Assortment Optimization under the Multinomial Logit Model with Sequential Offerings

Online Supplement

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A Appendix: Upper Bound on State Variable

In this section, we give a proof for Lemma 6. First, we show that if $\widehat{f}^k > \lceil n R_{\max} \rceil$, then we have $f^k_{n+1} > 0$. Since $f^k_{i+1} = \lceil f^k_i - r_i\, v^k_i\, x^k_i \rceil$, we have $f^k_{i+1} \geq f^k_i - r_i\, v^k_i\, x^k_i$. Adding this inequality over all $i \in N$ and noting that $f^k_1 = \widehat{f}^k$, along with the definition of $R_{\max}$, we obtain $f^k_{n+1} \geq \widehat{f}^k - \sum_{i \in N} r_i\, v^k_i\, x^k_i \geq \widehat{f}^k - n R_{\max} \geq \widehat{f}^k - \lceil n R_{\max} \rceil$. In this case, the last inequality implies that if $\widehat{f}^k > \lceil n R_{\max} \rceil$, then we have $f^k_{n+1} > 0$. Second, we show that if $\widehat{h}^k \geq \lceil \Delta(\rho, n)\, V_{\max} \rceil$, then we have $h^k_{n+1} \geq 0$. We claim that if $h^k_i \geq \Delta(\rho, n+1-i)\, V_{\max}$, then $h^k_{i+1} \geq \Delta(\rho, n-i)\, V_{\max}$. To see the claim, if $h^k_i \geq \Delta(\rho, n+1-i)\, V_{\max}$, then we have

$$\begin{aligned} h^k_{i+1} = \lfloor h^k_i - v^k_i\, x^k_i \rfloor \geq \frac{1}{1+\rho}\,(h^k_i - v^k_i\, x^k_i) \geq \frac{1}{1+\rho}\,(\Delta(\rho, n+1-i) - 1)\, V_{\max} \\ = \frac{\frac{(1+\rho)^{n+1-i}-1}{\rho} - 1}{1+\rho}\, V_{\max} = \frac{(1+\rho)^{n-i} - 1}{\rho}\, V_{\max} = \Delta(\rho, n-i)\, V_{\max}, \end{aligned}$$

where the first inequality holds since $h_i - v^k_i\, x^k_i \geq \Delta(\rho, n+1-i)\, V_{\max} - V_{\max} \geq 0$ and $\lfloor x \rfloor \geq x/(1+\rho)$ for any $x \in \Re_+$ and the second equality is by the definition of $\Delta(\rho, n)$. The chain of inequalities above establishes the claim. Using the claim, if $h^k_1 \geq \Delta(\rho, n)\, V_{\max}$, then $h^k_2 \geq \Delta(\rho, n-1)\, V_{\max}$, but using the claim once more, if $h^k_2 \geq \Delta(\rho, n-1)\, V_{\max}$, then $h^k_3 \geq \Delta(\rho, n-2)\, V_{\max}$. Using the claim successively, if $h^k_1 \geq \Delta(\rho, n)\, V_{\max}$, then $h^k_{n+1} \geq \Delta(\rho, 0)\, V_{\max}$. Since $h^k_1 = \widehat{h}^k$, if $\widehat{h}^k \geq \lceil \Delta(\rho, n)\, V_{\max} \rceil$, then $h^k_1 = \widehat{h}^k \geq \Delta(\rho, n)\, V_{\max}$, but if $h^k_1 \geq \Delta(\rho, n)\, V_{\max}$, then $h^k_{n+1} \geq \Delta(\rho, 0)\, V_{\max}$. By the definition of $\Delta(\rho, n)$, we have $\Delta(\rho, 0) = 0$, so it follows that if $\widehat{h}^k \geq \lceil \Delta(\rho, n)\, V_{\max} \rceil$, then $h^k_{n+1} \geq 0$.

B Appendix: Computation of Thresholds

In the next lemma, we show the relationship between the value functions $\{V_i(\cdot, \cdot) : i \in N\}$ and $\{J_i(\cdot, \cdot) : i \in N\}$ that are computed through the dynamic programs in (13) and (15).

Lemma 8 *For any* $(\boldsymbol{f}_i, \boldsymbol{h}_i) \in \mathsf{Dom}^m_+ \times \mathsf{Dom}^m_+$ *and* $i \in N$, *if* $f^1_i \leq \lfloor J_i(\boldsymbol{f}^{-1}_i, \boldsymbol{h}_i) \rfloor$, *then we have* $V_i(\boldsymbol{f}_i, \boldsymbol{h}_i) = 0$. *Similarly, if* $f^1_i > \lfloor J_i(\boldsymbol{f}^{-1}_i, \boldsymbol{h}_i) \rfloor$, *then we have* $V_i(\boldsymbol{f}_i, \boldsymbol{h}_i) = -\infty$.

Proof. We use induction over the products to show that if $f^1_i \leq \lfloor J_i(\boldsymbol{f}^{-1}_i, \boldsymbol{h}_i) \rfloor$, then we have $V_i(\boldsymbol{f}_i, \boldsymbol{h}_i) = 0$ for any $(\boldsymbol{f}_i, \boldsymbol{h}_i) \in \mathsf{Dom}^m_+ \times \mathsf{Dom}^m_+$ and $i \in N \cup \{n+1\}$. For any $(\boldsymbol{f}_{n+1}, \boldsymbol{h}_{n+1}) \in \mathsf{Dom}^m_+ \times \mathsf{Dom}^m_+$, since $f^1_{n+1} \geq 0$ and $J_{n+1}(\cdot, \cdot)$ takes only the value zero or $-\infty$, if $f^1_{n+1} \leq \lfloor J_{n+1}(\boldsymbol{f}^{-1}_{n+1}, \boldsymbol{h}_{n+1}) \rfloor$, then we must have $J_{n+1}(\boldsymbol{f}^{-1}_{n+1}, \boldsymbol{h}_{n+1}) = 0$ and $f^1_{n+1} \leq 0$. By the boundary condition of the dynamic program in (15), if $J_{n+1}(\boldsymbol{f}^{-1}_{n+1}, \boldsymbol{h}_{n+1}) = 0$, then we must have $f^k_{n+1} \leq 0$ for all $k \in M^{-1}$ and $h^k_{n+1} \geq 0$ for all $k \in M$. Thus, if $f^1_{n+1} \leq \lfloor J_{n+1}(\boldsymbol{f}^{-1}_{n+1}, \boldsymbol{h}_{n+1}) \rfloor$, then we must have $f^1_{n+1} \leq 0$, $f^k_{n+1} \leq 0$ for all $k \in M^{-1}$ and $h^k_{n+1} \geq 0$ for all $k \in M$, in which case, by the boundary condition of the dynamic program in (13), we have $V_{n+1}(\boldsymbol{f}_{n+1}, \boldsymbol{h}_{n+1}) = 0$. Therefore, the result holds for product $n+1$. Next, we assume that the result holds for product $i+1$ and we show that the result holds for product i. Consider $(\boldsymbol{f}_i, \boldsymbol{h}_i) \in \mathsf{Dom}^m_+ \times \mathsf{Dom}^m_+$ such that $f^1_i \leq \lfloor J_i(\boldsymbol{f}^{-1}_i, \boldsymbol{h}_i) \rfloor$. We use $\widehat{\boldsymbol{x}}_i$ to denote an optimal solution to the problem on the right side of (15). Since $f_i^1 \leq \lfloor J_i(\boldsymbol{f}_i^{-1}, \boldsymbol{h}_i) \rfloor$, noting the dynamic program in (15), we have

$$f_i^1 \leq \left\lfloor r_i\, v_i^1\, \widehat{x}_i^1 + \left\lfloor J_{i+1}\left(\left\lceil \boldsymbol{f}_i^{-1} - \sum_{k \in M^{-1}} \boldsymbol{e}^k\, r_i\, v_i^k\, \widehat{x}_i^k \right\rceil, \left\lfloor \boldsymbol{h}_i - \sum_{k \in M} \boldsymbol{e}^k\, v_i^k\, \widehat{x}_i^k \right\rfloor \right) \right\rfloor \right\rfloor.$$

By a simple lemma, given as Lemma 9 below, for any $a, b \in \mathsf{Dom}$ and $\alpha \in \Re$, we have $a \leq \lfloor \alpha + b \rfloor$ if and only if $\lceil a - \alpha \rceil \leq b$. Thus, the inequality above implies that we have

$$\lceil f_i^1 - r_i\, v_i^1\, \widehat{x}_i^1 \rceil \leq \left\lfloor J_{i+1}\left(\left\lceil \boldsymbol{f}_i^{-1} - \sum_{k \in M^{-1}} \boldsymbol{e}^k\, r_i\, v_i^k\, \widehat{x}_i^k \right\rceil, \left\lfloor \boldsymbol{h}_i - \sum_{k \in M} \boldsymbol{e}^k\, v_i^k\, \widehat{x}_i^k \right\rfloor \right) \right\rfloor.$$

For notational brevity, we let $f_{i+1}^k = \lceil f_i^k - r_i\, v_i^k\, \widehat{x}_i^k \rceil$ and $h_{i+1}^k = \lfloor h_i^k - v_i^k\, \widehat{x}_i^k \rfloor$ for all $k \in M$, in which case, the inequality above is equivalent to $f_{i+1}^1 \leq \lfloor J_{i+1}(\boldsymbol{f}_{i+1}^{-1}, \boldsymbol{h}_{i+1}) \rfloor$, but by the induction argument, if $f_{i+1}^1 \leq \lfloor J_{i+1}(\boldsymbol{f}_{i+1}^{-1}, \boldsymbol{h}_{i+1}) \rfloor$, then $V_{i+1}(\boldsymbol{f}_{i+1}, \boldsymbol{h}_{i+1}) = 0$. Therefore, noting the definitions of f_{i+1}^k and h_{i+1}^k, we have $V_{i+1}(\boldsymbol{f}_{i+1}, \boldsymbol{h}_{i+1}) = V_{i+1}(\lceil \boldsymbol{f}^k - \sum_{k \in M} \boldsymbol{e}^k\, r_i\, v_i^k\, \widehat{x}_i^k \rceil, \lfloor \boldsymbol{h}^k - \sum_{k \in M} \boldsymbol{e}^k\, v_i^k\, \widehat{x}_i^k \rfloor) = 0$. By last equality, the solution $\widehat{\boldsymbol{x}}_i$ provides an objective value of zero for the problem on the right side of (13). Since $V_i(\cdot, \cdot)$ takes only the value zero or $-\infty$ and there exists a solution to the problem on the right side of (13) that provides an objective value of zero, we must have $V_i(\boldsymbol{f}_i, \boldsymbol{h}_i) = 0$, completing the induction argument. The discussion so far shows that if $f_i^1 \leq \lfloor J_i(\boldsymbol{f}_i^{-1}, \boldsymbol{h}_i) \rfloor$, then we have $V_i(\boldsymbol{f}_i, \boldsymbol{h}_i) = 0$ for any $(\boldsymbol{f}_i, \boldsymbol{h}_i) \in \mathsf{Dom}_+^m \times \mathsf{Dom}_+^m$ and $i \in N \cup \{n+1\}$, establishing the first statement in the lemma. The second statement uses a similar reasoning. $\square$

In the next lemma, we show a result that we use in the proof of Lemma 8.

Lemma 9 *For any $a, b \in \mathsf{Dom}$ and $\alpha \in \Re$, we have $a \leq \lfloor \alpha + b \rfloor$ if and only if $\lceil a - \alpha \rceil \leq b$.*

Proof. First, we show that if $a \leq \lfloor \alpha + b \rfloor$, then we have $\lceil a - \alpha \rceil \leq b$. If $a \leq \lfloor \alpha + b \rfloor$, then $a \leq \alpha + b$, so that $a - \alpha \leq b$. Since $b \in \mathsf{Dom}$, having $a - \alpha \leq b$ implies that $\lceil a - \alpha \rceil \leq b$, as desired. Second, we show that if $\lceil a - \alpha \rceil \leq b$, then we have $a \leq \lfloor \alpha + b \rfloor$. If $\lceil a - \alpha \rceil \leq b$, then $a - \alpha \leq b$, so that $a \leq \alpha + b$. Since $a \in \mathsf{Dom}$, having $a \leq \alpha + b$ implies that $a \leq \lfloor \alpha + b \rfloor$, as desired. $\square$

C Appendix: Nested by Revenue Sets

We show that there exists an optimal solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$ to problem (1) that satisfies $\cup_{k \in M} \widehat{S}^k = \{i \in N : r_i \geq \widehat{\zeta}\}$ for some constant $\widehat{\zeta}$. In other words, the union of the sets offered over all stages is a nested by revenue set. Therefore, if we index the products such that $r_1 \geq r_2 \geq \ldots \geq r_n$, then an optimal solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$ to problem (1) is of the form $\cup_{k \in M} \widehat{S}^k = \{1, \ldots, i\}$ for some $i \in N$. Although this result gives some insight into the structure of the optimal solution, it does not allow us to obtain an optimal solution to problem (1) efficiently, since this result does not specify the stage in which each product should be offered. To show that there exists an optimal solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$ to problem (1) that satisfies $\cup_{k \in M} \widehat{S}^k = \{i \in N : r_i \geq \widehat{\zeta}\}$ for some constant $\widehat{\zeta}$, we use a recursive version of the objective function of problem (1). We use $R^\nu(S^\nu, \ldots, S^m)$ to denote the expected revenue obtained from a customer starting her choice process in stage ν when we offer the sets $S^\nu, \ldots, S^m$ in stages $\nu, \ldots, m$. Thus, noting the expected revenue expression in (1) and focusing only on the stages $\nu, \ldots, m$, $R^\nu(S^\nu, \ldots, S^m)$ is given by

$$R^\nu(S^\nu, \ldots, S^m) = \sum_{k=\nu}^{m} \left\{ \prod_{\ell=\nu}^{k} \frac{1}{1 + \sum_{i \in S^\ell} v_i^\ell} \right\} \sum_{i \in S^k} r_i \, v_i^k. \tag{18}$$

Comparing the expression above with (1), note that $R^1(S^1, \ldots, S^m)$ corresponds to the objective function of problem (1). In the next proposition, we show that there exists an optimal solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$ to problem (1) that satisfies $\cup_{k \in M} \widehat{S}^k = \{i \in N : r_i \geq \widehat{\zeta}\}$ for some constant $\widehat{\zeta}$ and the constant $\widehat{\zeta}$ is given by $\min\{R^k(\widehat{S}^k, \ldots, \widehat{S}^m) : k \in M\}$.

Proposition 10 *There exists an optimal solution* $(\widehat{S}^1, \ldots, \widehat{S}^m)$ *to problem* (1) *that satisfies* $\cup_{k \in M} \widehat{S}^k = \{i \in N : r_i \geq \widehat{\zeta}\}$, *where* $\widehat{\zeta} = \min\{R^k(\widehat{S}^k, \ldots, \widehat{S}^m) : k \in M\}$.

Proof. Let $(\widehat{S}^1, \ldots, \widehat{S}^m)$ be an optimal solution to problem (1) with the largest cardinality, so that if $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$ is another optimal solution, then $| \cup_{k \in M} \widehat{S}^k| \geq | \cup_{k \in M} \widetilde{S}^k|$. By (18), we have

$$\begin{aligned} R^\nu(S^\nu, \ldots, S^m) &= \frac{\sum_{i \in S^\nu} r_i \, v_i^\nu}{1 + \sum_{i \in S^\nu} v_i^\nu} + \frac{1}{1 + \sum_{i \in S^\nu} v_i^\nu} \sum_{k=\nu+1}^{m} \left\{ \prod_{\ell=\nu+1}^{k} \frac{1}{1 + \sum_{i \in S^\ell} v_i^\ell} \right\} \sum_{i \in S^k} r_i \, v_i^k \\ &= \frac{\sum_{i \in S^\nu} r_i \, v_i^\nu + R^{\nu+1}(S^{\nu+1}, \ldots, S^m)}{1 + \sum_{i \in S^\nu} v_i^\nu}. \end{aligned} \tag{19}$$

First, we show that $\cup_{k \in M} \widehat{S}^k \supseteq \{i \in N : r_i \geq \widehat{\zeta}\}$. To get a contradiction assume that there exists a product j such that $j \in \{i \in N : r_i \geq \widehat{\zeta}\}$ and $j \notin \cup_{k \in M} \widehat{S}^k$. For notational brevity, we let $\widehat{R}^k = R^k(\widehat{S}^k, \ldots, \widehat{S}^m)$. Since $j \in \{i \in N : r_i \geq \widehat{\zeta}\}$ and $\widehat{\zeta} = \min\{\widehat{R}^k : k \in M\}$, we have $r_j \geq \widehat{R}^\ell$ for some $\ell \in M$. Furthermore, since $j \notin \cup_{k \in M} \widehat{S}^k$, we have $j \notin \widehat{S}^\ell$. We define a solution $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$ to problem (1) as $\widetilde{S}^k = \widehat{S}^k$ for all $k \in M \setminus \{\ell\}$ and $\widetilde{S}^\ell = \widehat{S}^\ell \cup \{j\}$. Since $j \notin \cup_{k \in M} \widehat{S}^k$, the product j is not offered in any stage in the solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$. Therefore, the product j is only offered in stage ℓ in the solution $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$, so $(\widetilde{S}^1, \ldots, \widetilde{S}^m) \in \mathcal{F}$. Letting $\widetilde{R}^k = R(\widetilde{S}^k, \ldots, \widetilde{S}^m)$ for notational brevity, we observe that $\widehat{R}^k = \widetilde{R}^k$ for all $k = \ell+1, \ldots, m$, since $R^k(S^k, \ldots, S^m)$ depends on $S^k, \ldots, S^m$ and $\widehat{S}^k = \widetilde{S}^k$ for all $k = \ell+1, \ldots, m$. In this case, we have

$$\begin{aligned} \widetilde{R}^\ell - \widehat{R}^\ell = \frac{\sum_{i \in \widetilde{S}^\ell} r_i \, v_i^\ell + \widetilde{R}^{\ell+1}}{1 + \sum_{i \in \widetilde{S}^\ell} v_i^\ell} - \widehat{R}^\ell &= \frac{\sum_{i \in \widehat{S}^\ell} r_i \, v_i^\ell + r_j \, v_j^\ell + \widehat{R}^{\ell+1}}{1 + \sum_{i \in \widehat{S}^\ell} v_i^\ell + v_j^\ell} - \widehat{R}^\ell \\ &= \frac{\widehat{R}^\ell \, (1 + \sum_{i \in \widehat{S}^\ell} v_i^\ell) + r_j \, v_j^\ell}{1 + \sum_{i \in \widehat{S}^\ell} v_i^\ell + v_j^\ell} - \widehat{R}^\ell = \frac{(r_j - \widehat{R}^\ell) \, v_j^\ell}{1 + \sum_{i \in \widehat{S}^\ell} v_i^\ell + v_j^\ell} \geq 0, \end{aligned}$$

where the second equality uses the fact that $\widetilde{S}^\ell = \widehat{S}^\ell \cup \{j\}$ and $\widetilde{R}^{\ell+1} = \widehat{R}^{\ell+1}$, the third equality uses the fact $\widehat{R}^\ell \, (1 + \sum_{i \in \widehat{S}^\ell} v_i^\ell) = \sum_{i \in \widehat{S}^\ell} r_i \, v_i^\ell + \widehat{R}^{\ell+1}$ by (19) and the inequality follows from the fact that $r_j \geq \widehat{R}^\ell$. Therefore, we obtain $\widetilde{R}^\ell \geq \widehat{R}^\ell$. By (19), for all $k = 1, \ldots, \ell - 1$, we have $\widetilde{R}^k = (\sum_{i \in \widetilde{S}^k} r_i \, v_i^k + \widetilde{R}^{k+1})/(1 + \sum_{i \in \widetilde{S}^k} v_i^k) = (\sum_{i \in \widehat{S}^k} r_i \, v_i^k + \widetilde{R}^{k+1})/(1 + \sum_{i \in \widehat{S}^k} v_i^k)$, where we use the fact that $\widetilde{S}^k = \widehat{S}^k$. Similarly, we have $\widehat{R}^k = (\sum_{i \in \widehat{S}^k} r_i \, v_i^k + \widehat{R}^{k+1})/(1 + \sum_{i \in \widehat{S}^k} v_i^k)$ for all $k = 1, \ldots, \ell - 1$. Subtracting the two equalities, we obtain $\widetilde{R}^k - \widehat{R}^k = (\widetilde{R}^{k+1} - \widehat{R}^{k+1})/(1 + \sum_{i \in \widehat{S}^k} v_j^k)$ for all $k = 1, \ldots, \ell - 1$. In this case, having $\widetilde{R}^\ell \geq \widehat{R}^\ell$ implies that $\widetilde{R}^1 \geq \widehat{R}^1$. Therefore, the objective value provided by the solution $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$ for problem (1) is at least as large as the one provided by the solution $(\widehat{S}^1, \ldots, \widehat{S}^m)$. Furthermore, $\cup_{k \in M} \widetilde{S}^k = \cup_{k \in M} \widehat{S}^k \cup \{j\}$, which contradicts the fact that $(\widehat{S}^1, \ldots, \widehat{S}^m)$ is an optimal solution to problem (1) with the largest cardinality.

Second, we show that $\cup_{k \in M} \widehat{S}^k \subseteq \{i \in N : r_i \geq \widehat{\zeta}\}$. To get a contradiction, assume that there exists a product j such that $j \in \cup_{k \in M} \widehat{S}^k$ and $j \notin \{i \in N : r_i \geq \widehat{\zeta}\}$. Since $j \in \cup_{k \in M} \widehat{S}^k$, we have $j \in \widehat{S}^\ell$ for some $\ell \in M$. Also, noting that $j \notin \{i \in N : r_i \geq \widehat{\zeta}\}$, we have $r_j < \widehat{\zeta} = \min\{\widehat{R}^k : k \in M\}$, which implies that $r_j < \widehat{R}^\ell$. We define a solution $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$ to problem (1) as $\widetilde{S}^k = \widehat{S}^k$ for all $k \in M \setminus \{\ell\}$ and $\widetilde{S}^\ell = \widehat{S}^\ell \setminus \{j\}$. Since $(\widehat{S}^1, \ldots, \widehat{S}^m) \in \mathcal{F}$ and $\widetilde{S}^k \subseteq \widehat{S}^k$ for all $k \in M$, each product is offered in at most one stage in the solution $(\widetilde{S}^1, \ldots, \widetilde{S}^m)$, so $(\widetilde{S}^1, \ldots, \widetilde{S}^m) \in \mathcal{F}$. Using the same argument in the previous paragraph and noting that $r_j < \widehat{R}^\ell$, we can show that $\widetilde{R}^1 > \widehat{R}^1$, contradicting the fact that $(\widehat{S}^1, \ldots, \widehat{S}^m)$ is an optimal solution to problem (1). ☐

The proposition above indicates that there exists an optimal solution to problem (1) that has a structure similar to that of an optimal solution when there is a single stage in the choice process. This structure is adequate to obtain an optimal solution efficiently when there is a single stage, but it is not adequate even when there are as few as two stages.

D Appendix: Structure of Data Files

In the file `test_problems_section_9.2.csv`, we list all of the 400 randomly-generated test problems used in Section 9.2. There are eight parameter configurations, which are labeled by $(N, 0.05)$, $(O, 0.05)$, $(N, 0.10)$, $(O, 0.10)$, $(N, 0.20)$, $(O, 0.20)$, $(N, 0.30)$, $(O, 0.30)$. In each parameter configuration, we randomly generate 50 problem instances. In the file, we give the parameter configuration first, then list all 50 problem instances in the parameter configuration. Note that each block of 252 lines gives the problem instances in one parameter configuration. One needs to scroll down in the file to reach different parameter configurations. Each problem instance includes the following information in four lines.

- Line 1: Instance number;
- Line 2: 18 values giving the revenues of the products;
- Line 3: 18 values giving the preference weights of the products;
- Line 4: 5 values giving the revenues or bounds from different approaches: (1) upper bound from the linear program in Section 9.1, (2) revenue from the FPTAS with $\epsilon = 1/4$, (3) revenue from the FPTAS with $\epsilon = 1/2$, (4) revenue from the iterative exchange heuristic, (5) revenue

when all products are offered in one stage.

In the file `test_problems_section_9.3.csv`, we list all of the 12 test problems used in Section 9.3. There are 12 test problems, which are captured by the parameter configurations $(U, 1)$, $(E, 1)$, $(U, 2)$, $(E, 2)$, $(U, 3)$, $(E, 3)$, $(U, 4)$, $(E, 4)$, $(U, 5)$, $(E, 5)$, $(U, 6)$, $(E, 6)$. Each problem instance includes the following information in three lines.

- Line 1: 18 values giving the revenues of the products;
- Line 2: 18 values giving the preference weights of the products;
- Line 3: 5 values giving the revenues or bounds from different approaches: (1) upper bound from the linear program in Section 9.1, (2) revenue from the FPTAS with $\epsilon = 1/4$, (3) revenue from the FPTAS with $\epsilon = 1/2$, (4) revenue from the iterative exchange heuristic, (5) revenue when all products are offered in one stage.