Online Supplement for : Planning for overtime – the value of shift extensions in physician scheduling

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This online supplement contains proofs and few additional analyses of the underlying data.

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1. Proofs

Proposition 1. *The variable shift extension problem with flexible shift starting times and durations is NP-complete in the strong sense.*

Proof: Our proof is based on the principle that if any instance of a known NP-complete problem can be reduced to an instance of a problem in polynomial time then the considered problem is NP-complete as well. Similar to Brunner et al. (2013), we consider the circulant problem with intermittently available resources. In the circulant problem, the number of workers required to cover a (cyclic) demand is minimized. The eligible shift rosters follow the same pattern of working and rest periods, but start at different starting times. Thus, the model may be formulated as $\min\{\mathbf{1}x : Ax \geq b, x \geq 0, \text{integer}\}$, where A is an $m \times n$ circulant matrix with a wraparound structure (m being the number of periods of the time horizon, and n the number of eligible shift rosters). As the Exact Cover by 3-sets problem (NP-complete in the strong sense) can be transformed into an instance of

the circulant problem, the latter is NP-complete in the strong sense as well (Bartholdi 1981). We now consider the following parameters for our problem:
$\overline{P}^{reg-shift} = \underline{P}^{reg-shift} = P^{reg-shift}$, $\underline{P}^{reg-rest} = 0.5|\mathcal{P}| - P^{reg-shift} = P^{reg-rest}$, $\overline{P}^{ext-shift} = 0$, $|\mathcal{S}| = 1$, $W = 2P^{reg-shift}$, and let $D_{p,1}$ be a non-negative integer number for each time period.

We now construct a circulant problem with the following parameters. Let $b = D_{p,1}$ for $p \in \mathcal{P}$. We construct A with $m = n = 2P^{reg-shift} + 2P^{reg-rest} = |\mathcal{P}| = |\mathcal{P}^{start}|$, in a way that the first column starts with $P^{reg-shift}$ values of one and $m - P^{reg-shift}$ values of zero, followed by again $P^{reg-shift}$ values of one and $m - P^{reg-shift}$ values of zero. Columns $i \in \{2; \cdots; m - P^{reg-shift} + 1\}$ start with $i-1$ values of zero, followed by $P^{reg-shift}$ values of one, $m - P^{reg-shift}$ values of zero, $P^{reg-shift}$ values of one, and $m - P^{reg-shift} - i + 1$ values of zero. Columns $i \in \{m - P^{reg-shift} + 2; \cdots; m\}$ start with $i - P^{reg-shift}$ values of one, followed by $m - P^{reg-shift}$ values of zero, $P^{reg-shift}$ values of one, $m - P^{reg-shift}$ values of zero, and $2P^{reg-shift} - i$ values of one. Note that we drop shift rosters with only one shift as they are dominated by shift rosters with two shifts. An optimal solution of the resulting circulant problem is also an optimal solution of the special case described above of the variable shift extension problem with flexible shift starting times and durations. □

PROPOSITION 2. *LB_1 is a lower bound for the objective function.*

Proof: The proposition follows directly from (2), (9), and (12). The maximum number of physicians that may have to be present at the same time period multiplied by the costs for one physician provides a lower bound for the objective function.

$$\sum_{i \in \mathcal{I}} (x^{reg}_{i,p} + x^{ext}_{i,p}) \geq D_{p,s} \qquad p \in \mathcal{P}, s \in \mathcal{S} \tag{41}$$

$$\rightarrow \sum_{i \in \mathcal{I}} y_i \geq D_{p,s} \qquad p \in \mathcal{P}, s \in \mathcal{S} \tag{42}$$

$$\rightarrow \sum_{i \in \mathcal{I}} y_i \geq \operatorname*{Max}_{p \in \mathcal{P}, S \in \mathcal{S}} \{D_{ps}\} \tag{43}$$

□

PROPOSITION 3. *LB_2 is a lower bound for the objective function.*

Proof: Determination of Reg_p. Note that $1 - F_p(i)$ is the probability that the realized demand exceeds i physicians in period p. We reformulate (3) and (4) to (44) and (45) and combine those to (46). We further reformulate (12) to (47). Extending (47) to all physicians, we obtain (48) with the objective function on the left and a lower bound based on Reg_p and Ext_p on the right side of the inequality. Replacing Ext_p from (46) leads to (49), where the lower bound only includes Reg_p as a variable.

$$o_{p,s} \geq D_{p,s} - Reg_p \qquad p \in \mathcal{P}, s \in \mathcal{S} \tag{44}$$

$$Ext_p \geq \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s o_{p,s} \qquad p \in \mathcal{P} \tag{45}$$

$$\rightarrow Ext_p \geq \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s \text{Max}\{0, D_{p,s} - Reg_p\} \qquad p \in \mathcal{P} \tag{46}$$

$$y_i \geq \frac{1}{W} \sum_{p \in \mathcal{P}} (x_{i,p}^{reg} + T x_{i,p}^{ext}) \qquad i \in \mathcal{I} \tag{47}$$

$$\rightarrow \sum_{i \in \mathcal{I}} y_i \geq \frac{1}{W} \sum_{p \in \mathcal{P}} (Reg_p + T Ext_p) \tag{48}$$

$$\rightarrow \sum_{i \in \mathcal{I}} y_i \geq \frac{1}{W} \sum_{p \in \mathcal{P}} (Reg_p + T \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s \text{Max}\{0, D_{p,s} - Reg_p\}) \tag{49}$$

The smallest possible value of the right hand side of (49) defines a valid lower bound for the optimal objective function. Thus, we are looking for the value of Reg_p that minimizes this term. To simplify (49), we drop $\frac{1}{W}$ as it is not relevant for the minimization, and decompose the term into periods p and obtain (50).

$$Reg_p + T \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s \text{Max}\{0, D_{p,s} - Reg_p\} \qquad p \in \mathcal{P}. \tag{50}$$

We differentiate (50) with respect to Reg_p and obtain (51). Using $\sum_{s \in \mathcal{S}} \Pi_s \mathbb{1}_{D_{p,s} > Reg_p} = 1 - F_p(Reg_p)$, we obtain (52). Dissolving (52) after Reg_p leads to (53) indicating Reg_p as in LB_2.

$$1 - T \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s \mathbb{1}_{D_{p,s} > Reg_p} = 0 \qquad p \in \mathcal{P} \tag{51}$$

$$\rightarrow \frac{1}{U}(1 - F_p(Reg_p)) = \frac{1}{T} \qquad p \in \mathcal{P} \tag{52}$$

$$\rightarrow Reg_p = F_p^{-1}(1 - \frac{U}{T}) \qquad p \in \mathcal{P} \tag{53}$$

Determination of Ext_p^{exp}: Each demand that is not covered by regular workers needs to be covered by shift extensions. The number of required physicians on shift extension to cover the workload are directly derived from (46) by rounding up to the next integer in (54).

$$Ext_p^{exp} = \lceil \frac{1}{U} \sum_{s \in \mathcal{S}} \Pi_s \text{Max}\{0, D_{p,s} - Reg_p\} \rceil \qquad p \in \mathcal{P} \qquad (54)$$

Determination of Ext_p^{max}: As demand needs to be covered in all scenarios, at least the difference between the maximum demand and the number of physicians assigned to a regular working period needs to be covered by shift extensions. We reformulate (2) and round up to the next integer in (55) to determine Ext_p^{max}.

$$Ext_p^{max} = \lceil \underset{s \in \mathcal{S}}{\text{Max}}\{D_{p,s}\} - Reg_p \rceil \qquad p \in \mathcal{P} \qquad (55)$$

Determination of Ext_p: The higher number of Ext_p^{exp} and Ext_p^{max} defines the minimum number of physicians assigned to a shift extension in period p.

Determination of LB_2: The lower bound based on Reg_p and Ext_p was already derived in (48). We further increase it by rounding up the number of required physicians to the next integer and obtain (56).

$$\rightarrow \sum_{i \in \mathcal{I}} y_i \geq \lceil \frac{1}{W} \sum_{p \in \mathcal{P}} (Reg_p + TExt_p) \rceil \qquad (56)$$

□

2. Demand scenarios

We chose 12 demand scenarios selected by hospital management and claim that those 12 scenarios are quite representative for the 52 week horizon of the historic data: The average OR load in the 12 scenarios is 445.6 hours (445.7 over all 52 weeks) with a standard deviation of 82.7 (92.5 over all 52 weeks, 70.1 excluding the Christmas week, where no scheduled surgeries took place); the 25th percentile is 427.5 (428.5 over all 52 weeks) and the 75th percentile is 499.0 (500.3 over all 52 weeks). Please find the average OR load over all 168 periods (one week) for all 52 weeks and the 12 scenarios in Figure 1.

3. Seasonality

Our data does not show a strong yearly seasonal pattern (see Figure 2). However, some drops in demand may be observed, especially during the winter vacation periods (first and last week of the year). Thus, a specific holiday work schedule could be defined for those weeks.

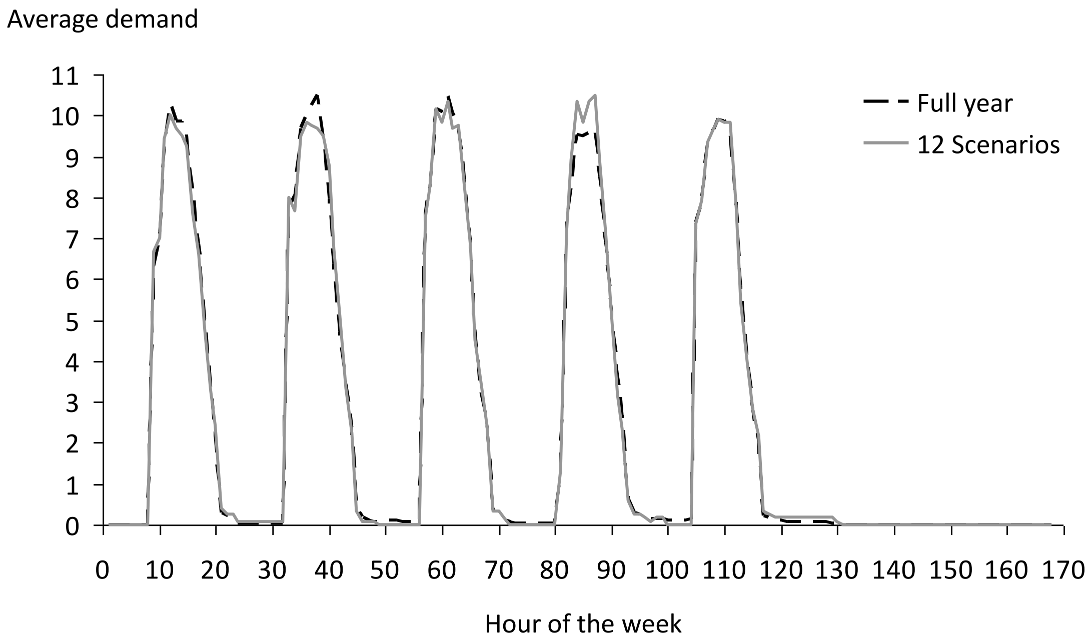

Figure 1 **Comparison average demand per hour of the week: 12 scenarios versus 52 weeks**

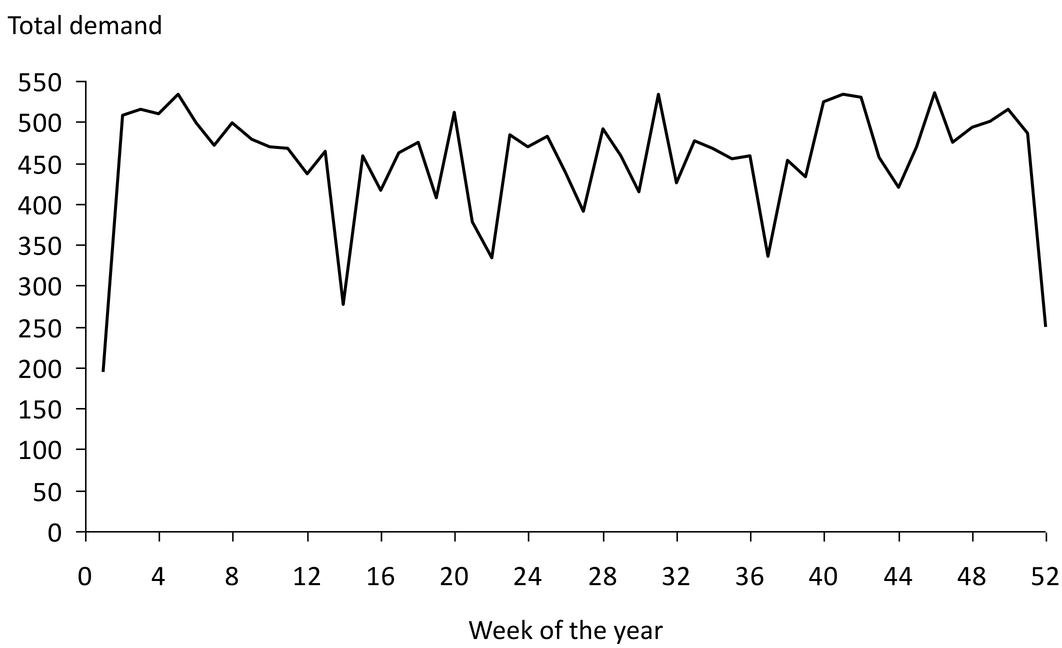

Figure 2 **Total weekly demand over the year**

References

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