0.01	303.73	0.0
UK10B-3	301.89	0.01	301.89	0.0	301.89	0.01	301.89	0.0	301.89	0.01	301.89	0.0
UK10B-4	273.90	0.01	273.90	0.0	273.90	0.01	273.90	0.0	273.90	0.02	273.90	0.0
UK10B-5	255.07	0.01	255.07	0.0	255.47	0.00	255.10	0.0	255.47	0.00	255.07	0.0
UK10B-6	332.34	0.01	332.34	0.0	332.34	0.01	332.34	0.0	332.34	0.01	332.34	0.0
UK10B-7	308.85	0.00	314.64	0.1	308.85	0.00	314.64	0.1	304.91	0.00	314.64	0.1
UK10B-8	333.12	0.00	339.32	0.0	333.11	0.00	339.32	0.0	333.12	0.00	339.32	0.0
UK10B-9	261.10	0.01	261.10	0.0	261.10	0.00	261.10	0.0	261.10	0.01	261.10	0.0
UK10B-10	285.20	0.01	285.20	0.0	285.20	0.01	285.20	0.0	285.20	0.00	285.20	0.0
UK10C-1	210.21	0.02	210.21	0.0	210.18	0.05	210.18	0.1	209.66	0.10	210.18	0.2
UK10C-2	271.93	0.03	271.93	0.0	268.93	0.10	271.93	0.3	265.77	0.20	271.93	1.1
UK10C-3	229.18	0.03	229.18	0.0	229.18	0.05	229.18	0.1	227.41	0.20	229.18	0.5
UK10C-4	230.52	0.01	230.52	0.0	227.31	0.00	230.52	0.2	222.99	0.10	230.52	6.2
UK10C-5	205.49	0.02	205.49	0.0	205.49	0.08	205.49	0.1	203.33	0.30	205.49	0.8
UK10C-6	255.82	0.02	255.82	0.0	255.82	0.03	255.82	0.0	254.48	0.10	255.82	0.1
UK10C-7	217.79	0.03	217.79	0.0	217.79	0.07	217.79	0.1	217.79	0.33	217.79	0.3
UK10C-8	251.29	0.03	251.29	0.0	251.29	0.02	251.29	0.0	251.29	0.10	251.29	0.1
UK10C-9	186.04	0.06	186.04	0.1	186.04	0.12	186.04	0.1	186.04	0.09	186.04	0.1
UK10C-10	231.62	0.04	231.62	0.0	231.62	0.02	231.62	0.0	230.93	0.10	231.62	0.2

Table 1: Results of the BCP algorithms under different route choices for the PRP instances with 10 customers

	elementary				2-cycle-free				q-route			
	LP				LP				LP			
Instance	Value	Time	BUB	Time	Value	Time	BUB	Time	Value	Time	BUB	Time/ Gap
UK20A-1	351.82	26.5	351.82	26.5	351.69	41.6	351.82	64.0	350.46	14.5	351.82	19.4
UK20A-2	365.77	31.5	365.77	31.5	365.77	16.7	365.77	16.7	365.77	4.68	365.77	4.7
UK20A-3	230.49	1204	230.49	1204	230.49	556	230.49	556	230.49	52.9	230.49	52.9
UK20A-4	347.04	2370	347.04	2370	347.04	659	347.04	659	347.04	59.4	347.04	59.4
UK20A-5	323.44	154	323.44	154	323.44	134	323.44	134	323.31	17.3	323.44	28.7
UK20A-6	364.11	437	364.23	757	364.11	40.5	364.23	94.6	363.67	6.00	364.23	17.0
UK20A-7	-	-	3318.70	-	-	-	3016.70	-	253.43	2429	253.61	0.1%
UK20A-8	301.51	151	301.51	151	301.51	50.1	301.51	50.1	301.51	15.9	301.51	15.9
UK20A-9	362.56	38.6	362.56	38.6	362.56	98.3	362.56	98.3	362.56	11.4	362.56	11.4
UK20A-10	313.33	455	313.33	455	313.33	231	313.33	231	313.33	16.7	313.33	16.7
UK20B-1	469.35	0.04	469.35	0.0	469.35	0.04	469.35	0.0	469.35	0.04	469.35	0.0
UK20B-2	477.05	0.18	477.05	0.2	477.05	0.06	477.05	0.1	477.05	0.10	477.05	0.1
UK20B-3	354.46	0.11	354.46	0.1	354.46	0.11	354.46	0.1	354.46	0.11	354.46	0.1
UK20B-4	523.59	0.12	523.59	0.1	523.59	0.08	523.59	0.1	523.59	0.15	523.59	0.1
UK20B-5	447.33	0.08	447.33	0.1	447.33	0.07	447.33	0.1	439.38	0.20	447.33	1.2
UK20B-6	511.11	0.40	511.78	1.4	511.11	0.10	511.78	0.4	511.78	1.34	511.78	1.3
UK20B-7	377.94	1.60	379.02	4.0	379.02	0.62	379.02	0.6	377.57	0.30	379.02	0.4
UK20B-8	418.00	0.10	431.31	0.3	418.00	0.10	431.31	0.6	416.64	0.10	431.31	1.7
UK20B-9	545.37	0.10	548.68	0.1	545.37	0.10	548.68	0.1	548.68	0.23	548.68	0.2
UK20B-10	410.32	0.04	410.32	0.0	410.32	0.03	410.32	0.0	407.96	0.10	410.32	0.2
UK20C-1	432.35	0.73	432.35	0.7	424.06	1.30	432.35	90.3	420.38	1.60	432.35	0.9%
UK20C-2	445.64	2.00	448.29	12.9	444.96	1.30	448.29	4.4	442.07	1.00	448.29	7.7
UK20C-3	287.48	4.39	287.48	4.4	287.20	2.90	287.48	4.1	281.39	3.40	287.04	76.2
UK20C-4	432.55	0.90	434.23	2.9	431.69	0.90	434.23	4.4	422.25	1.5	434.23	0.7%
UK20C-5	380.28	1.70	381.70	6.1	379.85	1.40	381.70	4.5	375.64	1.60	381.70	41.9
UK20C-6	441.43	2.50	444.35	23.4	439.28	1.10	444.35	28.9	438.56	0.80	444.35	16.3
UK20C-7	317.73	800	317.73	800	317.44	52.0	317.73	63.1	308.47	34.8	317.73	0.9%
UK20C-8	410.35	0.57	410.35	0.6	410.35	0.52	410.35	0.5	410.16	1.00	410.35	1.4
UK20C-9	421.39	0.25	421.39	0.2	421.39	0.49	421.39	0.5	410.48	1.30	421.39	44.3
UK20C-10	384.88	0.54	384.88	0.5	380.30	1.20	384.88	13.7	373.95	1.90	384.88	130

Table 2: Results of the BCP algorithms under different route choices for the PRP instances with 20 customers

	Elementary				2-cycle-free				q-route			
	LP				LP				LP			
Instance	Value	Time	BUB	Time/ Gap	Value	Time	BUB	Time/ Gap	Value	Time	BUB	Time/ Gap
UK25A-1	-	-	3316.14	-	-	-	389.46	-	316.18	304	316.18	304
UK25A-2	-	-	397.10	-	-	-	373.92	-	373.92	1244	373.92	1562
UK25A-3	-	-	268.46	-	-	-	273.08	-	-	-	3591.44	-
UK25A-4	-	-	677.11	-	-	-	321.14	-	-	-	880.80	-
UK25A-5	-	-	3932.00	-	365.00	2925	365.00	2925	365.00	169	365.00	169
UK25A-6	-	-	397.30	-	-	-	313.72	-	312.22	552	316.83	1.2%
UK25A-7	-	-	467.97	-	-	-	3431.12	-	350.72	610	350.72	610
UK25A-8	359.08	1432	359.08	1432	359.08	1107	359.08	1107	358.69	84.1	359.08	118
UK25A-9	-	-	346.40	-	-	-	380.68	-	325.91	184	330.60	0.9%
UK25A-10	-	-	411.33	-	-	-	411.33	-	411.33	3503	411.33	3503
UK25B-1	474.35	2.20	475.13	6.5	474.35	0.60	475.13	1.0	473.63	0.50	475.13	1.3
UK25B-2	530.00	8.60	533.59	141	530.00	0.50	533.59	10.6	533.59	63.6	533.59	63.6
UK25B-3	390.17	15.7	390.27	37.9	390.17	1.80	390.27	2.6	388.45	1.90	390.27	9.1
UK25B-4	467.49	239	467.49	239	458.56	0.80	467.49	132	449.56	0.80	467.49	1.4%
UK25B-5	489.16	192	491.69	1286	489.16	2.50	491.69	11.2	488.80	1.60	491.69	10.0
UK25B-6	515.46	1.00	516.71	16.0	515.46	0.30	516.71	1.5	514.11	0.40	516.71	6.5
UK25B-7	525.68	120	526.12	164	524.90	0.70	526.12	2.3	525.69	0.40	526.12	2.0
UK25B-8	534.38	1.60	537.93	126	533.59	0.40	537.15	3.5	533.38	0.50	537.15	3.1
UK25B-9	436.23	218	436.84	464	436.23	1.70	436.84	2.8	434.89	1.40	436.84	3.5
UK25B-10	531.86	13.5	533.88	126	531.86	1.00	533.88	5.0	531.10	0.90	533.88	3.4
UK25C-1	417.59	1826	417.59	1826	416.88	52.9	417.59	163	409.86	61.9	417.59	0.7%
UK25C-2	467.87	1506	468.95	0.1%	467.51	53.7	468.95	548	453.86	25.3	481.59	4.0%
UK25C-3	-	-	425.39	-	326.31	329	328.47	0.1%	323.41	55.0	329.34	1.1%
UK25C-4	388.98	48.4	389.56	77.8	386.21	3.60	391.34	1.3%	367.54	33.4	391.11	4.4%
UK25C-5	-	-	1444.90	-	-	-	458.55	-	-	-	456.58	-
UK25C-6	415.31	436	418.12	0.3%	415.29	8.80	418.12	135	412.09	5.30	418.12	0.4%
UK25C-7	-	-	727.00	-	478.25	48.1	482.31	0.4%	473.62	13.8	481.75	1.2%
UK25C-8	497.49	158	499.18	560	496.31	8.00	499.18	30.5	481.75	4.40	499.18	1.1%
UK25C-9	406.22	109	413.28	1.0%	405.99	7.50	412.51	520	400.25	6.20	412.51	1.1%
UK25C-10	509.00	1441	551.17	8.3%	508.78	40.6	513.37	0.4%	497.67	4.80	513.47	1.5%

Table 3: Results of the BCP algorithms under different route choices for the PRP instances with 25 customers

	Elementary				2-cycle-free				q-routes			
	LP				LP				LP			
Instance	Value	Time	BUB	Time/ Gap	Value	Time	BUB	Time/ Gap	Value	Time	BUB	Time/ Gap
UK50A-1	-	-	2748.2	-	-	-	689.6	-	673.3	1884	6425.2	846%
UK50A-2	-	-	3119.9	-	-	-	929.3	-	668.5	3583	668.5	3583
UK50A-3	-	-	7512.3	-	-	-	7750.3	-	688.2	544	988.6	43%
UK50A-4	-	-	3653.8	-	-	-	4085.5	-	-	-	3845.7	-
UK50A-5	-	-	2847.0	-	-	-	7434.9	-	702.2	421	706.3	0.4%
UK50A-6	-	-	3464.4	-	-	-	6697.3	-	631.3	1621	631.3	1621
UK50A-7	-	-	833.9	-	-	-	7774.5	-	-	-	6758.5	-
UK50A-8	-	-	650.6	-	-	-	638.9	-	636.9	1052	638.5	0.1%
UK50A-9	-	-	7292.9	-	-	-	3728.1	-	-	-	786.0	-
UK50A-10	-	-	2182.7	-	-	-	7369.2	-	-	-	6533.2	-
UK50B-1	-	-	3055.38	-	984.90	6.80	992.15	0.3%	978.67	4.70	992.00	0.7%
UK50B-2	-	-	1887.07	-	1021.91	5.90	1028.01	0.0%	1018.19	4.80	1028.01	0.4%
UK50B-3	1013.47	1807	8857.58	774.0%	1013.44	8.40	1022.40	0.5%	1009.38	10.1	1022.69	0.8%
UK50B-4	1141.88	596	2624.56	129.8%	1141.85	5.50	1149.08	1985	1140.73	3.30	1149.08	3408
UK50B-5	963.84	61.6	964.72	185	963.84	4.00	964.72	6.8	960.57	1.70	964.72	14.8
UK50B-6	-	-	8962.33	-	911.64	14.5	921.95	0.8%	907.76	7.50	924.94	1.4%
UK50B-7	-	-	3407.17	-	847.84	16.2	852.33	1022	839.92	8.20	852.33	0.8%
UK50B-8	924.43	50.5	924.43	50.5	924.43	3.62	924.43	3.6	923.90	6.20	924.43	13.8
UK50B-9	1027.56	2472	1592.42	55.0%	1027.41	3.40	1034.92	2252	1023.36	2.50	1036.14	0.4%
UK50B-10	-	-	8504.07	-	1022.60	83.2	1026.80	0.2%	1017.29	17.9	1033.92	1.2%
UK50C-1	-	-	8400.51	-	900.91	308	1198.26	32.8%	886.64	57.3	1243.50	38.8%
UK50C-2	-	-	1846.43	-	884.84	771	1227.79	38.6%	872.69	73.1	1387.62	58.2%
UK50C-3	-	-	4949.76	-	851.70	680	1015.20	19.0%	844.60	65.8	869.18	2.5%
UK50C-4	-	-	7966.25	-	996.01	200	1003.43	0.5%	990.32	38.5	1002.20	0.8%
UK50C-5	916.44	2018	8394.03	815.9%	914.41	48.6	925.85	1.0%	902.56	22.1	1177.71	29.5%
UK50C-6	-	-	7557.40	-	792.78	1033	2607.89	228.5%	784.36	50.6	1569.29	99.4%
UK50C-7	-	-	1754.22	-	851.59	508	852.67	0.0%	834.81	64.0	895.22	6.4%
UK50C-8	-	-	826.66	-	789.03	463	1233.17	56.0%	782.84	87.6	2643.45	236.6%
UK50C-9	959.55	331	962.69	0.2%	957.10	32.5	962.69	0.2%	951.46	12.2	963.40	0.8%
UK50C-10	-	-	4547.35	-	-	-	8330.19	-	959.03	152	985.50	2.5%

Table 4: Results of the BCP algorithms under different route choices for the PRP instances with 50 customers

References

[1] Ricardo Fukasawa, Qie He, Fernando Santos, and Yongjia Song. A joint vehicle routing and speed optimization problem. To appear in *INFORMS Journal on Computing*, 2018.