Online Supplement to "Efficient Convexification Strategy for Generalized Geometric Programming Problems"

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Several examples in this Appendix are solved by Lingo 14 (2013) on an Intel(R) Core i7 930 CPU with 12 GB RAM. Such examples demonstrate the effectiveness and efficiency of the proposed convexification strategy.

EXAMPLE 1

Min $ax_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}+bx_1+cx_2+dx_3$

s.t. $x_1-x_2+x_3\le 30$, (28)

$x_1+x_2-x_3\le 0$, (29)

$-x_1+x_2+x_3\ge 0$, (30)

$1\le x_i\le 100, i=1,2,3$, (31)

where $\alpha_1,\alpha_2,\alpha_3,a,b,c,$ and d are constants.

Only one signomial term, $ax_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$, exists in this objective function, whereas other terms are linear. Four parameter sets, $\{\alpha_1,\alpha_2,\alpha_3,a,b,c,d\}$, specified in Table 5, are used to form four subproblems in this example. On the basis of Proposition 1, the signomial terms in all subproblems are nonconvex; they must be convexified as an underestimator to obtain global optimization. Given that different underestimators with varying conditions cause distinct results, this example conducts an extensive experimental study for verifying the effectiveness of the proposed convexification strategy. This example gathers the computational efficiencies and accuracies of each underestimator (ET, NPT, and PCT) to verify the rules of the proposed convexification strategy.

In subproblem 1, the nonconvex signomial term $180x_1^{-1}x_2^{-1}x_3^2$ can be converted into convex underestimator $180x_1^{-1}x_2^{-1}e^{2z_3}$ on the basis of Proposition 3, where $z_3 = L(\ln x_3)$ and $L(\ln x_3)$ is the linearization function of $\ln x_3$. Given r equidistant break points ($a_1 = 1$,…, $a_w = a_1 + (100-1)(w-1)/(r-1), \ldots, a_r = 100$) for linearizing $\ln x_3$, the nonlinear term z_3 can be linearized by the piecewise linearization technique proposed by Beale and Tomlin (1970), which is expressed as follows:

$$z_3 = \sum_{w=1}^{r} \lambda_w \ln a_w \,,\; x_3 = \sum_{w=1}^{r} \lambda_w a_w \,,\; \sum_{w=1}^{r} \lambda_w = 1, \tag{32}$$

$$\sum_{w=1}^{r-1} u_w = 1, \tag{33}$$

$$\lambda_1 \le u_1,\; \lambda_r \le u_{r-1},\; \lambda_w \le u_{w-1} + u_w, w = 2,\ldots,r-1, \tag{34}$$

$$0 \le \lambda_w \le 1, w = 1,\ldots,r;\; u_w \in \{0,1\}, w = 1,\ldots,r-1. \tag{35}$$

Subproblem 1 is converted into a mixed-integer convex program with the ET technique as follows:

Min $\quad 180x_1^{-1}x_2^{-1}e^{2z_3} - x_1 + 8x_2 - 6x_3$

s.t. $\quad$ (28)−(35).

Subproblem 1 can also be convexified by the NPT (Propositions 4) and PCT techniques (Propositions 5). All subproblems with each convex underestimator are presented in Table 5.

Table 5 Parameters of Example 1 and its related transformations

Subproblem	Parameters $\{\alpha_1, \alpha_2, \alpha_3, a, b, c, d\}$	Original term	ET	NPT	PCT
1	{−1, −1, 2, 180, −1, 8, −6}	$180x_1^{-1}x_2^{-1}x_3^2$	$180x_1^{-1}x_2^{-1}e^{2z_3}$, $z_3 = L(\ln x_3)$.	$180x_1^{-1}x_2^{-1}z_3^{-2/\beta}$, $z_3 = L(x_3^{-\beta})$.	$180x_1^{-1}x_2^{-1}z_3^{3}$, $z_3 = L(x_3^{2/3})$.
2	(−1, 0.1, 3, 2, 0.7, −29, −23)	$2x_1^{-1}x_2^{0.1}x_3^3$	$2x_1^{-1}e^{0.1z_2+3z_3}$, $z_2 = L(\ln x_2)$, $z_3 = L(\ln x_3)$.	$2x_1^{-1}z_2^{-0.1/\beta}z_3^{-3/\beta}$, $z_2 = L(x_2^{-\beta})$, $z_3 = L(x_3^{-\beta})$.	$2x_1^{-1}z_2^{-1}x_3^{3}$, $z_2 = L(x_2^{-0.1})$.
3	(−3, 0.1, 1, 455, −2.9, −3, 2.78)	$455x_1^{-3}x_2^{0.1}x_3$	$455x_1^{-3}e^{0.1z_2+z_3}$, $z_2 = L(\ln x_2)$,	$455x_1^{-3}z_2^{-0.1/\beta}z_3^{-1/\beta}$, $z_2 = L(x_2^{-\beta})$,	$455x_1^{-3}z_2^{-0.1/\beta}z_3^{4+0.1/\beta}$, $z_2 = L(x_2^{-\beta})$,

			$z_3 = L(\ln x_3)$.	$z_3 = L(x_3^{-\beta})$.	$z_3 = L(x_3^{\frac{\beta}{4\beta+0.1}})$.
4	(0.9, 1, 1.1, 5, −55, −62, 26)	$5x_1^{0.9}x_2x_3^{1.1}$	$5e^{0.9z_1+z_2+1.1z_3}$, $z_1 = L(\ln x_1)$, $z_2 = L(\ln x_2)$, $z_3 = L(\ln x_3)$.	$5z_1^{-0.9/\beta}z_2^{-1/\beta}z_3^{-1.1/\beta}$, $z_1 = L(x_1^{-\beta})$, $z_2 = L(x_2^{-\beta})$, $z_3 = L(x_3^{-\beta})$.	$z_1^{-0.9/\beta}z_2^{-1/\beta}z_3^{1+1.9/\beta}$, $z_1 = L(x_1^{-\beta})$, $z_2 = L(x_2^{-\beta})$, $z_3 = L(x_3^{\frac{1.1\beta}{\beta+1.9}})$.

Tables 6 to 9 provide the results of subproblems 1 to 4, respectively. These tables present the solution, objective value, CPU time, and error rate $err\%$ of each convex underestimator, in which ET has two different β values (1 and the minimal values obtained by Propositions 4 and 5 with computer accuracy $\varepsilon = 2.55*10^{-6}$). Each subproblem has four different equidistant break points ($r = 8,\ 16,\ 32,$ and 64) for experimenting. The CPU times are executed 10 times consecutively, and the median of the 10 results is considered the computation time (in seconds). Error rate (%) expressed in Equation (36) is used to calculate the gap of the experimental objective value and global optimization, where Obj is the experimental objective value, and Obj^* is the near global optimization obtained from the ET technique with 2048 break points.

$$\text{Error rate(\%)}=\left|\frac{Obj - Obj^*}{Obj^*}\right| \times 100\% \,. \tag{36}$$

Table 6 Experimental results of subproblem 1 in Example 1

Global optimization (x_1, x_2, x_3)	r	Transformation	Solution (x_1, x_2, x_3)	Objective value	CPU time (sec.)	Error rate (%)
643.8444 (15, 13.887, 28.887)	8	ET	(2.674, 2.315, 4.989)	140.6749	0.30	78.15%
		NPT(β=1)	(5.261, 2.822, 8.083)	23.6015	0.32	96.33%
		NPT(β=1/480)	(2.678, 2.316, 4.995)	140.202	0.36	78.22%
		PCT	**(1.755, 1.692, 3.447)**	**417.7526**	**0.27**	**35.12%**
	16	ET	(1.631, 1.552, 3.183)	277.6712	0.75	56.87%
		NPT(β=1)	(2.238, 1.944, 4.182)	121.4399	1.18	81.14%
		NPT(β=1/240)	(1.633, 1.553, 3.187)	276.6151	0.86	57.04%
		PCT	**(1.310, 1.281, 2.591)**	**520.8603**	**0.67**	**19.10%**
	32	ET	(1.117, 1.092, 2.209)	438.3986	0.71	31.91%
		NPT(β=1)	(1.306, 1.255, 2.562)	280.2766	0.65	56.47%
		NPT(β=1/120)	(1.118, 1.093, 2.212)	436.6528	0.74	32.18%
		PCT	**(1.009, 1, 2.009)**	**607.2811**	**0.66**	**5.68%**
	64	ET	(1, 1, 2)	597.8533	1.44	7.14%
		NPT(β=1)	(1, 1, 2)	482.9523	1.49	24.99%
		NPT(β=1/60)	(1, 1, 2)	595.8467	1.48	7.45%
		PCT	**(15, 14.047, 29.047)**	**643.7934**	**0.95**	**0.01%**

Table 7 Experimental results of subproblem 2 in Example 1

Global optimization (x_1, x_2, x_3)	r	Transformation	Solution (x_1, x_2, x_3)	Objective value	CPU time (sec.)	Error rate (%)
−82.34646 (2.94136, 1.761818, 4.703178)	8	ET	(2.856, 5.53, 8.387)	−271.232	0.34	229.38%
		NPT(β=1)	(2.07, 8.813, 10.883)	−445.168	0.33	440.60%
		NPT(β=1/480)	(2.855, 5.538, 8.393)	−271.65	0.30	229.89%
		PCT	**(2.947, 1.985, 4.932)**	**−86.013**	**0.28**	**4.45%**
	16	ET	(1.878, 3.13, 5.007)	−151.990	0.47	84.57%
		NPT(β=1)	(1.419, 4.374, 5.794)	−218.479	0.59	165.32%
		NPT(β=1/240)	(1.875, 3.135, 5.011)	−152.292	0.54	84.94%
		PCT	**(2.951, 1.922, 4.873)**	**−85.178**	**0.50**	**3.44%**
	32	ET	(1.314, 1.984, 3.298)	−95.663	1.22	16.17%
		NPT(β=1)	(1.008, 2.386, 3.394)	−117.662	0.90	42.89%
		NPT(β=1/120)	(1.31, 1.988, 3.298)	−95.844	1.22	16.39%
		PCT	**(2.964, 1.845, 4.81)**	**−84.198**	**0.55**	**2.25%**
	64	ET	(3.038, 1.818, 4.857)	−86.064	3.72	4.51%
		NPT(β=1)	(3.03, 1.891, 4.921)	−89.819	4.62	9.07%
		NPT(β=1/60)	(3.038, 1.819, 4.858)	−86.127	3.40	4.59%
		PCT	**(2.989, 1.765, 4.754)**	**−83.223**	**1.37**	**1.06%**

Table 8 Experimental results of subproblem 3 in Example 1

Global optimization (x_1, x_2, x_3)	r	Transformation	Solution (x_1, x_2, x_3)	Objective value	CPU time (sec.)	Error rate (%)
0.11863 (15, 35.37001, 50.37001)	8	ET	(15, 5.907, 20.907)	−0.1221	0.55	202.93%
		NPT(β=1)	(15, 7.411, 22.411)	−0.4651	0.90	492.06%
		NPT(β=1/480)	(15, 5.911, 20.911)	−0.1229	0.82	203.60%
		PCT(β=1)	(15, 8.189, 23.189)	−0.2998	0.47	352.72%
		PCT(β=1/480)	**(15, 5.911, 20.911)**	**−0.1197**	**0.83**	**200.90%**
	16	ET	(15, 2.143, 17.143)	0.0858	1.21	27.67%
		NPT(β=1)	(15, 3.243, 18.243)	−0.0215	0.89	118.12%
		NPT(β=1/240)	(15, 2.153, 17.153)	0.0854	1.65	28.01%
		PCT(β=1)	(15, 3.882, 18.882)	0.0127	0.89	89.29%
		PCT(β=1/240)	**(15, 2.158, 17.158)**	**0.087**	**1.70**	**26.66%**
	32	ET	(15, 33.995, 48.995)	0.1134	2.01	4.41%
		NPT(β=1)	(15, 2.388, 17.388)	0.1028	1.39	13.34%
		NPT(β=1/120)	(15, 33.995, 48.995)	0.1134	1.89	4.41%
		PCT(β=1)	(15, 2.252, 17.252)	0.1116	1.17	5.93%
		PCT(β=1/120)	**(15, 33.996, 48.996)**	**0.1137**	**1.98**	**4.16%**
	64	ET	(15, 34.704, 49.704)	0.1172	5.08	1.21%
		NPT(β=1)	(15, 34.68, 49.68)	0.1161	6.16	2.13%
		NPT(β=1/60)	(15, 34.703, 49.703)	0.1174	5.52	1.04%
		PCT(β=1)	(15, 34.626, 49.626)	0.1175	6.65	0.95%
		PCT(β=1/60)	**(15, 34.7, 49.7)**	**0.1176**	**5.72**	**0.87%**

Table 9 Experimental results of subproblem 4 in Example 1

Global optimization (x_1, x_2, x_3)	r	Transformation	Solution (x_1, x_2, x_3)	Objective value	CPU time (sec.)	Error rate (%)
−69.42404 (1, 2.54483,	8	**ET**	**(1, 6.827, 7.827)**	**−192.95**	**0.45**	**177.93%**
		NPT(β=1)	(2.473, 8.817, 11.29)	−322.033	0.47	363.86%
		NPT(β=1/480)	(1, 6.835, 7.835)	−193.263	0.72	178.38%
		PCT(β=1)	(3.01, 7.249, 10.256)	−233.721	0.47	236.66%

3.54483)		PCT(β=1/480)	(1, 6.829, 7.829)	−193.117	0.44	178.17%
		ET	**(1, 4.025, 5.025)**	**−120.009**	**4.31**	**72.86%**
		NPT(β=1)	(1.359, 4.769, 6.128)	−167.56	1.83	141.36%
	16	NPT(β=1/240)	(1, 4.03, 5.03)	−120.331	3.82	73.33%
		PCT(β=1)	(2.058, 4.144, 6.202)	−139.219	1.31	100.53%
		PCT(β=1/240)	(1, 4.027, 5.027)	−120.16	3.70	73.08%
		ET	**(1, 2.586, 3.586)**	**−84.431**	**2.55**	**21.62%**
		NPT(β=1)	(2.135, 2.706, 4.841)	−98.266	3.78	41.54%
	32	NPT(β=1/120)	(1, 2.587, 3.587)	−84.585	3.73	21.84%
		PCT(β=1)	(1.99, 2.664, 4.654)	−96.041	3.03	38.34%
		PCT(β=1/120)	(1, 2.588, 3.588)	−84.521	2.73	21.75%
		ET	**(1, 2.264, 3.264)**	**−72.22**	**17.32**	**4.03%**
		NPT(β=1)	(1.512, 1.968, 3.48)	−78.006	19.01	12.36%
	64	NPT(β=1/60)	(1, 2.262, 3.262)	−72.331	14.70	4.19%
		PCT(β=1)	(1.507, 1.953, 3.46)	−76.461	12.48	10.14%
		PCT(β=1/60)	(1, 2.261, 3.261)	−72.3	18.08	4.14%

Tables 6 to 9 show the enormous computing results of subproblems 1 to 4 and obtain the following conclusion: the PCT technique yields a better solution for subproblems 1 to 3 (Tables 6 to 8), whereas the ET technique yields a better solution for subproblem 4 (Table 9). The proposed convexification strategy can be employed to obtain the most appropriate variable transformation technique, which does not require enormous computations. Table 10 shows the effectiveness of the proposed convexification strategy. In subproblem 1, the nonconvex signomial, $180x_1^{-1}x_2^{-1}x_3^2$, satisfies Rule 3 in Table 2, and the PCT technique with Proposition 5 is selected for obtaining the underestimator. Similarly, the nonconvex term, $2x_1^{-1}x_2^{0.1}x_3^3$, in subproblem 2 satisfies Rule 4 in Table 2, and the PCT technique is selected. Subproblems 3 and 4 respectively satisfy Rules 5 and 6 and necessitate the estimator with Equation (21) for selecting the most appropriate variable transformation technique. The values of estimator in Equation (21) for terms $455x_1^{-3}x_2^{0.1}x_3$ and $5x_1^{0.9}x_2x_3^{1.1}$ with different β values are listed in Table 10. The values of estimators in subproblem 3 are always positive, whereas the values in subproblem 4 are always negative. According to Rules 5 and 6 in Table 2, the suggested convex underestimators of $455x_1^{-3}x_2^{0.1}x_3$ and $5x_1^{0.9}x_2x_3^{1.1}$ can be constructed by the PCT and ET techniques, respectively. Based on the experiments of Example 1, the proposed convexification strategy offers a guide for selecting the most appropriate transformation technique on any condition of signomial term for obtaining the tightest convex underestimator.

Table 10 Effectiveness of the proposed convexification strategy in Example 1

Sub-problem	β value	Experiment results	Proposed convexification strategy (Table 2) Rule	Value of estimator in Equation (21)	Suggesting transformation technique
1	n/a	**PCT (Table 6)**	3	n/a	**PCT**
2	n/a	**PCT (Table 7)**	4	n/a	**PCT**
3	1/480	**PCT (Table 8)**	5 (Estimator)	6882.6	**PCT**
3	1/240	**PCT (Table 8)**	5 (Estimator)	12825.4	**PCT**
3	1/120	**PCT (Table 8)**	5 (Estimator)	22546.7	**PCT**
3	1/60	**PCT (Table 8)**	5 (Estimator)	36210.9	**PCT**
4	1/480	**ET (Table 9)**	6 (Estimator)	−36250635.6	**ET**
4	1/240	**ET (Table 9)**	6 (Estimator)	−72495016.9	**ET**
4	1/120	**ET (Table 9)**	6 (Estimator)	−144959707.8	**ET**
4	1/60	**ET (Table 9)**	6 (Estimator)	−289756133.9	**ET**

EXAMPLE 2 This example compares the efficiency of three piecewise linearization techniques in GGP problems with the same subproblems (subproblems 1 to 4) in Example 1. Given that the most appropriate variable transformation technique for each subproblem is obtained in Example 1, this example only compares the different piecewise linearization techniques with the most appropriate variable transformation technique in each subproblem. The first piecewise linearization technique is derived from Beale and Tomlin (1970), who used an SOS2 model to represent piecewise linear functions. The features of this technique are reasonably easy to implement and use; thus, this technique is applied in Example 1 (Constraints (32)−(35)). Li et al. (2009) proposed a superior approach to expressing the piecewise linear function with logarithmic number of binary variables and constraints, and it is employed as the second piecewise linearization technique in this example. The last piecewise linearization technique is proposed by Vielma and Nemhauser (2011), and it is an efficient SOS2 constraint with a logarithmic number of binary variables and constraints. For example, the piecewise linear functions of the nonlinear term $z_3 = L(\ln x_3)$ with 9 break points in Example 1 (Constraints (32)−(35)) can be reduced through this technique (Proposition 8), as shown below.

(32),

$$\lambda_{3,3} + \lambda_{3,7} \le u_{3,1}, \ \lambda_{3,1} + \lambda_{3,5} + \lambda_{3,9} \le 1 - u_{3,1}, \tag{37}$$

$$\lambda_{3,4} + \lambda_{3,5} + \lambda_{3,6} \le u_{3,2}, \ \lambda_{3,1} + \lambda_{3,2} + \lambda_{3,8} + \lambda_{3,9} \le 1 - u_{3,2}. \tag{38}$$

$$\lambda_{3,6} + \lambda_{3,7} + \lambda_{3,8} + \lambda_{3,9} \le u_{3,3}, \ \lambda_{3,1} + \lambda_{3,2} + \lambda_{3,3} + \lambda_{3,4} \le 1 - u_{3,3}. \tag{39}$$

$$0 \le \lambda_{3,w} \le 1, w = 1,\ldots,r; \ u_{3,s} \in \{0,1\}, s = 1,\ldots,\lceil \log_2(r-1) \rceil. \tag{40}$$

The experimental results of all subproblems with three piecewise linear techniques and r equidistant break points ($r = 32,\ 64,\ 128, 256, 512, \text{and } 1024$) are presented in Table 11, which include the solution, accuracy rate, and CPU time. The CPU time is executed 10 times consecutively, and the median of the 10 results is considered the computation time (in seconds). To examine the accuracy rate of the piecewise linear functions, we use Equation (41) to evaluate the gap between the original nonlinear term and the related convex underestimator. In Equation (41), $\mathbf{x}^*$ is the solution derived from P1 model, $S(\mathbf{x}^*)$ is the value of the original nonlinear term with solution $\mathbf{x}^*$, and $S^C(\mathbf{x}^*,\hat{\mathbf{z}})$ is the value of convex underestimator of $S(\mathbf{x}^*)$ with solution $\mathbf{x}^*$ and its related piecewise linear functions $\hat{\mathbf{z}}$.

$$accuracy\% = \frac{S(\mathbf{x}^*) - S^C(\mathbf{x}^*,\hat{\mathbf{z}})}{S(\mathbf{x}^*)} \times 100\% . \tag{41}$$

In Table 11, the piecewise linearization technique of Vielma and Nemhauser (2011) always yields the shortest solving time, especially for $r \geq 512$. Based on this example, an efficient piecewise linearization technique can significantly reduce the solving time of GGP problems with PCT and ET variable transformations.

Table 11 Comparison of the results of piecewise linear techniques in Example 2

Subproble	Trans-for-mation	r	Solution (x_1, x_2, x_3)	Accuracy%	CPU Time (second) Piecewise Tech. Beale &Tomlin	Li et al.	Vielma & Nemhauser
1	PCT	32	(1.009, 1, 2.009)	14.95%	0.66	0.55	**<0.01**
		64	(15, 14.047, 29.047)	0.01%	0.95	1.11	**<0.01**
		128	(15, 13.963, 28.963)	<0.01%	2.67	4.27	**1.01**
		256	(15, 13.92, 28.92)	<0.01%	5.13	11.57	**0.91**
		512	(15, 13.899, 28.899)	<0.01%	25.51	64.13	**2.21**
		1024	(15, 13.888, 28.888)	<0.01%	125.56	297.68	**6.63**
2	PCT	32	(2.964, 1.845, 4.81)	2.44%	0.55	0.32	**0.03**
		64	(2.989, 1.765, 4.754)	1.16%	1.37	1.22	**0.55**
		128	(3.023, 1.693, 4.716)	0.15%	6.1	1.85	**0.89**
		256	(2.971, 1.735, 4.706)	0.03%	13.27	3.26	**1.12**
		512	(2.954, 1.75, 4.704)	0.01%	42.87	11.73	**2.44**
		1024	(2.947, 1.757, 4.703)	<0.01%	171.86	53.6	**4.39**
3	PCT	32	(15, 33.996, 48.996)	0.06%	1.98	6.02	**1.09**
		64	(15, 34.7, 49.7)	0.01%	5.72	15.3	**1.54**
		128	(15, 35.19, 50.19)	<0.01%	24.36	86.07	**3.77**
		256	(15, 35.297, 50.297)	<0.01%	847.24	338.97	**11.38**
		512	(15, 35.383, 50.383)	<0.01%	2644.89	2533.6	**27.55**
		1024	(15, 35.36, 50.36)	<0.01%	>86400	25730.12	**97.45**
4	ET	32	(1, 2.586, 3.586)	28.57%	2.55	9.19	**1.98**
		64	(1, 2.264, 3.264)	8.07%	17.32	29.82	**2.78**

128	(1, 2.698, 3.698)	1.18%	108.94	130.2	**7.17**
256	(1, 2.48, 3.48)	0.35%	1418.71	817.9	**13.34**
512	(1, 2.587, 3.587)	0.08%	8935.74	6879.43	**34.58**
1024	(1, 2.564, 3.564)	0.02%	>86400	76413.71	**123.21**

EXAMPLE 3 This example does a comprehensive comparison among the proposed convexification strategy with popular referencing convexification strategies. The referencing convexification strategies include those proposed by Pörn et al. (2008), Li and Lu (2009), Tsai and Lin (2011) as well as Lundell et al. (2009). Considering that only one transformation technique can be used for negative signomial term, all convexification strategies always follow the rules of Table 3 to convexify the negative term with the PT technique (described in Proposition 6). For positive signomial term, more freedom exists regarding the selection of transformation techniques. The concepts of comparing convexification strategies for positive signomial terms can be briefly described as follows.

(i) Pörn et al. (2008) used the ET technique (described in Proposition 3) to transform only the variables with positive exponents.

(ii) Li and Lu (2009) and Tsai and Lin (2011) used the NPT technique (described in Proposition 4) to transform only the variables with positive exponents.

(iii) Lundell et al. (2009) used a MILP model to select either the NPT technique (described in Proposition 4) or the PCT technique to transform related variables.

(iv) The proposed convexification strategy utilizes an estimator in Proposition 7 to determine the use of ET (described in Proposition 3) or PCT technique (described in Proposition 5) to transform related variables. Table 2 lists the detailed rules.

This example is tested on a pressure vessel design problem proposed by Sandgren (1990), as shown in Figure 1, to minimize the total cost of materials used to form and weld the pressure vessel. The four decision variables are x_1 (i.e., shell thickness), x_2 (i.e., spherical head thickness), x_3 (i.e., radius of the shell), and x_4 (i.e., length of the shell). All the variables are changed into continuous variables to demonstrate the effectiveness of the proposed convexification strategy and referencing convexification strategies. The problem is formulated as follows:

$$\textbf{Min} \quad 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3 \tag{42}$$

$$\textbf{s.t.} \quad 0.0193x_3 - x_1 \le 0,$$

$$0.00954x_3 - x_2 \le 0,$$
$$1296000 - \pi x_3^2 x_4 - \frac{4}{3}\pi x_3^3 \le 0, \tag{43}$$
$$x_4 - 240 \le 0,$$
$$0.0625 \le x_i \le 6.1875, i = 1, 2,$$
$$10 \le x_i \le 200, i = 3, 4.$$

[Insert Figure 1]

Figure 1. Tube and pressure vessel

Four nonconvex posynomial terms in objective function (42) and two nonconvex signomial terms in constraint (43) are considered. The convex underestimators of the nonconvex terms in each convexification strategy are expressed in Table 12, and the rules for the proposed strategy are presented in Tables 2 and 3. All piecewise linear functions ($z_{\square} = L(f(x_{\square}))$) utilize the linearization technique proposed by Vielma and Nemhauser (2011).

Table 12 Convex underestimators with each convexification strategy in Example 3

Original Terms	Pörn et al (2008)	Li & Lu(2009), Tsai & Lin (2011)	Lundell et al. (2009)	Proposed strategy
$0.6224x_1x_3x_4$	$0.6224e^{z_1+z_3+z_4}$	$0.6224z_1^{-1}z_3^{-1}z_4^{-1}$	$0.6224z_1^{3}z_3^{-1}z_4^{-1}$	$0.6224e^{z_1+z_3+z_4}$, rule 6
$1.7781x_2x_3^2$	$1.7781e^{z_2+2z_3}$	$1.7781z_2^{-1}z_3^{-2}$	$1.7781z_2^{-1}x_3^{2}$	$1.7781e^{z_2+2z_3}$, rule 6
$3.1661x_1^2x_4$	$3.1661e^{2z_1+z_4}$	$3.1661z_1^{-2}z_4^{-1}$	$3.1661x_1^{2}z_4^{-1}$	$3.1661x_1^{2}z_{41}^{-1}$, rule 6
$19.84x_1^2x_3$	$19.84e^{2z_1+z_3}$	$19.84z_1^{-2}z_3^{-1}$	$19.84x_1^{2}z_3^{-1}$	$19.84x_1^{2}z_{31}^{-1}$, rule 6
	$z_i = L(\ln x_i)$,	$z_i = L(x_i^{-1})$,	$z_1 = L(x_1^{1/3})$,	$z_i = L(\ln x_i), i = 1, \ldots, 4,$
	$i = 1, \ldots, 4.$	$i = 1, \ldots, 4.$	$z_i = L(x_i^{-1})$,	$z_{31} = L(x_3^{-1})$,
			$i = 2, 3, 4.$	$z_{41} = L(x_4^{-1})$.
$-\pi x_3^2 x_4$	$-\pi z_{32}^{2/3} z_{42}^{1/3}$, rule 13			
$-4/3\,\pi x_3^3$	$-4/3\,\pi z_{32}$, rule 13			
	$z_{32} = L(x_3^3)$, $z_{42} = L(x_4^3)$			

The global optimal objective value is 5885.333, with a feasibility tolerance of $2.55 * 10^{-6}$. Table 13 provides the computation results of the proposed convexification strategy and referencing convexification strategies. In each case of r equidistant break points (r =8, 16, 32, 64, and 128), Table 13 lists the solution, objective value, CPU time, and error rate $err\%$. The CPU time is executed 10 times consecutively, and the median of the 10 results is considered the computation time (in seconds). The *error rate* (%) is expressed in Equation (36).

From the computation results, the proposed convexification strategy outperforms the referencing strategies (Pörn et al. (2008), Li and Lu (2009), Tsai and Lin (2011), and Lundell et al. (2009)). Specifically, the proposed convexification strategy utilizes an estimator to determine the most appropriate variable transformation technique for obtaining the tightest convex underestimator. Figure 2 illustrates the solution quality derived by each strategy in each case of different equidistant break points. The proposed convexification strategy outperforms the referencing strategies in terms of both solution quality and computation efficiency.

Table 13 Computation results of Example 3

Global optimization (x_1, x_2, x_3, x_4)	r	Convexification strategy	Solution (x_1, x_2, x_3, x_4)	Objective value	CPU time (sec.)	Error rate (%)
5885.333 (0.7781686, 0.3846492, 40.31962, 200)	8	Pörn et al. (2008)	(0.7333, 0.36247, 37.9942, 200)	3695.201	3	37.21%
		Li & Lu(2009), Tsai & Lin (2011)	(0.7333, 0.36246, 37.9942, 200)	1870.305	2	68.22%
		Lundell et al. (2009)	(1.1751, 0.5808, 60.8846, 18.3368)	3365.461	3	42.82%
		Proposed strategy	**(0.7369,0.3643,38.1809,196.2757)**	**3927.136**	**3**	**33.27%**
	16	Pörn et al. (2008)	(0.76126, 0.37629, 39.44327, 200)	5229.291	4	11.15%
		Li & Lu(2009), Tsai & Lin (2011)	(0.76126, 0.37629, 39.44327, 200)	4740.568	6	19.45%
		Lundell et al. (2009)	(0.76126, 0.37628, 39.44327, 200)	4994.263	13	15.14%
		Proposed strategy	**(0.76125, 0.37629, 39.44327, 200)**	**5262.756**	**4**	**10.58%**
	32	Pörn et al. (2008)	(0.7796, 0.3854, 40.3936,197.6235)	5781.977	8	1.756%
		Li & Lu(2009), Tsai & Lin (2011)	(0.7796, 0.3853, 40.3911, 197.661)	5699.324	22	3.161%
		Lundell et al. (2009)	(0.7809, 0.386, 40.4612, 196.5927)	5759.954	12	2.130%
		Proposed strategy	**(0.7802, 0.3857, 40.4254,197.1352)**	**5792.173**	**8**	**1.583%**
	64	Pörn et al. (2008)	(0.77748, 0.38431, 40.28401, 200)	5852.527	13	0.557%
		Li & Lu(2009), Tsai & Lin (2011)	(0.77748, 0.38431, 40.28401, 200)	5832.095	11	0.905%
		Lundell et al. (2009)	(0.77748, 0.38431, 40.28401, 200)	5850.111	13	0.598%
		Proposed strategy	**(0.77748, 0.3843, 40.284, 200)**	**5855.590**	**11**	**0.505%**
	128	Pörn et al. (2008)	(0.77791, 0.38452, 40.30631, 200)	5877.551	29	0.132%
		Li & Lu(2009), Tsai & Lin (2011)	(0.77791, 0.38452, 40.30631, 200)	5874.099	30	0.191%
		Lundell et al. (2009)	(0.77791, 0.38452, 40.30631, 200)	5875.646	30	0.165%
		Proposed strategy	**(0.77791, 0.38452, 40.30631, 200)**	**5877.732**	**29**	**0.129%**

Number of break points (r)

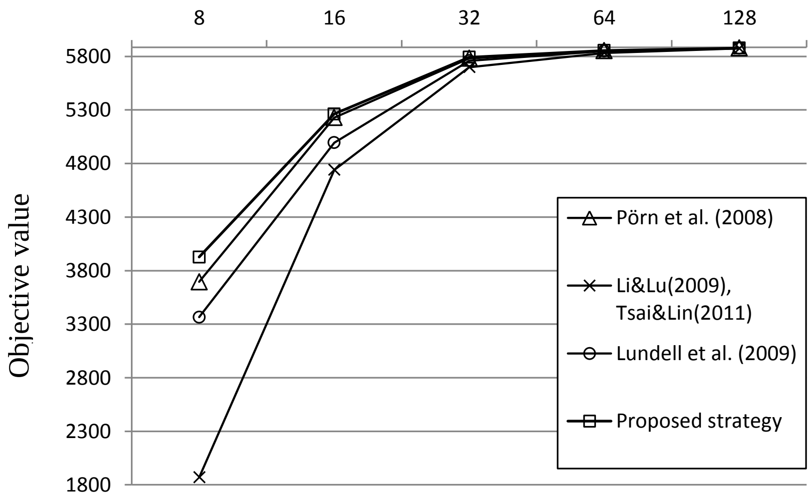

Figure 2 Objective Value Comparison of Example 3