Online Supplement to
Permutations in the factorization of simplex bases*

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1 Proof of Theorem 1 and Corollary 1

Theorem 1 states that Algorithm 1 yields a pseudo triangular permutation of B with a nucleus of minimum size. The proof is detailed here.

Proof of Theorem 1. We apply Algorithm 1 on B and obtain the matrix $B^{(\lambda)}$. For conciseness, we denote by $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{G}$ the index subsets corresponding to the upper triangular, lower triangular, and nucleus parts of $B^{(\lambda)}$, respectively: $\mathcal{U} := \{1, \dots, \kappa\}$, $\mathcal{L} := \{\kappa + 1, \dots, \lambda\}$ and $\mathcal{G} := \{\lambda + 1, \dots, m\}$. Let B^* be a pseudo triangular permutation of B with a nucleus of minimum size. The sets $\mathcal{U}^*$, $\mathcal{L}^*$, and $\mathcal{G}^*$ are defined for B^* similarly to their counterparts for B. Note that, for $\mathcal{U}^*$ fixed, $\mathcal{L}^*$ is maximal: there are no singleton-rows in the submatrix formed with rows and columns $\mathcal{G}^*$ of B^*. The contrary would immediately contradict the assumption that $\mathcal{G}^*$ is minimum. Furthermore, we may assume without loss of generality that $\mathcal{U}^*$ is maximal too: there are no singleton-columns in the submatrix formed with rows and columns $\mathcal{L}^* \cup \mathcal{G}^*$ of B^*. Indeed, any such singleton-column can be moved to $\mathcal{U}^*$ without affecting the size of $\mathcal{G}^*$. Algorithm 1 ensures that $\mathcal{U}$ and $\mathcal{L}$ are also maximal in the same sense. We then define the functions $r^*, c^* : \{1, \dots, m\} \to \{1, \dots, m\}$, such that the row $r^*(i)$ of B^* corresponds to the row i of B, and the column $c^*(j)$ of B^* corresponds to the column j of B. Recall that by Lemma 1, every diagonal element in the triangular blocks of $B^{(\lambda)}$ and B^* is nonzero. Thus, in the following, "diagonal" will always imply "nonzero". The proof works in three steps.

(i) We show that every column in $\mathcal{L} \cup \mathcal{G}$ maps to a column in $\mathcal{L}^* \cup \mathcal{G}^*$, i.e., $j \in \mathcal{L} \cup \mathcal{G}$ implies $c^*(j) \in \mathcal{L}^* \cup \mathcal{G}^*$. Let $j_a = \operatorname{argmin}_{j \in \mathcal{L} \cup \mathcal{G}} c^*(j)$. Suppose that the claim is not true: suppose that there exists $j \in \mathcal{L} \cup \mathcal{G}$ such that $c^*(j) \in \mathcal{U}^*$. Then, $c^*(j_a) \in \mathcal{U}^*$ (Figure 1). Since $j_a \in \mathcal{L} \cup \mathcal{G}$ and $\mathcal{U}$ is maximal, there are at least two nonzero elements in column j_a of $B^{(\lambda)}$. All of them correspond to elements of $\mathcal{U}^*$, so at least one lies above the diagonal of $\mathcal{U}^*$. Let that element

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be in row i_a of $B^{(\lambda)}$, corresponding to the element of B^* in row $r^*(i_a)$ and column $c^*(j_a)$, with $r^*(i_a) < c^*(j_a)$. The diagonal element in that row of $\mathcal{U}^*$ is in column $c^*(j_b) = r^*(i_a)$ for some j_b. Since it is a nonzero in the row i_a of $B^{(\lambda)}$, we know that $j_b \in \mathcal{L} \cup \mathcal{G}$. However, because $c^*(j_b) = r^*(i_a) < c^*(j_a)$, this contradicts the construction of j_a as $\text{argmin}_{j \in \mathcal{L} \cup \mathcal{G}} c^*(j)$.

(ii) We can reverse the roles of $B^{(\lambda)}$ and B^* in the proof of (i). Therefore, every column in $\mathcal{L}^* \cup \mathcal{G}^*$ maps to a column in $\mathcal{L} \cup \mathcal{G}$, i.e., $c^*(j) \in \mathcal{L}^* \cup \mathcal{G}^*$ implies $j \in \mathcal{L} \cup \mathcal{G}$. Together, (i) and (ii) prove that $\mathcal{U}$ is a permutation of $\mathcal{U}^*$.

(iii) We transpose the reasoning that led to (i) and (ii) and apply it to the submatrices formed with the columns $\mathcal{L}^* \cup \mathcal{G}^*$ of B^* and $\mathcal{L} \cup \mathcal{G}$ of $B^{(\lambda)}$ (Figure 2). This yields $|\mathcal{L}| = |\mathcal{L}^*|$ and hence $|\mathcal{G}| = |\mathcal{G}^*|$, completing our proof. □

Corollary 1 states that Algorithm 1 finds a permutation of the rows and columns of B that is upper triangular whenever one exists.

Proof of Corollary 1. By Theorem 1, Algorithm 1 yields a pseudo triangular matrix $B^{(m)}$ with no nucleus. Assume that $\kappa < m$ when the algorithm starts Step 2. Then it means that $G^{(\kappa)}$ does not contain a column-singleton. However, since the final result $B^{(m)}$ has no nucleus, there exists a permutation $L^{(m)}$ of $G^{(\kappa)}$ that is lower triangular, contradicting the absence of a column-singleton in $G^{(\kappa)}$. Hence $\kappa < m$ is impossible. □

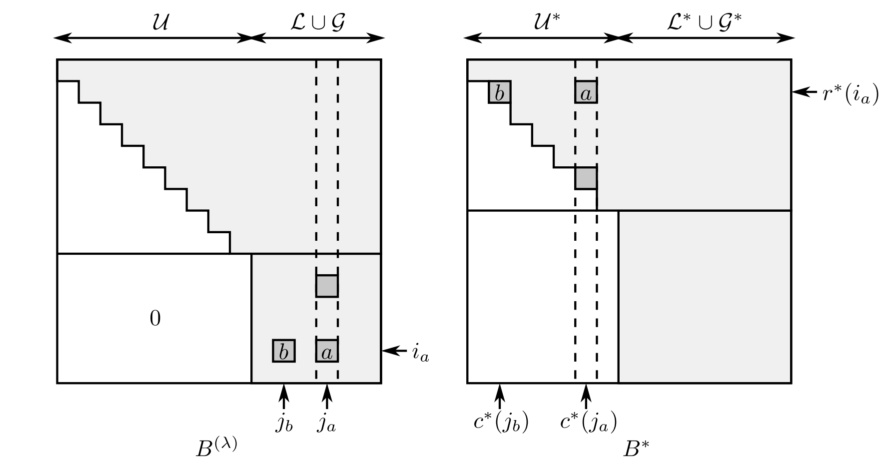

Figure 1: Proof of Theorem 1, step (i).

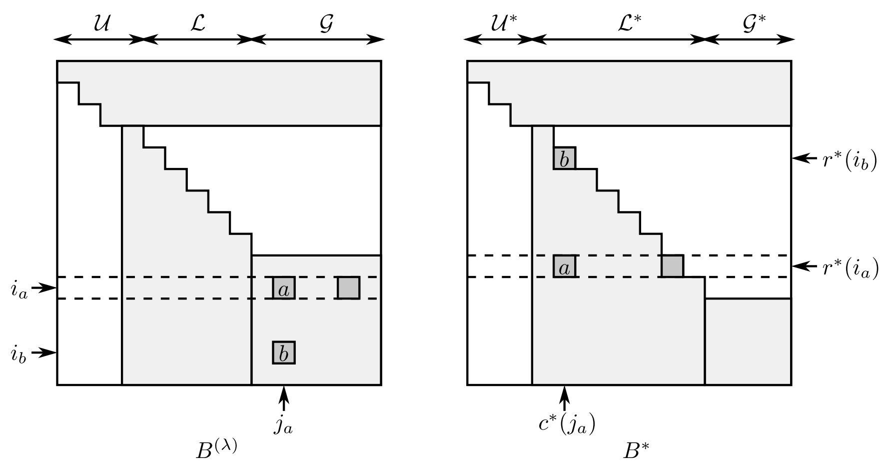

Figure 2: Proof of Theorem 1, step (iii).