Online Supplement for Paper: Multi-Product Newsvendor Problem with Customer-driven Demand Substitution: A Stochastic Integer Program Perspective

Jie Zhang
Department of Industrial and Systems Engineering, Virginia Tech, jiezhang@vt.edu
Weijun Xie
Department of Industrial and Systems Engineering, Virginia Tech, wxie@vt.edu
Subhash C. Sarin
Department of Industrial and Systems Engineering, Virginia Tech, sarins@vt.edu

A. Proofs

A.1 Proof of Theorem 1

Theorem 1 *Let $\boldsymbol{Q}^*$ be optimal order quantities for the MPNP-CDS(D)* (4)*. Then,*

(i)

$$Q_j^* = \begin{cases} 0, & \text{if } \bar{P}_j - \sum_{i\in[n]\setminus\Gamma^*} \alpha_{ji}\bar{P}_i < 0 \\ D_j^s(\boldsymbol{Q}^*) = D_j + \sum_{i\in\Gamma^*} \alpha_{ij} D_i, & \text{otherwise.} \end{cases} \tag{5}$$

for each $j \in [n]$; and

(ii)

$$v_D^* = \max_{\Gamma\subseteq[n]} \left\{ f(\Gamma) := \sum_{j\in\Gamma} \sum_{i\in[n]\setminus\Gamma} \alpha_{ji}\bar{P}_i D_j + \sum_{i\in[n]\setminus\Gamma} \bar{P}_i D_i \right\} := f(\Gamma^*), \tag{6}$$

where $[n] \setminus \operatorname{supp}(\boldsymbol{Q}^*) = \Gamma^*$, i.e., $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$.

Proof: We prove the result by using the following three arguments.

(1) Let $x_i = Q_i - D_i$ denotes the unsold units of ith product for each $i \in [n]$. Then, Model (4) is equivalent to

$$v_D^* = \max_{\boldsymbol{x}\geq -\boldsymbol{D}} \left\{ g(\boldsymbol{x}) := \sum_{i\in[n]} \bar{P}_i x_i - \sum_{i\in[n]} \bar{S}_i \left(x_i - \sum_{j\in[n]} \alpha_{ji} (-x_j)_+ \right)_+ + \sum_{i\in[n]} \bar{P}_i D_i \right\}, \tag{27}$$

To simplify function $g(\boldsymbol{x})$, we classify n products into the following three sets according to the value of $\boldsymbol{x}$, i.e.,

$$I_+ = \{i : x_i \geq 0\}, I_- = \{i : x_i \leq 0\}, I_{++} = \left\{ i \in I_+ : x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0 \right\}.$$

Consequently, we can remove $(\cdot)_+$ from (27), and we have

$$g(\boldsymbol{x}) = \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i x_i + \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i\right) x_i + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that in the above function, for each $i \in I_+ \setminus I_{++}$, the coefficient of x_i is positive as $\bar{P}_i = p_i - c_i > 0$, and by definition, $x_i \leq -\sum_{j \in I_-} \alpha_{ji} x_j$. Therefore, by letting $x_i = -\sum_{j \in I_-} \alpha_{ji} x_j$ for each $i \in I_+ \setminus I_{++}$, function $g(\boldsymbol{x})$ is upper bounded by

$$\begin{aligned} g(\boldsymbol{x}) &\leq \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i \left(-\sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i\right) x_i + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i \\ &= \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i\right) \left(x_i + \sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i. \end{aligned}$$

For each $i \in I_{++}$, we have $\bar{P}_i - \bar{S}_i = s_i - c_i < 0$ by definition, and $x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0$ by definition of set I_{++}. Thus, by letting $I_{++} = \emptyset$, function $g(\boldsymbol{x})$ is further upper bounded by

$$g(\boldsymbol{x}) \leq \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that for each $j \in I_-$, we note that $x_j \in [-D_j, 0]$. Hence, for each $j \in I_-$, let $x_j = 0$ if $\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \geq 0$, and $-D_j$, otherwise. Then $g(\boldsymbol{x})$ is further upper bounded by

$$g(\boldsymbol{x}) \leq \sum_{j \in I_-} \left(\sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i,$$

where the equality is achieved when $I_{++} = \emptyset$ and for each $j \in [n]$,

$$x_j = \begin{cases} 0, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \leq 0, j \in I_- \\ -D_j, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j > 0, j \in I_- \\ -\sum_{i \in I_-} \alpha_{ij} x_i, & \text{otherwise} \end{cases}. \tag{28}$$

Note that $I_+ = [n] \setminus I_-$. Therefore, Model (27) is further equivalent to the following combinatorial optimization problem

$$v_D^* = \max_{I_- \subseteq [n]} \left\{ \widehat{g}(I_-) := \sum_{j \in I_-} \left(\sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i \right\}. \tag{29}$$

(2) Next, we prove the following property of Model (29).

Claim 1 *In the Model (29), for any subset $I_- \subseteq [n]$, let $J_0 = \left\{ j \in I_- : \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i \leq \bar{P}_j \right\}$, then $\widehat{g}(I_-) \leq \widehat{g}(I_- \setminus J_0)$.*

Proof: Let us define $\widehat{I}_- = I_- \setminus J_0$. By definitions of sets I_-, J_0 and $\widehat{I}_-$, we have

$$\begin{aligned}
\widehat{g}(I_-) &= \sum_{j \in I_-} \left(\sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i \\
&= \sum_{j \in I_- \setminus J_0} \left(\sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i \\
&= \sum_{j \in I_- \setminus J_0} \left(\sum_{i \in [n] \setminus (I_- \setminus J_0)} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i - \sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \\
&= \widehat{g}(\widehat{I}_-) - \sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \\
&\leq \widehat{g}(\widehat{I}_-)
\end{aligned}$$

where the inequality holds due to $\sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \geq 0$.

◇

By Claim 1 and equation (28), we note that there exists an optimal solution to Model (27) $\boldsymbol{x}^*$ with subset $I_-^* := \left\{ j : \sum_{i \in [n] \setminus I_-^*} \alpha_{ji} \bar{P}_i > \bar{P}_j \right\}$ such that

$$x_j^* = \begin{cases} -D_j, & \text{if } j \in I_-^* \\ -\sum_{i \in I_-} \alpha_{ij} x_i^*, & \text{otherwise.} \end{cases}.$$

Let $\boldsymbol{Q}^* = \boldsymbol{x}^* + \boldsymbol{D}$, and $\Gamma^* = I_-^*$. Clearly, $\boldsymbol{Q}^*$ satisfies (5) and is an optimal solution to Model (4) and $v_D^* = f(\Gamma^*)$.

(3) Finally, by Claim 1 and letting $\Gamma = I_-$, Model (29) reduces to

$$v_D^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j\in\Gamma} \left(\sum_{i\in[n]\setminus\Gamma} \alpha_{ji}\bar{P}_i - \bar{P}_j \right) D_j + \sum_{i\in[n]} \bar{P}_i D_i \right\},$$

which is equivalent to (6).

□

A.2 Proof of Proposition 1

Proposition 1 *The set function $f(\Gamma)$, defined in (6), is submodular.*

Proof: Let $A \subseteq B \subseteq [I], k \in [n] \setminus B$. By Definition 1, we only need to show that

$$f(A\cup\{k\}) - f(A) \geq f(B\cup\{k\}) - f(B).$$

Note that

$$\begin{aligned} f(B\cup\{k\}) - f(B) &= \sum_{j\in B\cup\{k\}} \sum_{i\in[n]\setminus(B\cup\{k\})} \alpha_{ji}\bar{P}_i D_j - \sum_{j\in B}\sum_{i\in[n]\setminus B} \alpha_{ji}\bar{P}_i D_j - \bar{P}_k D_k \\ &= \sum_{j\in B} \sum_{i\in[n]\setminus(B\cup\{k\})} \alpha_{ji}\bar{P}_i D_j - \sum_{j\in B}\sum_{i\in[n]\setminus B} \alpha_{ji}\bar{P}_i D_j + \sum_{i\in[n]\setminus(B\cup\{k\})} \alpha_{ki}\bar{P}_i D_k - \bar{P}_k D_k \\ &= -\sum_{j\in B} \alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(B\cup\{k\})} \alpha_{ki}\bar{P}_i D_k - \bar{P}_k D_k \end{aligned}$$

Similarly,

$$f(A\cup\{k\}) - f(A) = -\sum_{j\in A} \alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(A\cup\{k\})} \alpha_{ki}\bar{P}_i D_k - \bar{P}_k D_k.$$

Hence,

$$\begin{aligned} &(f(B\cup\{k\}) - f(B)) - (f(A\cup\{k\}) - f(A)) \\ =& -\sum_{j\in B} \alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(B\cup\{k\})} \alpha_{ki}\bar{P}_i D_k - \left(-\sum_{j\in A} \alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(A\cup\{k\})} \alpha_{ki}\bar{P}_i D_k \right) \\ =& -\sum_{j\in B\setminus A} \alpha_{jk}\bar{P}_k D_j - \sum_{i\in B\setminus A} \alpha_{ki}\bar{P}_i D_k \leq 0 \end{aligned}$$

where the inequality follows because $\sum_{j\in B\setminus A} \alpha_{jk}\bar{P}_k D_j \geq 0$ and $\sum_{i\in B\setminus A} \alpha_{ki}\bar{P}_i D_k \geq 0$. Thus, $f(\Gamma)$ is submodular. □

A.3 Proof of Theorem 2

Theorem 2 *The MPNP-CDS(D) is strongly NP-hard, so is the MPNP-CDS* (3).

Proof: We prove this result by showing that the weighted max-cut problem (WMCP) is a special case of the MPNP-CDS(D).

(Weighted Max-Cut Problem) Given an undirected graph $G=(V,E)$ with $|V|=n$, and a nonnegative integer weight w_{ij} associated with each edge $(i,j)\in E$ in the graph, (and $w_{ij}=0$ if there is no edge between nodes i,j.) find a subset $\Lambda\subseteq V$ which maximizes the total weights of edges between subsets Λ and $[n]\setminus\Lambda$.

Clearly, this problem can be formulated as:

$$v_w=\max_{\Lambda\subseteq[n]}\left\{\sum_{j\in\Lambda}\sum_{i\in[n]\setminus\Lambda}w_{ji}\right\}. \tag{30}$$

Without loss of generality, we assume that there is at lease one edge $(i,j)\in E$ such that $w_{ij}>0$, otherwise, the weighted max-cut problem is trivial.

Consider a special instance of MPNP-CDS(D), where $\alpha_{ji}=\alpha_{ij}=(n+1)w_{ji}$ and $\bar{P}_i=D_i=1$ for all $i,j\in[n]$. Under this setting, Model (6) reduces to

$$v_{DW}=\max_{\Lambda\subseteq[n]}\left\{(n+1)\sum_{j\in\Lambda}\sum_{i\in[n]\setminus\Lambda}w_{ji}+n-|\Lambda|\right\}. \tag{31}$$

Let $\lfloor x\rfloor$ denote the floor function of number x. It remains to show that

Claim 2 $\lfloor\frac{v_{DW}}{n+1}\rfloor=v_w$.

Proof: We separate the proof into two steps.

$v_w\le\lfloor\frac{v_{DW}}{n+1}\rfloor$ Let Λ^* be an optimal solution to (30). Clearly, Λ^* is feasible to (31), thus

$$(n+1)v_w\le(n+1)\sum_{j\in\Lambda^*}\sum_{i\in[n]\setminus\Lambda^*}w_{ji}+n-|\Lambda^*|\le v_{DW}.$$

Due to our assumption that all the weights are integral, we have $v_w\le\lfloor\frac{v_{DW}}{n+1}\rfloor$. Next we show that

$v_w\ge\lfloor\frac{v_{DW}}{n+1}\rfloor$ Suppose that $v_w<\lfloor\frac{v_{DW}}{n+1}\rfloor$, i.e., $v_w\le\lfloor\frac{v_{DW}}{n+1}\rfloor-1$, which implies that

$$(n+1)v_w\le v_{DW}-(n+1).$$

Let $\widehat{\Lambda}$ be an optimal solution to (31). We have

$$(n+1)v_w \le v_{DW} - (n+1) = (n+1)\sum_{j\in\widehat{\Lambda}}\sum_{i\in[n]\setminus\widehat{\Lambda}} w_{ji} + n - |\widehat{\Lambda}| - (n+1)$$

which implies that

$$\sum_{j\in\widehat{\Lambda}}\sum_{i\in[n]\setminus\widehat{\Lambda}} w_{ji} \ge v_w + \frac{1+|\widehat{\Lambda}|}{n+1} > v_w$$

a contradiction that v_w is the optimal value to (30).

$\diamond$

Hence, it follows that we can solve the MPNP-CDS(D) efficiently, only if we can solve the weighted max-cut problem (30) efficiently. However, the weighted max-cut problem is strongly NP-hard. Therefore, the MPNP-CDS is also NP-hard, and consequently, so is the MPNP-CDS. $\square$

A.4 Proof of Proposition 2

Proposition 2 *Model* (6) *is equivalent to*

$$v_D^* = \max_{\boldsymbol{y}} \left\{ \sum_{i\in[n]}\sum_{j\in[n+1]} \frac{1}{4} w_{ij}\left(1 - y_i y_{n+1} + y_j y_{n+1} - y_i y_j\right) : y_i \in \{-1,1\}, \forall i \in [n+1] \right\}. \quad (7)$$

Proof: Let $\widehat{v}_D^*$ be the optimal value of Model (7). We need to show $\widehat{v}_D^* = v_D^*$.

($\widehat{v}_D^* \le v_D^*$) Given an optimal solution $\boldsymbol{y}^*$ of Model (7), we define a set $\widehat{\Gamma} = \{j \in [n] : y_j^* = y_{n+1}^*\}$. Clearly, $\widehat{\Gamma}$ is a feasible solution of Model (6) and

$$\begin{aligned}
\widehat{v}_D^* &= \sum_{i\in[n]}\sum_{j\in[n+1]} \frac{1}{4} w_{ij}\left(1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*\right), \\
&= \sum_{j\in[n]}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*\right) + \sum_{i\in[n]} \frac{1}{2} w_{i(n+1)}\left(1 - y_i^* y_{n+1}^*\right), \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(2 - y_i^* y_{n+1}^* - y_i^* y_j^*\right) + \sum_{j\in[n]\setminus\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(-y_i^* y_{n+1}^* - y_i^* y_j^*\right) + \\
&\quad \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)}, \qquad (32) \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(2 - 2y_i^* y_{n+1}^*\right) + \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)}, \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]\setminus\widehat{\Gamma}} w_{ij} + \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)},
\end{aligned}$$

$$= \sum_{j\in\widehat{\Gamma}} \sum_{i\in[n]\setminus\widehat{\Gamma}} \alpha_{ji}\bar{P}_i D_j + \sum_{i\in[n]\setminus\widehat{\Gamma}} \bar{P}_i D_i,$$

where the third equality is because $y_j^* y_{n+1}^* = 1$ for all $j \in \widehat{\Gamma}$, and -1, otherwise; the fourth equality is due to the definition of set $\widehat{\Gamma}$, we have $y_i^* y_{n+1}^* + y_i^* y_j^* = 2y_i^* y_{n+1}^*$ if $j \in \widehat{\Gamma}$, and 0, otherwise; the fifth equality is due to that $y_i^* y_{n+1}^* = 1$ for all $i \in \widehat{\Gamma}$, -1, otherwise; and the last equality is due to the definition of w_{ij}. Thus, $\widehat{v}_D^* \leq v_D^*$.

($\widehat{v}_D^* \geq v_D^*$) Given an optimal solution Γ^* of Model (6), let us construct vector $\widehat{\boldsymbol{y}} \in \{-1,1\}^{n+1}$ as follows: If $j \in \Gamma^*$, then $\widehat{y}_j = \widehat{y}_{n+1} = 1$, otherwise, $\widehat{y}_j = -1 \neq \widehat{y}_{n+1}$. Clearly, $\widehat{\boldsymbol{y}}$ is feasible to Model (7) and following the same derivation as (32), we have

$$v_D^* = \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4} w_{ij} \left(1 - \widehat{y}_i\widehat{y}_{n+1} + \widehat{y}_j\widehat{y}_{n+1} - \widehat{y}_i\widehat{y}_j\right).$$

Thus, $\widehat{v}_D^* \geq v_D^*$.

□

A.5 Proof of Theorem 3

Theorem 3 *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model* (10)*. Then,*

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \geq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n], \tag{11a}$$

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n]. \tag{11b}$$

Proof: For notational convenience, given a vector $\boldsymbol{Q}$, let $(\boldsymbol{Q}|Q_i \leftarrow q)$ denote a new vector that is the same as $\boldsymbol{Q}$ except that the ith entry is q. Note that $\boldsymbol{Q}^*$ is optimal to (10), i.e.,

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q}\in\mathbb{R}_n^+} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \mathbb{E}\left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+\right] \right\}.$$

We note that in the above optimization model, since $0 < \bar{P}_i < \bar{S}_i$ for each $i \in [n]$ and demand $\tilde{\boldsymbol{D}}$ is nonnegative, thus, $\Pi(\boldsymbol{Q}) < \Pi((\boldsymbol{Q})_+)$ if $\boldsymbol{Q} \notin \mathbb{R}_n^+$ is not a nonnegative vector. Therefore, we can relax the domain of $\boldsymbol{Q}$ to be $\mathbb{R}^n$ as below:

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q}} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \mathbb{E}\left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+\right] \right\}.$$

By the optimality of $\boldsymbol{Q}^*$, for each $i \in [n]$, Q_i^* is also optimal to the following mathematical program

$$Q_i^* \in \operatorname{argmax} \left\{ G(Q_i) := \bar{P}_i Q_i - \bar{S}_i \mathbb{E}\left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}^* | Q_i^* \leftarrow Q_i)\right)_+ \right.$$
$$\left. - \sum_{j\in[n], j\neq i} \mathbb{E}\left(\bar{S}_j \left(Q_j^* - \tilde{D}_j^s\left(\boldsymbol{Q}^* | Q_i^* \leftarrow Q_i\right)\right)_+\right) \right\}.$$

Let $Q_i^1 := Q_i^* + \varepsilon, Q_i^2 := Q_i^* - \varepsilon$, where $\varepsilon > 0$ is a sufficiently small positive constant. Simple calculation yields

$$G\left(Q_i^1\right) - G\left(Q_i^*\right) = \bar{P}_i\varepsilon - \mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*)\right)\bar{S}_i\varepsilon - \sum_{j\in[n], j\neq i} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right)\bar{S}_j\alpha_{ji}\varepsilon \leq 0,$$
$$G\left(Q_i^2\right) - G\left(Q_i^*\right) = -\bar{P}_i\varepsilon + \mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right)\bar{S}_i\varepsilon + \sum_{j\in[n], j\neq i} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right)\bar{S}_j\alpha_{ji}\varepsilon \leq 0,$$

where $G\left(Q_i^1\right) - G\left(Q_i^*\right) \leq 0$, and $G\left(Q_i^2\right) - G\left(Q_i^*\right) \leq 0$ are due to the optimality of Q_i^*. Thus, we arrive at (11). $\square$

A.6 Proof of Proposition 3

Proposition 3 *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model* (10)*. Then, $\boldsymbol{Q}^*$ is upper and lower bounded by $\overline{\boldsymbol{Q}}$ and $\underline{\boldsymbol{Q}}$, respectively, i.e., for each product $i \in [n]$, $\overline{Q}_i \geq Q_i^* \geq \underline{Q}_i$ with*

$$\underline{Q}_i = \begin{cases} F_{\tilde{D}_i}^{-1}\left(\frac{\bar{P}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}\right), & \text{if} \quad \bar{S}_i > \sum_{j\in[n]}\alpha_{ij}\bar{S}_j \\ 0, & \text{otherwise} \end{cases}, \tag{12a}$$
$$\overline{Q}_i = \bar{F}_{\tilde{D}_i + \sum_{j\in[n]}\alpha_{ji}\tilde{D}_j}^{-1}\left(\frac{\bar{P}_i}{\bar{S}_i}\right), \tag{12b}$$

where $F_{\tilde{X}}^{-1}, \bar{F}_{\tilde{X}}^{-1}$ denote the lower and upper inverse distribution function of random variable $\tilde{X}$, respectively, i.e., $F_{\tilde{X}}^{-1}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} \leq \kappa\right) \geq t\right\}$ and $\bar{F}_{\tilde{X}}^{-1}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} < \kappa\right) \geq t\right\}$.

Proof: Note that by definition, for each product $i \in [n]$, we have

$$\tilde{D}_i \leq \tilde{D}_i^s(\boldsymbol{Q}^*) = \tilde{D}_i + \sum_{j\in[n]}\alpha_{ji}\left(\tilde{D}_j - Q_j^*\right)_+ \leq \tilde{D}_i + \sum_{j\in[n]}\alpha_{ji}\tilde{D}_j,$$

where the second inequality holds because $\tilde{D}_j, Q_j^*$ are nonnegative for all $j \in [n]$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\tilde{D}_i \le Q_i^*\right) \ge \mathbb{P}\left(\tilde{D}_i^s(\boldsymbol{Q}^*) \le Q_i^*\right) \ge \mathbb{P}\left(\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j \le Q_i^*\right),\\
&\mathbb{P}\left(\tilde{D}_i < Q_i^*\right) \ge \mathbb{P}\left(\tilde{D}_i^s(\boldsymbol{Q}^*) < Q_i^*\right) \ge \mathbb{P}\left(\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j < Q_i^*\right),
\end{aligned} \tag{33}$$

Now, we separate the rest of the proof into two parts.

(1) Clearly, by (10), $\boldsymbol{Q}^* \in \mathbb{R}_+^n$. According to (11a) in Theorem 3, we have

$$\begin{aligned}
&\mathbb{P}\left(Q_i^* \ge \tilde{D}_i\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_i^* < \tilde{D}_i\right) \ge \mathbb{P}\left(Q_i^* \ge \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j \ge \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right)\\
&\ge \frac{\bar{P}_i}{\bar{S}_i}
\end{aligned}$$

where the first inequality follows because (33) and $\mathbb{P}\left(Q_j \ge \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \le \mathbb{P}\left(Q_i^* < \tilde{D}_i\right)$. Since $\mathbb{P}\left(Q_i^* < \tilde{D}_i\right) = 1 - \mathbb{P}\left(Q_i^* \ge \tilde{D}_i\right)$ and multiply $\bar{S}_i$ on both sides of $\mathbb{P}\left(Q_i^* \ge \tilde{D}_i\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_i^* < \tilde{D}_i\right) \ge \frac{\bar{P}_i}{\bar{S}_i}$, we obtain

$$\mathbb{P}\left(Q_i^* \ge \tilde{D}_i\right) \ge \frac{\bar{P}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}$$

given that $\bar{S}_i > \sum_{j\in[n]}\alpha_{ij}\bar{S}_j$, i.e., we arrive at (12a).

(2) According to (11b) in Theorem 3, we have

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i + \sum_{j\in[n]}\alpha_{ji}\tilde{D}_j\right) \le \mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \le \tilde{D}_i\right) \le \frac{\bar{P}_i}{\bar{S}_i}$$

where the first inequality is due to (33) and $\sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \le \tilde{D}_i\right) \ge 0$. Thus, we arrive at (12b).

□

A.7 Proof of Proposition 4

Proposition 4 *The profit function* $\Pi(\boldsymbol{Q})$ *defined in* (10) *is continuous submodular.*

Proof:

$$(10) = \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - D_i^{sk}(\boldsymbol{Q})\right)_+\right] \tag{34a}$$

$$= \sum_{i\in[n]} \bar{P}_i Q_i + \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \min\left(D_i^{sk}(\boldsymbol{Q}) - Q_i, 0\right) \right] \tag{34b}$$

In (34b), $D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{i\neq j} \alpha_{ij}\left(D_j^k - Q_j\right)_+ = D_i^k + \sum_{i\neq j}\alpha_{ij}\min\left(D_j^k - Q_j, 0\right) = D_i^k - \sum_{i\neq j}\alpha_{ij}Q_j + \sum_{i\neq j}\alpha_{ij}\min\left(D_i^k, Q_j\right)$. As proved in **?**, $D_i^{sk}(\boldsymbol{Q})$ is submodular and supermodular on $\boldsymbol{D}$. Thus, $D_i^s(\boldsymbol{Q})$ is submodular and supermodular on $\boldsymbol{Q}$, since $\boldsymbol{Q}$ and $\boldsymbol{D}$ are symmetric in the function $\min(D_i, Q_i)$, and also the summation of linear funciton are still submodular or supermodular. So $Q_i - D_i^{sk}(\boldsymbol{Q})$ is submodular on $\boldsymbol{Q}$. Since $\min\{t, 0\}$ is non-decreasing and concave on t, according to **?**, $\min\{f(\boldsymbol{Q}), 0\}$ is submodular if $f(\boldsymbol{Q})$ is submodular. Therefore, $\min\left(D_i^{sk}(\boldsymbol{Q}) - Q_i, 0\right)$ is submodualr on $\boldsymbol{Q}$. The first term $\sum_{i\in[n]}\bar{P}_iQ_i$ in (34b) is linear function and $m_k, \bar{P}_i \geq 0,$ for all $k\in[N]$ and $i\in[n]$. Thus, (10) is submodular on $\boldsymbol{Q}$. □

A.8 Proof of Theorem 4

Theorem 4 *The MILP Model 2 is stronger than MILP Model 1, i.e., their continuous relaxation values satisfy* $\bar{v}_M^1 \leq \bar{v}_M^2$*, where* $\bar{v}_M^1, \bar{v}_M^2$ *are defined in* (16a), (16b), *respectively.*

Proof: Let $(\boldsymbol{Q}^*, \boldsymbol{\chi}^*, \boldsymbol{u}^*, \boldsymbol{w}^*, \boldsymbol{y}^*)$ be an optimal solution to relaxed Model (16b). For each $i\in[n], k\in[N]$, define

$$z_i^{(k)*} = 1 - \sum_{\tau\in[k]} \chi_i^{(\tau)*}.$$

Clearly, $\boldsymbol{z}^* \in [0,1]^{n\times N}$. We need to show that $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ is feasible to relaxed Model (16a). Note that $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ satisfies constraints (14b) and (14e).

According to (15d), for each $i\in[n]$ and $k\in[N]$, we have

$$\begin{aligned} u_i^{(k)*} + Q_i^* - D_i^{(k)} &= D_i^{(k)}\left[\sum_{\tau\in[k]} \chi_i^{(\tau)^*} - 1\right] + Q_i^* - \sum_{\tau\in[k]} w_i^{(\tau)*} \\ &\geq D_i^{(k)}\left[\sum_{\tau\in[k]} \chi_i^{(\tau)^*} - 1\right] \geq -M_i z_i^{(k)*} \end{aligned}$$

where the first inequality is due to $Q_i^* \geq \sum_{\tau\in[k]} w_i^{(\tau)*}$ and the second inequality is due to $z_i^{(k)*} = 1 - \sum_{\tau\in[k]}\chi_i^{(\tau)*} = 0$ if $D_i^{(k)} > M_i$, and $z_i^{(k)*} = 1 - \sum_{\tau\in[k]}\chi_i^{(\tau)*} \in [0,1]$, otherwise. On the other hand,

$$u_i^{(k)*} + Q_i^* - D_i^{(k)} = D_i^{(k)}\left[\sum_{\tau\in[k]} \chi_i^{(\tau)^*} - 1\right] + Q_i^* - \sum_{\tau\in[k]} w_i^{(\tau)*} = D_i^{(k)}\left[\sum_{\tau\in[k]} \chi_i^{(\tau)^*} - 1\right] + \sum_{\tau\in[N+1]\setminus[k]} w_i^{(\tau)*}$$

$$\leq M_i \sum_{\tau\in[N+1]\setminus[k]} \chi_i^{(\tau)*} := M_i z_i^{(k)*}$$

where the second equality follows because $Q_i^* = \sum_{\tau\in[N+1]} w_i^{(\tau)*}$, the first inequality is due to $D_i^{(k)}\left[\sum_{\tau\in[k]}\chi_i^{(\tau)^*}-1\right]\leq 0$ and $w_i^{(\tau)*}\leq \widehat{D}_i^{(\tau)}\chi_i^{(\tau)*}\leq M_i\chi_i^{(\tau)*}$ for each $\tau\in[N+1]\setminus[k]$. Therefore, $(\boldsymbol{Q}^*,\boldsymbol{z}^*,\boldsymbol{u}^*,\boldsymbol{y}^*)$ satisfies constraints (14c).

Finally,we note that $u_i^{(k)*}\geq 0$ for each $i\in[n]$ and $k\in[N]$. In addition, by (15d), we have

$$u_i^{(k)*} = D_i^{(k)}\sum_{\tau\in[k]}\chi_i^{(\tau)^*} - \sum_{\tau\in[k]} w_i^{(\tau)*} \leq D_i^k \sum_{\tau\in[k]}\chi_i^{(\tau)*} := D_i^k(1-z_i^{(k)*})$$

where the inequality because $\sum_{\tau\in[k]} w_i^{(\tau)*}\geq 0$. Thus, $(\boldsymbol{Q}^*,\boldsymbol{z}^*,\boldsymbol{u}^*,\boldsymbol{y}^*)$ satisfies constraints (14d). □

A.9 Proof of Proposition 7

Proposition 7 *Suppose that $\boldsymbol{Q}\in\mathbb{R}_+^n$ is known. Then,*

(i) the following optimization model is efficiently solvable,

$$\max_{q\in\left[\underline{Q}_i,\overline{Q}_i\right]} \Pi\left(\boldsymbol{Q}|Q_i\leftarrow q\right) \tag{18}$$

for each $i\in[n]$; and

(ii) an optimal solution to Model (18) *belongs to set $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$, where*

$$\mathcal{R}_1 = \left\{D_i^k : D_i^k\in\left[\underline{Q}_i,\overline{Q}_i\right], \forall k\in[N+1]\right\}, \tag{19a}$$

$$\mathcal{R}_2 = \left\{D_i^{sk} : D_i^{sk}\in\left[\underline{Q}_i,\overline{Q}_i\right], \forall k\in[N]\right\}, \tag{19b}$$

$$\mathcal{R}_3 = \left\{D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} : D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} \in\left[\underline{Q}_i, D_i^k\right], \forall j\in[n], k\in[N]\right\}. \tag{19c}$$

Proof: First of all, we can simplify Model (18) to an equivalent form by eliminating all of the constant terms, i.e., the following optimization problem has the same optimal solutions as Model (18):

$$\max_{q\in\left[\underline{Q}_i,\overline{Q}_i\right]} \bar{P}_i q - \sum_{k\in[N]} m_k \bar{S}_i\left(q - D_i^{sk}\left(\boldsymbol{Q}|Q_i\leftarrow q\right)\right)_+ - \sum_{k\in[N]} m_k \sum_{j\in[n],j\neq i}\bar{S}_j\left(Q_j - D_j^{sk}\left(\boldsymbol{Q}|Q_i\leftarrow q\right)\right)_+,$$

which is further equivalent to

$$\max_{q\in\left[\underline{Q}_i,\overline{Q}_i\right]} \bar{P}_i q - \sum_{k\in[N]} m_k \bar{S}_i\left(q - D_i^{sk}\left(\boldsymbol{Q}\right)\right)_+ - \sum_{k\in[N]} m_k \sum_{j\in[n],j\neq i}\bar{S}_j\left(Q_j - D_j^{sk}\left(\boldsymbol{Q}|Q_i\leftarrow q\right)\right)_+, \tag{35}$$

since $\alpha_{ii}=0$ and $D_i^{sk}(\boldsymbol{Q}|Q_i \leftarrow q) = D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{j\in[n]} \alpha_{ji}(D_j^k - Q_j)_+$ is a constant.

Notice that

$$D_j^{sk}(\boldsymbol{Q}|Q_i \leftarrow q) = D_j^k + \sum_{\tau\in[n],\tau\neq i} \alpha_{\tau j}\left(D_\tau^k - Q_\tau\right)_+ + \alpha_{ij}\left(D_i^k - q\right)_+ := D_{j,-i}^{sk}(\boldsymbol{Q}) + \alpha_{ij}\left(D_i^k - q\right)_+$$

where $D_{j,-i}^{sk}(\boldsymbol{Q}) = D_j^k + \sum_{\tau\in[n],\tau\neq i} \alpha_{\tau j}\left(D_\tau^k - Q_\tau\right)_+$.

From Property 2, we know that the demand of product i is sorted as

$$D_i^{(1)} \leq \ldots \leq D_i^{(N)}.$$

Now let $\widehat{D}_i^{(k)} = \max\left\{\min\left\{D_i^{(k)}, \overline{Q}_i\right\}, \underline{Q}_i\right\}$. Hence, the optimal order quantity q^* of Model (35) must belong to one of the following $N+1$ intervals:

$$\left[\widehat{D}_i^{(0)}, \widehat{D}_i^{(1)}\right], \left[\widehat{D}_i^{(1)}, \widehat{D}_i^{(2)}\right], \ldots, \left[\widehat{D}_i^{(N)}, \widehat{D}_i^{(N+1)}\right].$$

where $\widehat{D}_i^{(0)} = \underline{Q}_i, \widehat{D}_i^{(N+1)} = \overline{Q}_i$. Let us set $I_r = \left\{\tau : D_i^\tau \geq \widehat{D}_i^{(r)}\right\}$ for each $r \in [N]$. By removing constant terms, Model (35) further reduces to

$$\max_{r\in[N+1]} \max_{q\in[\widehat{D}_i^{(r-1)}, \widehat{D}_i^{(r)}]} \bar{P}_i q - \sum_{k\in[N]} m_k \bar{S}_i\left(q - D_i^{sk}(\boldsymbol{Q})\right)_+ - \sum_{k\in I_r} m_k \sum_{\substack{j\in[n]\\ j\neq i}} \bar{S}_j\left(\alpha_{ij} q - D_{j,-i}^{sk}(\boldsymbol{Q}) - \alpha_{ij} D_i^k + Q_j\right)_+.$$

Note that in the above optimization model, the inner optimization is to maximize a piecewise linear concave function with optimal value achieved by one of its extreme points, which are included in the set of all the breaking points of the piecewise linear concave function. Therefore, one of the optimal solution to the above maximization model is contained in a set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are defined in (19). There are at most $2N + nN$ points in set $\mathcal{R}$, thus, Model (18) is efficiently solvable. $\square$

A.10 Proof of Theorem 5

Theorem 5 *Let v^*, v^{LD}, v_R^{LD} denote the optimal values obtained for Models* (10), (22b), *and* (25), *respectively. Then,*

(i) $v^* \leq v^{LD} \leq v_R^{LD}$; *and*

(ii)

$$v_R^{LD} = \max_{\boldsymbol{\pi},\boldsymbol{\beta}} \quad \sum_{i\in[n]} \bar{P}_i \beta_i - \sum_{k\in[N]} m_k \sum_{i\in[n]} \bar{S}_i \pi_i^k, \tag{26a}$$

$$s.t.\quad \beta_i - \pi_i^k = \sum_{j=1}^{n+1} \frac{1}{4} w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right), \forall i \in [n], \forall k \in [N], \tag{26b}$$

$$\pi_i^k \geq 0, \forall i \in [n], \forall k \in [N], \tag{26c}$$

$$\beta_i \geq 0, \forall i \in [n] \tag{26d}$$

$$Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N] \tag{26e}$$

$$\boldsymbol{Y}^k \succeq 0, \forall k \in [N] \tag{26f}$$

(i) Clearly, by the discussion above, we have $v^* \leq v^{LD} \leq v_R^{LD}$.

(ii) Notice that in Model (25), the inner maximization problem are seperable, thus, we can swap summation and max operators as below,

$$v_R^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \max_{\boldsymbol{Y} \in C_R^N} \sum_{k \in [N]} m_k \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\},$$

where C_R^N denotes n-fold Cartesian product of set C_R and $\boldsymbol{Y} = \{\boldsymbol{Y}^k\}_{k \in [N]}$. Note that C_R is a bounded convex set and Ω is a nonempty polyhedral set, The above function is bilinear in $\boldsymbol{\lambda}$ and $\boldsymbol{Y}$. According to the well-known Sion's minimax theorem (cf., **?**), we can switch the inf and max operators, i.e., Model (25) is equivalent to

$$v_R^{LD} = \max_{\boldsymbol{Y} \in C_R^N} \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\}.$$

By the strong duality of linear program, we can reformulate the inner infimum in the above formulation as an equivalent maximization problem with dual variables $\boldsymbol{\beta} = \{\beta_i\}_{i \in [n]}, \boldsymbol{\pi} = \{\pi_i^k\}_{i \in [n], k \in [N]}$ corresponding to the constraints in Ω, i.e., Model (25) is equivalent to

$$\begin{aligned}
v_R^{LD} = &\max_{\boldsymbol{\pi}, \boldsymbol{\beta}, \boldsymbol{Y}} \quad \sum_{i \in [n]} \bar{P}_i \beta_i - \sum_{k \in [N]} \sum_{i \in [n]} \bar{S}_i \pi_i^k, \\
\text{s.t.} \quad & \beta_i m_k - \pi_i^k = \sum_{j \in [n+1]} \frac{1}{4} m_k w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right), \forall i \in [n], \forall k \in [N], \\
& \pi_i^k \geq 0, \forall i \in [n], \forall k \in [N], \\
& \beta_i \geq 0, \forall i \in [n] \\
& Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N] \\
& Y^k \succeq 0, \forall k \in [N]
\end{aligned}$$

Redefining $\pi_i^k := \frac{\pi_i^k}{m_k}$ for each $i \in [n], k \in [N]$, we arrive at Model (26). $\square$

A.11 Proof of Theorem 6

Theorem 6 *Let v^*, v^{LD}, v_R^{LD} denote the optimal value of Models* (10), (22b), *and* (25), *respectively. Then,*

(i) $v_R^{LD} \le \frac{v^{LD}}{0.79607}$*; and*

(ii) if Assumption 3 holds, then

$$v_R^{LD} \le \frac{v^{LD}}{0.79607} \le \frac{(1+\overline{\delta})}{0.79607(1-\underline{\delta})} v^*$$

Proof:

(i) We first prove $v_R^{LD} \le \frac{v^{LD}}{0.79607}$. From (25), we have

$$\begin{aligned}
v_R^{LD} &= \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C_R} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\} \\
&\le \frac{1}{0.79607} \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\} \\
&= \frac{1}{0.79607} v^{LD}
\end{aligned}$$

where the inequality follows by the result of Corollary 2.

(ii) It remains to show that $v^{LD} \le \frac{1+\overline{\delta}}{1-\underline{\delta}} v^*$ under Assumption 3. By (24), we have

$$\begin{aligned}
v^{LD} &= \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\} \\
&\le \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\} \\
&\le (1+\overline{\delta}) \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij} \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\} \\
&= \left(1+\overline{\delta}\right) \sum_{k \in [N]} m_k v_D^* && (37) \\
&= \left(1+\overline{\delta}\right) v_D^*, && (38)
\end{aligned}$$

where the first inequality follows because we let $\boldsymbol{\lambda} = \mathbf{0}$, the second inequality holds because $D_i^k \le (1+\overline{\delta}) D_i$ for all $k \in [N]$, the second equality follows by the definition of v_D^* in (8), and the third equality is due to $\sum_{k \in [N]} m_k = 1$.

On the other hand, note that for any fixed $\boldsymbol{Q}\in\mathbb{R}^n_+$, $D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{j\in[n]}\alpha_{ji}(D_j^k - Q_j)_+$ is nondecreasing in $\boldsymbol{D}^k$. Since $(1-\underline{\delta})D_i \le D_i^k$ for all $k\in[N]$, we have

$$D_i^{sk}(\boldsymbol{Q}) \ge (1-\underline{\delta})D_i^s\left(\frac{\boldsymbol{Q}}{1-\underline{\delta}}\right) := (1-\underline{\delta})\left[D_i + \sum_{j\in[n]}\alpha_{ji}\left(D_j - \frac{Q_j}{1-\underline{\delta}}\right)_+\right].$$

Thus, by (10), we have

$$\begin{aligned}
v^* &= \max_{\boldsymbol{Q}\in\mathbb{R}^n_+}\left\{\sum_{i\in[n]}\bar{P}_iQ_i - \sum_{k\in[N]}m_k\left[\sum_{i\in[n]}\bar{S}_i\left(Q_i - D_i^{sk}(\boldsymbol{Q})\right)_+\right]\right\}\\
&\ge \max_{\boldsymbol{Q}\in\mathbb{R}^n_+}\left\{\sum_{i\in[n]}\bar{P}_iQ_i - \sum_{k\in[N]}m_k\left[\sum_{i\in[n]}\bar{S}_i\left(Q_i - (1-\underline{\delta})D_i^s\left(\frac{\boldsymbol{Q}}{1-\underline{\delta}}\right)\right)_+\right]\right\}\\
&= (1-\underline{\delta})\max_{\boldsymbol{Q}\in\mathbb{R}^n_+}\left\{\sum_{i\in[n]}\bar{P}_i\frac{Q_i}{1-\underline{\delta}} - \sum_{k\in[N]}m_k\left[\sum_{i\in[n]}\bar{S}_i\left(\frac{Q_i}{1-\underline{\delta}} - D_i^s\left(\frac{\boldsymbol{Q}}{1-\underline{\delta}}\right)\right)_+\right]\right\}\\
&= (1-\underline{\delta})\,v^*(\boldsymbol{D}) \qquad (39)
\end{aligned}$$

where the first inequality follows because $D_i^{sk}(\boldsymbol{Q}) \ge (1-\underline{\delta})D_i^s\left(\frac{\boldsymbol{Q}}{1-\underline{\delta}}\right)$ for each $k\in[N]$, and the third equality obtained by letting $Q_i := \frac{Q_i}{1-\underline{\delta}}$ for each $i\in[n]$.

Combining (38) and (39), we have

$$v^* \ge \frac{1-\underline{\delta}}{1+\overline{\delta}}v^{LD}.$$

□