

## 2D-Phase Unwrapping via Balanced Spanning Forests

### *Electronic Companion*

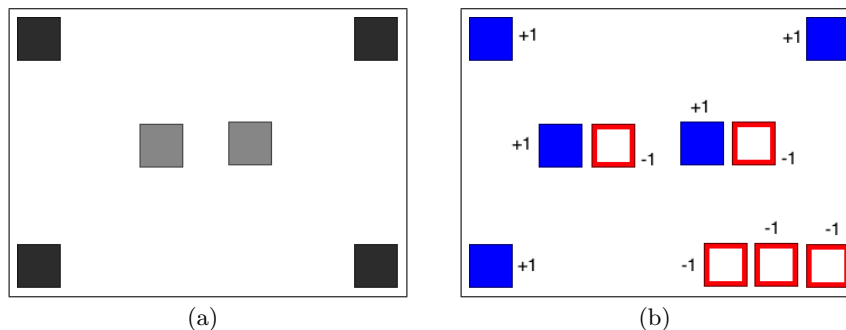
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### The MSFBCP is $\mathcal{NP}$ -hard

*Proof:* Let  $G = (V, E)$  be a graph with positive edge costs and a set of terminal vertices  $T \subseteq V$ . The Steiner-tree problem seeks to find a minimum-cost connected subgraph  $G' = (V', E')$  with  $T \subseteq V'$ . We can reduce an instance of the Steiner problem in graphs to an MSFBCP instance by assigning a positive weight of  $w = 1$  to all the terminal vertices, except for one, which is replaced by  $|T| - 1$  vertices with a negative weight of  $w = -1$ , connected to each other with cost zero. In addition, a pair of connected vertices with opposite signed weights is located for each of the  $(V - T)$  remaining vertices (Steiner points). The cost of connecting such a pair of vertices is also set to zero.

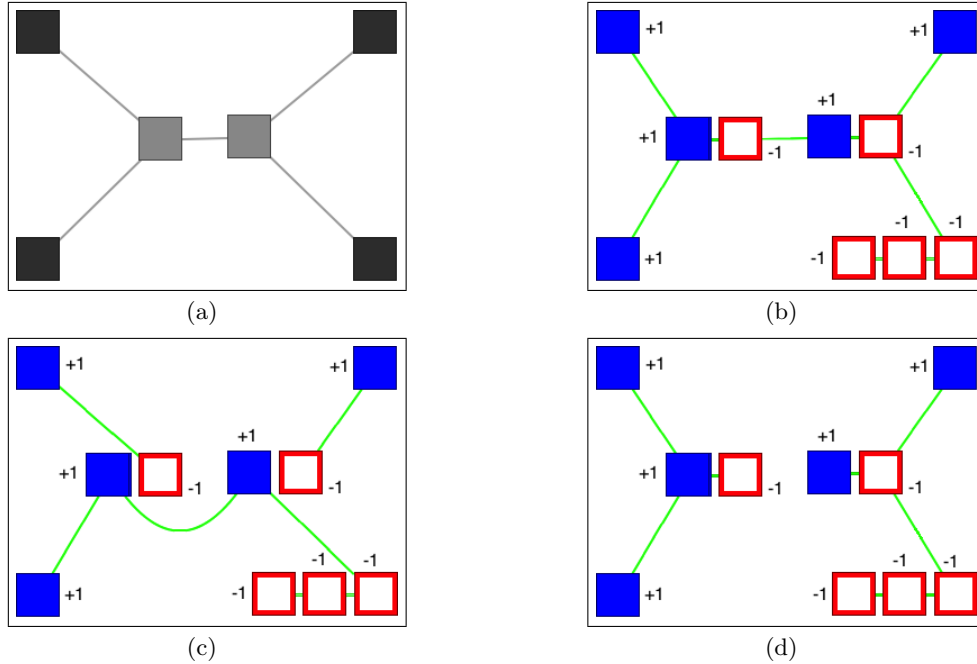


**Figure EC.1** Reduction of the Steiner-tree problem to the MSFBCP: (a) Steiner-tree instance, with four terminal vertices (black) and two Steiner-points (grey). (b) Equivalent instance for the MSFBCP. Positive residues are represented with solid blue squares, and negative residues are represented with empty red squares.

Figure EC.1 shows the reduction of a Steiner-tree instance with four terminal nodes and two Steiner nodes, while Figure 2(a) shows the optimal solution for the Steiner-tree problem. In the MSFBCP solution of the reduced instance, one of the following cases occurs:

1. A single balanced tree spans all the terminal vertices (Figure 2(b));
2. At least two disjoint balanced trees contain the terminal vertices (Figure 2(c)).

In the first case, the corresponding minimum spanning tree is an optimal Steiner-tree. All the remaining trees of the MSFBCP solution correspond to unused Steiner-points (2-node balanced trees with cost zero), which do not impact the solution cost ( $c(P) = w(G') + 0 = w(G')$ ). The MSFBCP solution is a feasible and optimal solution for the Steiner-tree problem.



**Figure EC.2** Optimal solution for the Steiner-tree problem and two feasible solutions for the MSFBCP: (a) Optimal solution for the Steiner-tree problem. (b) An optimal MSFBCP solution that is also feasible and optimal for the Steiner-tree problem. (c) An MSFBCP solution with the same cost that is infeasible for the Steiner-tree problem. (d) An infeasible MSFBCP solution.

In the second case, since there are disjoint trees containing the terminal vertices, we can assume that there is at least one unused edge of zero cost connecting a pair of vertices that replaces a Steiner point. Indeed, by contradiction, if all the zero-cost edges are used in the solution, then there must be at least one unbalanced tree in the disjoint set of trees. The connection of such a pair of vertices would add an excess of plus or minus one to that tree, making it unbalanced, as illustrated in Figure 2(d). Since the trees must be balanced, this solution would be infeasible for the MSFBCP.

We use this observation to generate in polynomial time another solution with the same cost, in which disjoint trees can be merged and at least one more zero-cost edge is included. Let  $T_1 = (V_1, E_1)$  and  $T_2 = (V_2, E_2)$  be two disjoint balanced trees containing terminal vertices and let  $E_{1,2}$  be the set of edges connecting vertices in  $T_1$  and  $T_2$ . We can use Kruskal’s algorithm to compute the minimum spanning tree over the merged component  $T_m = (V_m, E_m)$ , where  $V_m = \{V_1 \cup V_2\}$  and  $E_m = \{E_1 \cup E_2 \cup E_{1,2}\}$ . The algorithm enumerates the edge set in increasing order, and ties can be broken in such a way that zero-cost edges in  $E_{1,2}$  are treated first as well as edges in  $T_1$  and  $T_2$ . The addition of such edges will connect the disjoint components, and the cost of the new component will be given by

$$c(T_m) = c(T_1) + c(T_2) + 0 = c(T_1) + c(T_2). \quad (\text{EC.1})$$

This “merge” procedure can be executed iteratively until a single tree spans all the terminal vertices, resulting in the first case where  $c(P) = w(G')$ . Since we are able to build a valid and optimal Steiner-tree by solving the MSFBCP on the reduced instance and performing a limited number of calls to a polynomial “merge” algorithm, we prove by extension that the MSFBCP is also  $\mathcal{NP}$ -hard.

## Comparison of linear relaxations

Let  $Z_{\text{DIR}}$  and  $Z_{\text{UNDIR}}$  be the linear relaxations of the directed and undirected MSFBCP cut formulations, respectively. We will show that  $Z_{\text{DIR}} \geq Z_{\text{UNDIR}}$ , and that this inequality is strict for some instances. The undirected formulation of the MSFBCP is formulated in Equations (EC.2)–(EC.4). A binary variable  $\bar{x}_{ij}$  is associated to each edge  $\{i, j\} \in E$ , and for each cut  $S \subset V$ ,  $\delta(S) = \{\{i, j\} \in E : |\{i, j\} \cap S| = 1\}$  is the set of edges with exactly one endpoint in  $S$ .

$$\min \sum_{\{i,j\} \in E} d_{ij} \bar{x}_{ij} \quad (\text{EC.2})$$

$$\text{s.t.} \quad \sum_{\{i,j\} \in \delta(S)} \bar{x}_{ij} \geq 1 \quad \forall S \subset V : w(S) \neq 0 \quad (\text{EC.3})$$

$$\bar{x}_{ij} \in \{0, 1\} \quad \forall \{i, j\} \in E \quad (\text{EC.4})$$

First, observe that for every feasible solution  $\mathbf{x}$  of the directed formulation, the solution  $\bar{\mathbf{x}}$  defined as  $\bar{x}_{ij} = x_{ij} + x_{ji}$  for  $\{i, j\} \in E$  satisfies Equations (EC.2)–(EC.4), and has identical cost. As such,  $Z_{\text{DIR}} \geq Z_{\text{UNDIR}}$ . Second, to highlight an instance where  $Z_{\text{DIR}} > Z_{\text{UNDIR}}$ , we adapt a classic example in the literature, illustrated in left side of Figure EC.3. For this instance, the linear relaxation of the undirected formulation has value 1.5, and that of the directed formulation has value 2.

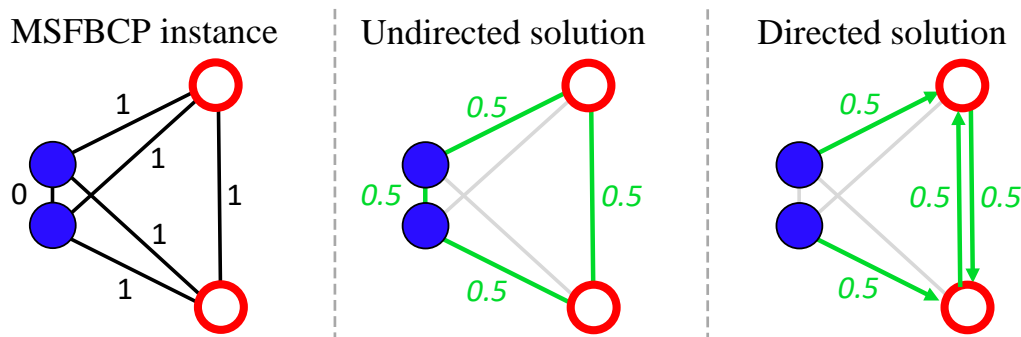


Figure EC.3 Example instance for which  $Z_{\text{dir}} > Z_{\text{undir}}$

## Detailed computational results

This section presents the detailed results for individual instances for all the benchmark sets considered in this paper. To evaluate the methods, we collected all the known optimal solutions and the best solutions found throughout our tests. These values are presented in the column **BKS**. We use asterisks to indicate optimal solutions. The best known primal solutions are highlighted in bold.

Tables EC.1 to EC.3 present the HILS results for each instance. The value  $\mathbf{Z}_{\text{Best}}$  is the best solution found in 10 runs, while  $\mathbf{Z}_{\text{Avg}}$  is the average solution over these runs. The columns  $\mathbf{GAP}_{\text{best}}$  (%) and  $\mathbf{GAP}_{\text{Avg}}$  (%) give the percentage gap between the solution and the BKS of the best and average solutions of the 10 runs. Finally, the column  $\mathbf{T}(\mathbf{s})$  gives the average CPU time per run.

Tables EC.4 to EC.8 present the branch-and-cut results for every instance. The time limit is set to 3600 seconds. The following information is reported:

- $\mathbf{LP}_{\mathbf{ROOT}}$ : the LP value at the root node;
- $\mathbf{GAP}_{\mathbf{ROOT}}$ : the percentage gap between the root node LP and the best upper bound
- $\mathbf{LB}$  and  $\mathbf{UB}$ : the best lower and upper bounds obtained;
- $\mathbf{GAP}_{\mathbf{FINAL}}$ : the percentage gap between the best upper and lower bounds;
- $\mathbf{T}(\mathbf{s})$ : the CPU time measured in seconds;
- $\mathbf{T}_{\mathbf{FLOW}}(\mathbf{s})$ : the total CPU time spent in the separation algorithm;
- $\mathbf{T}_{\mathbf{ROOT}}(\mathbf{s})$ : the total CPU time spent solving the root node;
- $\mathbf{N}$ : the number of nodes explored in the branch-and-bound tree;
- $\mathbf{N}_{\mathbf{TREE}}$ : the number of trees in the optimal solution (when available).
- $\mathbf{S}_{\mathbf{TREES}}$ : the average size of the trees in the optimal solution (when available).

### **Benchmark Instances**

The benchmark instances used in this study can be found in the online supplement or at the following address: <https://w1.cirrelt.ca/~vidalt/en/research-data.html>.

**Table EC.1 Detailed results for HILS algorithm**

Instance	$ V $	$ E $	$Z_{\text{BEST}}$	$\text{GAP}_{\text{BEST}}(\%)$	$Z_{\text{AVG}}$	$\text{GAP}_{\text{AVG}}(\%)$	BKS	Avg T(s)
PUC-8-1	10	90	<b>24.71</b>	0.00	<b>24.71</b>	0.00	24.71*	0.40
PUC-8-2	10	90	<b>16.46</b>	0.00	<b>16.46</b>	0.00	16.46*	0.25
PUC-8-3	10	90	<b>14.26</b>	0.00	<b>14.26</b>	0.00	14.26*	0.30
PUC-8-4	10	90	<b>24.99</b>	0.00	<b>24.99</b>	0.00	24.99*	0.23
PUC-8-5	10	90	<b>23.90</b>	0.00	27.29	12.42	23.90*	0.15
PUC-12-1	14	182	<b>54.54</b>	0.00	<b>54.54</b>	0.00	54.54*	0.86
PUC-12-2	14	182	<b>49.05</b>	0.00	<b>49.05</b>	0.00	49.05*	0.67
PUC-12-3	14	182	<b>53.59</b>	0.00	<b>53.59</b>	0.00	53.59*	0.72
PUC-12-4	14	182	<b>49.49</b>	0.00	<b>49.49</b>	0.00	49.49*	0.94
PUC-12-5	14	182	<b>46.94</b>	0.00	<b>46.94</b>	0.00	46.94*	0.66
PUC-16-1	18	306	<b>79.38</b>	0.00	<b>79.38</b>	0.00	79.38*	1.46
PUC-16-2	18	306	<b>69.68</b>	0.00	<b>69.68</b>	0.00	69.68*	1.46
PUC-16-3	18	306	<b>82.50</b>	0.00	82.86	0.43	82.50*	1.57
PUC-16-4	18	306	<b>79.01</b>	0.00	<b>79.01</b>	0.00	79.01*	1.46
PUC-16-5	18	306	<b>79.60</b>	0.00	81.38	2.19	79.60*	2.05
PUC-20-1	22	462	<b>128.16</b>	0.00	128.20	0.03	128.16*	2.15
PUC-20-2	22	462	<b>132.93</b>	0.00	<b>132.93</b>	0.00	132.93*	3.73
PUC-20-3	22	462	<b>112.17</b>	0.00	<b>112.17</b>	0.00	112.17*	3.59
PUC-20-4	22	462	<b>119.74</b>	0.00	120.97	1.02	119.74*	3.88
PUC-20-5	22	462	<b>121.93</b>	0.00	<b>121.93</b>	0.00	121.93*	4.64
PUC-24-1	26	650	<b>183.25</b>	0.00	183.97	0.39	183.25*	4.15
PUC-24-2	26	650	<b>124.07</b>	0.00	125.96	1.50	124.07*	6.30
PUC-24-3	26	650	<b>157.61</b>	0.00	157.64	0.02	157.61*	4.81
PUC-24-4	26	650	<b>163.07</b>	0.00	163.50	0.26	163.07*	5.67
PUC-24-5	26	650	<b>168.41</b>	0.00	170.16	1.03	168.41*	6.36
PUC-28-1	30	870	<b>203.86</b>	0.00	208.37	2.16	203.86*	8.87
PUC-28-2	30	870	<b>209.82</b>	0.00	215.03	2.42	209.82*	12.87
PUC-28-3	30	870	<b>215.69</b>	0.00	<b>215.69</b>	0.00	215.69*	7.89
PUC-28-4	30	870	<b>207.88</b>	0.00	208.13	0.12	207.88*	10.82
PUC-28-5	30	870	<b>209.48</b>	0.00	210.69	0.57	209.48*	12.04
PUC-32-1	34	1122	<b>259.67</b>	0.00	<b>259.67</b>	0.00	259.67*	12.33
PUC-32-2	34	1122	<b>237.03</b>	0.00	237.26	0.10	237.03*	16.99
PUC-32-3	34	1122	<b>222.04</b>	0.00	222.56	0.23	222.04*	16.07
PUC-32-4	34	1122	<b>234.29</b>	0.00	237.36	1.29	234.29*	13.36
PUC-32-5	34	1122	<b>284.18</b>	0.00	290.45	2.16	284.18*	15.66

**Table EC.2 Detailed results for HILS algorithm (continued)**

Instance	$ V $	$ E $	$Z_{\text{BEST}}$	$\text{GAP}_{\text{BEST}}(\%)$	$Z_{\text{AVG}}$	$\text{GAP}_{\text{AVG}}(\%)$	BKS	Avg T(s)
PUC-36-1	38	1406	<b>297.53</b>	0.00	297.70	0.06	297.53*	18.39
PUC-36-2	38	1406	<b>302.64</b>	0.00	303.97	0.44	302.64*	21.36
PUC-36-3	38	1406	<b>291.27</b>	0.00	292.30	0.35	291.27*	21.21
PUC-36-4	38	1406	<b>286.08</b>	0.00	296.15	3.40	286.08*	16.51
PUC-36-5	38	1406	<b>263.76</b>	0.00	264.96	0.45	263.76*	22.10
PUC-40-1	42	1722	<b>347.49</b>	0.00	351.70	1.20	347.49*	48.26
PUC-40-2	42	1722	<b>381.25</b>	0.00	<b>381.25</b>	0.00	381.25*	31.67
PUC-40-3	42	1722	<b>391.30</b>	0.00	394.76	0.88	391.30*	24.12
PUC-40-4	42	1722	<b>399.45</b>	0.00	402.28	0.70	399.45*	24.86
PUC-40-5	42	1722	<b>366.81</b>	0.00	368.18	0.37	366.81*	33.52
PUC-44-1	46	2070	462.14	0.68	473.89	3.14	459.01*	53.56
PUC-44-2	46	2070	<b>413.55</b>	0.00	417.59	0.97	413.55*	36.88
PUC-44-3	46	2070	<b>433.65</b>	0.00	439.33	1.29	434.67*	54.80
PUC-44-4	46	2070	<b>461.95</b>	0.00	464.01	0.44	461.95*	33.06
PUC-44-5	46	2070	<b>448.58</b>	0.00	454.66	1.34	448.58*	44.20
PUC-48-1	50	2450	<b>460.30</b>	0.00	464.82	0.97	460.75*	57.49
PUC-48-2	50	2450	<b>477.41</b>	0.00	480.90	0.73	477.41*	40.90
PUC-48-3	50	2450	<b>418.47</b>	0.00	421.27	0.66	418.47*	53.59
PUC-48-4	50	2450	<b>459.07</b>	0.00	463.97	1.06	459.07*	48.02
PUC-48-5	50	2450	<b>480.62</b>	0.00	481.82	0.25	480.62*	44.56
PUC-52-1	54	2862	<b>574.00</b>	0.00	584.11	1.73	574.00*	81.22
PUC-52-2	54	2862	<b>515.26</b>	0.00	526.52	2.14	515.26*	84.54
PUC-52-3	54	2862	<b>594.35</b>	0.00	595.50	0.19	594.35*	56.76
PUC-52-4	54	2862	<b>472.59</b>	0.00	475.03	0.51	472.59*	58.93
PUC-52-5	54	2862	<b>585.80</b>	0.00	598.28	2.09	585.80*	68.72
PUC-56-1	58	3306	<b>640.93</b>	0.00	657.44	2.51	640.93*	91.59
PUC-56-2	58	3306	<b>631.74</b>	0.00	636.72	0.78	631.74*	82.25
PUC-56-3	58	3306	<b>594.23</b>	0.00	594.43	0.03	594.23*	73.93
PUC-56-4	58	3306	<b>570.74</b>	0.00	576.15	0.94	570.74*	91.29
PUC-56-5	58	3306	<b>544.91</b>	0.00	558.62	2.45	544.91*	74.74
PUC-60-1	62	3782	<b>639.40</b>	0.00	648.90	1.46	639.40*	93.92
PUC-60-2	62	3782	<b>722.47</b>	0.00	736.34	1.88	722.47*	98.70
PUC-60-3	62	3782	<b>732.90</b>	0.00	746.47	1.82	732.90*	119.74
PUC-60-4	62	3782	<b>668.18</b>	0.00	669.94	0.26	668.18*	92.07
PUC-60-5	62	3782	<b>706.65</b>	0.00	709.07	0.34	706.65*	81.20
PUC-64-1	66	4290	802.77	2.03	834.17	5.68	786.81*	145.21

**Table EC.3 Detailed results for HILS algorithm (continued)**

Instance	$ V $	$ E $	$Z_{\text{BEST}}$	$\text{GAP}_{\text{BEST}}(\%)$	$Z_{\text{AVG}}$	$\text{GAP}_{\text{AVG}}(\%)$	BKS	Avg T(s)
PUC-64-2	66	4290	<b>748.66</b>	0.00	797.03	6.07	748.66*	126.61
PUC-64-3	66	4290	<b>754.85</b>	0.00	757.45	0.34	754.85*	131.41
PUC-64-4	66	4290	<b>805.77</b>	0.00	825.49	2.39	805.77*	141.08
PUC-64-5	66	4290	<b>789.85</b>	0.00	796.73	0.86	789.85*	127.95
PUC-80-1	82	6642	1089.77	1.20	1102.81	2.36	1076.83*	280.25
PUC-80-2	82	6642	1084.38	0.80	1124.80	4.36	1075.81*	295.59
PUC-80-3	82	6642	<b>1048.91</b>	0.00	1079.11	2.80	1048.91*	266.89
PUC-80-4	82	6642	<b>1114.46</b>	0.00	1135.72	1.87	1114.46*	329.72
PUC-80-5	82	6642	<b>1122.35</b>	0.00	1180.16	4.90	1122.35*	350.77
PUC-96-1	98	9506	<b>1348.94</b>	0.00	1421.62	5.11	1348.94*	643.02
PUC-96-2	98	9506	1613.75	1.22	1698.21	6.12	1594.22*	892.48
PUC-96-3	98	9506	1511.63	0.48	1569.20	4.13	1504.42*	514.21
PUC-96-4	98	9506	1495.14	0.51	1583.95	6.09	1487.52*	706.41
PUC-96-5	98	9506	<b>1422.95</b>	0.00	1485.31	4.20	1422.95*	494.14
PUC-128-1	130	16770	2450.87	1.38	2579.99	6.30	2417.43*	2482.93
PUC-128-2	130	16770	<b>2166.76</b>	0.00	2322.01	6.69	2166.76*	1099.27
PUC-128-3	130	16770	2576.42	7.56	2624.88	8.75	2395.29*	3348.55
PUC-128-4	130	16770	2293.61	0.10	2485.22	7.81	2291.24*	1735.70
PUC-128-5	130	16770	<b>2269.25</b>	0.00	2482.12	8.58	2269.25*	1791.81
PUC-256-1	258	66306	<b>7061.17</b>	0.00	7643.75	7.62	7061.17	3600.00
PUC-256-2	258	66306	<b>7748.26</b>	0.00	8008.76	3.25	7748.27	3600.00
PUC-256-3	258	66306	<b>7518.65</b>	0.00	7824.09	3.90	7518.65	3600.00
PUC-256-4	258	66306	<b>7530.24</b>	0.00	7539.17	0.12	7530.24	3600.00
PUC-256-5	258	66306	<b>6913.37</b>	0.00	8038.88	14.00	6913.38	3600.00
PUC-512-1	514	263682	<b>23517.25</b>	0.00	26140.36	10.03	23517.26	3600.00
PUC-512-2	514	263682	<b>23286.79</b>	0.00	23978.25	2.88	23286.79	3600.00
PUC-512-3	514	263682	<b>22874.35</b>	0.00	25420.15	10.01	22874.35	3600.00
PUC-512-4	514	263682	<b>23442.23</b>	0.00	24726.64	5.19	23442.23	3600.00
PUC-512-5	514	263682	<b>22863.50</b>	0.00	23672.32	3.42	22863.50	3600.00
PUC-1024-1	1026	1051650	<b>71011.17</b>	0.00	72375.26	1.88	71011.17	3600.00
PUC-1024-2	1026	1051650	<b>69510.15</b>	0.00	69890.54	0.54	69510.15	3600.00
PUC-1024-3	1026	1051650	<b>69401.36</b>	0.00	69523.12	0.18	69401.36	3600.00
PUC-1024-4	1026	1051650	<b>68040.88</b>	0.00	69141.38	1.59	68040.88	3600.00
PUC-1024-5	1026	1051650	<b>69884.32</b>	0.00	71384.11	2.10	69884.32	3600.00

**Table EC.4 Detailed results for B&C algorithm**

Group	LP <sub>ROOT</sub>	GAP <sub>ROOT</sub> (%)	LB	UB	GAP <sub>FINAL</sub> (%)	T(s)	T <sub>FLOW</sub> (s)	T <sub>ROOT</sub> (s)	N <sub>NODE</sub>	N <sub>TREE</sub>	S <sub>TREE</sub>
PUC-8-1	24.71	0.00	24.71	24.71	0.00	<0.01	<0.01	<0.01	1	2	5.00
PUC-8-2	16.46	0.00	16.46	16.46	0.00	<0.01	<0.01	<0.01	1	3	3.33
PUC-8-3	14.26	0.00	14.26	14.26	0.00	<0.01	<0.01	<0.01	1	2	5.00
PUC-8-4	24.99	0.00	24.99	24.99	0.00	<0.01	<0.01	<0.01	1	2	5.00
PUC-8-5	23.90	0.00	23.90	23.90	0.00	<0.01	<0.01	<0.01	1	2	5.00
PUC-12-1	54.54	0.00	54.54	54.54	0.00	<0.01	<0.01	<0.01	1	3	4.67
PUC-12-2	49.05	0.00	49.05	49.05	0.00	<0.01	<0.01	<0.01	1	5	2.80
PUC-12-3	53.59	0.00	53.59	53.59	0.00	<0.01	<0.01	<0.01	1	4	3.50
PUC-12-4	49.49	0.00	49.49	49.49	0.00	<0.01	<0.01	<0.01	1	3	4.67
PUC-12-5	46.94	0.00	46.94	46.94	0.00	<0.01	<0.01	<0.01	1	3	4.67
PUC-16-1	79.38	0.00	79.38	79.38	0.00	<0.01	<0.01	<0.01	1	4	4.50
PUC-16-2	69.68	0.00	69.68	69.68	0.00	<0.01	<0.01	<0.01	1	4	4.50
PUC-16-3	82.50	0.00	82.50	82.50	0.00	<0.01	<0.01	<0.01	1	5	3.60
PUC-16-4	79.01	0.00	79.01	79.01	0.00	<0.01	<0.01	<0.01	1	8	2.25
PUC-16-5	79.60	0.00	79.60	79.60	0.00	<0.01	<0.01	<0.01	1	3	6.00
PUC-20-1	128.16	0.00	128.16	128.16	0.00	<0.01	<0.01	<0.01	1	5	4.40
PUC-20-2	132.93	0.00	132.93	132.93	0.00	<0.01	<0.01	<0.01	1	6	3.67
PUC-20-3	112.17	0.00	112.17	112.17	0.00	<0.01	<0.01	<0.01	1	8	2.75
PUC-20-4	114.71	4.20	119.74	119.74	0.00	<0.01	<0.01	<0.01	1	7	3.14
PUC-20-5	119.73	1.81	121.93	121.93	0.00	<0.01	<0.01	<0.01	1	4	5.50
PUC-24-1	181.39	1.01	183.25	183.25	0.00	<0.01	<0.01	<0.01	1	6	4.33

Table EC.5 Detailed results for B&amp;C algorithm (continued)

Group	LP <sub>ROOT</sub>	GAP <sub>ROOT</sub> (%)	LB	UB	GAP <sub>FINAL</sub> (%)	T(s)	T <sub>FLOW</sub> (s)	T <sub>ROOT</sub> (s)	N <sub>NODE</sub>	N <sub>TREE</sub>	S <sub>TREE</sub>
PUC-24-2	124.07	0.00	124.07	124.07	0.00	<0.01	<0.01	<0.01	1	9	2.89
PUC-24-3	155.15	1.56	157.61	157.61	0.00	<0.01	<0.01	<0.01	1	9	2.89
PUC-24-4	160.97	1.28	163.07	163.07	0.00	<0.01	<0.01	<0.01	1	6	4.33
PUC-24-5	164.09	2.57	168.41	168.41	0.00	<0.01	<0.01	<0.01	1	7	3.71
PUC-28-1	194.91	4.39	203.86	203.86	0.00	<0.01	<0.01	<0.01	1	10	3.00
PUC-28-2	201.79	3.83	209.82	209.82	0.00	<0.01	<0.01	<0.01	1	8	3.75
PUC-28-3	213.65	0.94	215.69	215.69	0.00	<0.01	<0.01	<0.01	1	6	5.00
PUC-28-4	207.88	0.00	207.88	207.88	0.00	<0.01	<0.01	<0.01	1	7	4.29
PUC-28-5	208.35	0.54	209.48	209.48	0.00	<0.01	<0.01	<0.01	1	7	4.29
PUC-32-1	259.67	0.00	259.67	259.67	0.00	<0.01	<0.01	<0.01	1	10	3.40
PUC-32-2	233.93	1.31	237.03	237.03	0.00	<0.01	<0.01	<0.01	1	9	3.78
PUC-32-3	222.04	0.00	222.04	222.04	0.00	<0.01	<0.01	<0.01	1	12	2.83
PUC-32-4	226.04	3.52	234.29	234.29	0.00	<0.01	<0.01	<0.01	1	13	2.62
PUC-32-5	261.18	8.09	284.18	284.18	0.00	0.01	<0.01	0.01	1	9	3.78
PUC-36-1	280.47	5.74	297.53	297.53	0.00	<0.01	<0.01	<0.01	1	9	4.22
PUC-36-2	296.88	1.90	302.64	302.64	0.00	<0.01	<0.01	<0.01	1	10	3.80
PUC-36-3	282.69	2.94	291.27	291.27	0.00	<0.01	<0.01	<0.01	1	9	4.22
PUC-36-4	268.05	6.30	286.08	286.08	0.00	<0.01	<0.01	<0.01	1	13	2.92
PUC-36-5	257.21	2.49	263.76	263.76	0.00	<0.01	<0.01	<0.01	1	9	4.22
PUC-40-1	335.32	3.50	347.49	347.49	0.00	<0.01	<0.01	<0.01	1	11	3.82
PUC-40-2	369.86	2.99	381.25	381.25	0.00	<0.01	<0.01	<0.01	1	11	3.82
PUC-40-3	370.06	5.43	391.30	391.30	0.00	<0.01	<0.01	<0.01	1	11	3.82
PUC-40-4	394.30	1.29	399.45	399.45	0.00	2.34	0.19	0.05	55	14	3.00

**Table EC.6 Detailed results for B&C algorithm (continued)**

Group	LP <sub>ROOT</sub>	GAP <sub>ROOT</sub> (%)	LB	UB	GAP <sub>FINAL</sub> (%)	T(s)	T <sub>FLOW</sub> (s)	T <sub>ROOT</sub> (s)	N <sub>NODE</sub>	N <sub>TREE</sub>	S <sub>TREE</sub>
PUC-40-5	356.37	2.85	366.81	366.81	0.00	<0.01	<0.01	<0.01	1	16	2.63
PUC-44-1	451.81	1.57	459.01	459.01	0.00	9.52	0.75	0.06	113	10	4.60
PUC-44-2	405.29	2.00	413.55	413.55	0.00	<0.01	<0.01	<0.01	1	17	2.71
PUC-44-3	404.95	6.62	433.65	433.65	0.00	<0.01	<0.01	<0.01	1	13	3.54
PUC-44-4	446.29	3.39	461.95	461.95	0.00	<0.01	<0.01	<0.01	1	15	3.07
PUC-44-5	423.41	5.61	448.58	448.58	0.00	0.05	<0.01	0.05	1	14	3.29
PUC-48-1	437.59	4.93	460.30	460.30	0.00	<0.01	<0.01	<0.01	1	13	3.85
PUC-48-2	460.04	3.64	477.41	477.41	0.00	<0.01	<0.01	<0.01	1	19	2.63
PUC-48-3	411.55	1.65	418.47	418.47	0.00	<0.01	<0.01	<0.01	1	19	2.63
PUC-48-4	447.94	2.42	459.07	459.07	0.00	<0.01	<0.01	<0.01	1	16	3.13
PUC-48-5	470.43	2.12	480.62	480.62	0.00	<0.01	<0.01	<0.01	1	11	4.55
PUC-52-1	542.87	5.42	574.00	574.00	0.00	<0.01	<0.01	<0.01	1	20	2.70
PUC-52-2	501.23	2.72	515.26	515.26	0.00	<0.01	<0.01	<0.01	1	19	2.84
PUC-52-3	574.04	3.42	594.35	594.35	0.00	<0.01	<0.01	<0.01	1	17	3.18
PUC-52-4	447.91	5.22	472.59	472.59	0.00	<0.01	<0.01	<0.01	1	21	2.57
PUC-52-5	551.32	5.88	585.80	585.80	0.00	<0.01	<0.01	<0.01	1	17	3.17
PUC-56-1	605.68	5.50	640.93	640.93	0.00	0.02	<0.01	0.02	1	12	4.83
PUC-56-2	599.16	5.16	631.74	631.74	0.00	<0.01	<0.01	<0.01	1	18	3.22
PUC-56-3	566.63	4.64	594.23	594.23	0.00	<0.01	<0.01	<0.01	1	14	4.14
PUC-56-4	553.13	3.09	570.74	570.74	0.00	<0.01	<0.01	<0.01	1	22	2.64
PUC-56-5	530.04	2.73	544.91	544.91	0.00	<0.01	<0.01	<0.01	1	20	2.90
PUC-60-1	617.71	3.39	639.40	639.40	0.00	0.01	<0.01	0.01	1	20	3.10
PUC-60-2	717.76	0.65	722.47	722.47	0.00	1.07	0.09	0.19	9	17	3.65

Table EC.7 Detailed results for B&amp;C algorithm (continued)

Group	LP <sub>ROOT</sub>	GAP <sub>ROOT</sub> (%)	LB	UB	GAP <sub>FINAL</sub> (%)	T(s)	T <sub>FLOW</sub> (s)	T <sub>ROOT</sub> (s)	N <sub>NODE</sub>	N <sub>TREE</sub>	S <sub>TREE</sub>
PUC-60-3	732.62	0.04	732.90	732.90	0.00	0.17	0.01	0.09	3	22	2.82
PUC-60-4	642.54	3.84	668.18	668.18	0.00	<0.01	<0.01	<0.01	1	20	3.10
PUC-60-5	671.32	5.00	706.65	706.65	0.00	0.01	<0.01	0.01	1	19	3.26
PUC-64-1	786.81	0.00	786.81	786.81	0.00	2.68	0.11	0.40	13	17	3.88
PUC-64-2	712.84	4.78	748.66	748.66	0.00	0.02	<0.01	0.02	1	10	19.00
PUC-64-3	715.25	5.25	754.85	754.85	0.00	0.01	<0.01	0.01	1	9	18.00
PUC-64-4	798.87	0.86	805.77	805.77	0.00	12.75	0.46	0.33	33	18	3.67
PUC-64-5	786.26	0.45	789.85	789.85	0.00	0.61	0.05	0.14	3	10	25.00
PUC-80-1	1063.93	1.20	1076.83	1076.83	0.00	46.73	3.74	0.20	149	29	2.83
PUC-80-2	1049.38	2.46	1075.81	1075.81	0.00	405.18	37.57	0.21	1397	29	2.83
PUC-80-3	1043.09	0.56	1048.91	1048.91	0.00	2.13	0.25	0.22	9	25	3.28
PUC-80-4	1076.07	3.44	1114.46	1114.46	0.00	0.01	0.22	0.01	1	11	26.00
PUC-80-5	1110.42	1.06	1122.35	1122.35	0.00	11.22	0.79	0.41	43	29	2.83
PUC-96-1	1340.25	0.64	1348.94	1348.94	0.00	11.88	1.27	0.41	25	31	3.16
PUC-96-2	1566.29	1.75	1594.22	1594.22	0.00	1630.32	153.32	1.07	2251	29	3.38
PUC-96-3	1491.40	0.86	1504.42	1504.42	0.00	615.45	36.91	1.79	685	26	3.77
PUC-96-4	1467.85	1.32	1487.52	1487.52	0.00	49.71	5.48	0.71	103	32	3.06
PUC-96-5	1422.95	0.00	1422.95	1422.95	0.00	0.20	0.02	0.20	1	7	30.00
PUC-128-1	2369.39	1.99	2417.43	2417.43	0.00	3423.57	368.07	1.52	2386	38	3.42
PUC-128-2	2156.78	0.46	2166.77	2166.77	0.00	17.46	1.62	0.54	39	44	2.96
PUC-128-3	2374.39	0.87	2395.29	2395.29	0.00	3388.97	328.67	1.91	3039	44	2.96
PUC-128-4	2201.68	3.91	2291.24	2291.24	0.00	3101.67	311.98	1.19	2162	43	3.02
PUC-128-5	2269.26	0.00	2269.26	2269.26	0.00	0.19	0.01	0.19	1	53	2.45

**Table EC.8 Detailed results for B&C algorithm (continued)**

Group	LP <sub>ROOT</sub>	GAP <sub>ROOT</sub> (%)	LB	UB	GAP <sub>FINAL</sub> (%)	T(s)	T <sub>FLOW</sub> (s)	T <sub>ROOT</sub> (s)	N <sub>NODE</sub>	N <sub>TREE</sub>	S <sub>TREE</sub>
PUC-256-1	6559.38	7.11	6835.63	7061.17	3.19	3600.00	241.17	34.19	86	-	-
PUC-256-2	6927.09	10.60	7423.94	7748.26	4.19	3600.00	314.10	23.95	111	-	-
PUC-256-3	6533.74	13.10	7388.10	7518.65	1.74	3600.00	461.95	16.17	179	-	-
PUC-256-4	6724.89	10.69	7528.98	7530.24	0.02	3600.00	410.29	14.64	199	-	-
PUC-256-5	6618.98	4.26	6913.10	6913.38	<0.01	3600.00	286.83	47.27	260	-	-
PUC-512-1	19460.58	17.25	19506.06	23517.25	17.06	3600.00	354.10	380.68	7	-	-
PUC-512-2	19503.10	16.25	19549.84	23286.79	16.05	3600.00	296.51	529.26	7	-	-
PUC-512-3	18876.79	17.48	18961.38	22874.34	17.11	3600.00	286.42	591.55	7	-	-
PUC-512-4	19305.02	17.65	19359.15	23442.23	17.42	3600.00	187.68	1091.63	5	-	-
PUC-512-5	19510.50	14.67	19540.73	22863.50	14.53	3600.00	189.81	1098.31	4	-	-
PUC-1024-1	56045.21	21.08	56045.21	71011.17	21.08	3600.00	-	>3600 <sup>†</sup>	1	-	-
PUC-1024-2	54940.46	20.96	54940.46	69510.15	20.96	3600.00	-	>3600 <sup>†</sup>	1	-	-
PUC-1024-3	54861.55	20.95	54861.55	69401.36	20.95	3600.00	-	>3600 <sup>†</sup>	1	-	-
PUC-1024-4	53552.76	21.29	53552.76	68040.88	21.29	3600.00	-	>3600 <sup>†</sup>	1	-	-
PUC-1024-5	55196.42	21.02	55196.42	69884.32	21.02	3600.00	-	>3600 <sup>†</sup>	1	-	-

† – The solution of the root node did not terminate