

Online Supplement

Appendix A: Linearization

The standard linearization technique (SLT) is a known technique in the literature to transform the BQP model into its equivalent mixed integer program (MIP) model by substituting the quadratic terms with extra binary variables y_{ef} . For instance, linearized constraints for the MOT problem are as follows:

$$y_{ef} \geq x_e + x_f - 1 \quad \forall (e, f) \in \mathcal{A} \quad (46)$$

$$y_{ef} \leq x_e \quad \forall (e, f) \in \mathcal{A} \quad (47)$$

$$y_{ef} \leq x_f \quad \forall (e, f) \in \mathcal{A} \quad (48)$$

$$y_{ef} \geq 0 \quad \forall (e, f) \in \mathcal{A} \quad (49)$$

where it suggests $O(n^3)$ decision variables and constraints be added to the BQP model.

The reformulation linearization technique (RLT) is another mathematical programming technique used for linearization. In this study, we exploit a tighter linearization which is similar to the RLT to linearize the BQP formulation of MOT as well as to linearize the star-based reformulation of MOT. Considering (37) and multiplying it once by $x_f, \forall f \in \delta^{t'}(j)$, where t' represents any frame except frame t and then generating the same constraint for x_e , we obtain the following strong valid inequalities instead of the SLT constraints (47) and (48):

$$\sum_{e \in \delta^{t'}(j)} y_{ef} \leq x_f \quad \forall t \in T, \forall j \in H, \forall f \in \delta^{t'}(j) : t' \neq t \quad (50)$$

$$\sum_{f \in \delta^{t'}(j)} y_{ef} \leq x_e \quad \forall t \in T, \forall j \in H, \forall e \in \delta^{t'}(j) : t' \neq t \quad (51)$$

The RLT model applied to the quadratic semi-assignment problem of (14)-(16) is formulated as follows:

$$\min \sum_{e \in A} c_e x_e + \sum_{\substack{i, k \in N \\ i \neq k}} \sum_{\substack{e \in \delta(i) \\ f \in \delta(k)}} q_{ef} y_{ef} \quad (52)$$

$$\text{s.t.} \quad \sum_{e \in \delta(i)} x_e = 1 \quad \forall i \in N \quad (53)$$

$$\sum_{e \in \delta(i)} y_{ef} = x_f \quad \forall i, k \in N, i \neq k, \forall f \in \delta(k) \quad (54)$$

$$\sum_{f \in \delta(k)} y_{ef} = x_e \quad \forall i, k \in N, i \neq k, \forall e \in \delta(i) \quad (55)$$

$$y_{ef} \geq 0 \quad \forall i, k \in N, i \neq k, \forall e \in \delta(i), \forall f \in \delta(k) \quad (56)$$

$$x_e \in \{0, 1\} \quad \forall e \in A. \quad (57)$$

Appendix B: Instance-by-instance tables

In the following, we detail the computational results for each problem in two tables consisting of dual bounds and computational time. For each problem, one table demonstrates the experiments for the instances in which at least one of the compared methods stops within the time limit. The other table contains instances in which all the methods reach the time limit; thus, the three hours of fixed computation time is not reported in this table to avoid repetition. As an index of efficiency for each method, we report the ratio of the obtained lower bound to the best known feasible solution (BFS). The BFS is computed by comparing the obtained upper bounds of all methods in the three hour timeframe.

Table 1 AGSAP – Comparing different methods – At least one of the methods stops within the time limit

Instance (n-h)	BFS		BQP		CG+BQP		CG+HeuristicBQP		CG+HeuristicSLTPricing		SLT		CG+SLTPricing		RLT		Outer Approximation	
	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time
10-3	1	0	1	0	1	2	1	8	1	6	1	1	1	1	1	1	1	0.1
10-4	1	0	1	4	1	4	1	11	1	8	1	2	1	1	1	2	1	0.1
15-3	1	52	1	56	1	56	1	18	1	14	1	40	1	1	1	30	1	6
15-4	1	524	1	57	1	57	1	28	1	28	1	314	1	1	1	593	1	120
15-6	1	327	1	41	1	41	1	25	1	16	1	233	1	1	1	1395	1	95
18-3	1	43	1	298	1	298	1	35	1	24	1	25	1	1	1	61	1	13
18-4	1	1852	1	142	1	142	1	5	1	8	1	909	1	1	1	1910	1	535
18-6	0.82	10800	1	96	1	96	1	53	1	43	0.81	10800	1	147	0.73	10800	0.82	10800
18-8	0.85	10800	1	96	1	96	1	83	1	82	0.8	10800	1	160	0.74	10800	0.79	10800
20-3	1	1099	1	430	1	430	1	10	1	25	1	345	1	1	1	422	1	68
20-4	1	13657.3	1	333	1	333	1	58	1	25	0.94	10800	1	331	0.86	10800	1	7406
20-6	0.92	10800	1	227	1	227	1	101	1	52	0.68	10800	1	293	0.55	10800	0.66	10800
20-8	0.62	10800	1	105	1	105	1	10	1	31	0.62	10800	1	176	0.55	10800	0.62	10800
20-10	0.83	10800	1	151	1	151	1	18	1	59	0.81	10800	1	195	0.61	10800	0.79	10800
22-3	0.97	10800	1	1744	1	1744	1	51	1	74	1	7360	1	1612	0.98	10800	1.00	4807
22-4	24085.5	0.87	10800	500	1	500	1	27	1	76	0.84	10800	1	590	0.75	10800	0.80	10800
22-6	18316.3	0.72	10800	237	1	237	1	28	1	66	0.63	10800	1	404	0.46	10800	0.52	10800
22-8	7394.2	0.74	10800	205	1	205	1	13	1	31	0.67	10800	1	293	0.61	10800	0.67	10800
22-10	6017.8	0.78	10800	188	1	188	1	95	1	49	0.73	10800	1	284	0.59	10800	0.73	10800
25-3	42794.9	0.94	10800	3127	1	3127	1	100	1	73	0.76	10800	1	197	0.56	10800	0.80	10800
25-4	14012.7	0.9	10800	598	1	598	1	51	1	72	0.98	10800	1	2417	0.89	10800	0.92	10800
25-6	9478.1	0.68	10800	1937	1	1937	1	16	1	79	0.9	10800	1	656	0.82	10800	0.88	10800
25-8	8807.1	0.68	10800	581	1	581	1	684	1	174	0.59	10800	1	1477	0.46	10800	0.54	10800
25-10	6318.8	0.7	10800	975	1	975	1	566	1	252	0.64	10800	1	614	0.47	10800	0.57	10800
25-12	4744	0.61	10800	265	1	265	1	98	1	99	0.61	10800	1	591	0.43	10800	0.56	10800
30-3	48847.1	0.89	10800	1741	1	1741	1	228	1	179	0.52	10800	1	348	0.38	10800	0.47	10800
30-4	20369.8	0.74	10800	0.99	10800	0.99	10800	70	1	98	0.94	10800	1	1863	0.90	10800	1.00	8424
30-6	14167	0.52	10800	1	10800	1	10800	3665	1	1143	0.73	10800	1	9636	0.62	10800	0.65	10800
30-8	10322.5	0.58	10800	1	6378	1	2612	2001	1	2287	0.5	10800	1	10800	0.39	10800	0.45	10800
30-14	6680.8	0.47	10800	1	1875	1	214	964	1	831	0.41	10800	1	2953	0.31	10800	0.45	10800
40-14	10365.5	0.4	10800	0.98	10800	0.98	10800	10800	1	4062	0.34	10800	1	10800	0.21	10800	0.31	10800
50-4	49468.9	0.43	10800	1	10800	1	10800	6895	1	10800	0.37	10800	0.97	10800	0.29	10800	0.35	10800
50-6	40831	0.67	10800	1	10800	1	10800	4156	1	10800	0.25	10800	0.82	10800	0.17	10800	0.22	10800
50-8	34960.5	0.5	10800	1	10800	1	10800	7146	1	10800	0.24	10800	0.78	10800	0.18	10800	0.22	10800

Table 2 AQSAP – Comparing different methods – None of the methods stop within the time limit

Instance (n-h)	BFS	BQP	CG+BQPPricing	CG+Heuristic BQPPricing	CG+Heuristic SLTPricing	SLT	CG+SLTPricing	RLT	Outer Approximation
	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS
40-3	46075.8	0.66	0.94	1	1	0.58	0.93	0.51	0.58
40-4	37264	0.46	0.89	0.80	0.80	0.41	0.89	0.28	0.35
40-6	17322.9	0.36	0.86	0.86	0.86	0.28	0.88	0.20	0.26
40-8	16018.9	0.34	0.94	0.84	1	0.29	0.90	0.21	0.27
50-3	52448.4	0.61	0.99	1	1	0.50	0.94	0.47	0.50
50-14	11696.4	0.29	0.99	1	0.41	0.34	0.78	0.21	0.29

A closer look at Tables 3 to 10 proves that in some cases of the QSAP with out-of-star interactions, one has to choose the solution methodology by considering the trade-off between time and the quality of LB. The reason is that CG may stop in a shorter time with a slightly worse LB, which we do not consider in our performance analysis. In addition, for larger-size instances of different data sets in QSAP, a significant pattern is observed in the results. When the number of servers (h) is relatively large compared to the number of clients (n), one of the CG methods yields the best results, while instances with a smaller number of servers are solved by GUROBI and the OuterApproximation methods with better bounds. Interestingly, in certain cases where none of the techniques stops prior to the time limit, CG exhibits a superior LB. As various techniques (such as acceleration and stabilization techniques) can be applied to further improve the performance of CG, there is room for enhancements in our methodology to make it more effective in tackling large-scale problems.

According to the tables associated with the MOT problem, increasing the parameter d for a problem with a fixed number of frames (\mathcal{T}) results in more computational time, highlighting the effects of the density of the quadratic matrix in solving the problem. Table 12 demonstrates that in many cases CG algorithms converge to an optimal solution of the problem. Specifically, when we use CG+RLTPricing and the problem can be solved within the time limit, optimality is proved for almost all the instances.

Table 3 QAP – 10% out-of-star quadratic matrix density – Comparing different methods – At least one of the methods stops within the time limit

Instance (n-h)	BFS	BQP		CG+BQPPricing		CG+HeuristicBQPPricing		CG+HeuristicSLTPricing		SLT		CG+SLTPricing		RLT		OuterApproximation	
		LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time
10-3	3061.4	1	0.1	1	6	1	6	1	3	1	2	1	2	1	0.1	1	0.1
10-4	2829.3	1	1	1	5	1	5	1	5	1	1	1	1	1	0.1	1	0.1
15-3	33075.4	1	4	0.98	77	0.98	10	0.98	6	1	11	0.98	22	1	2	1	1
15-4	21665.4	1	21	0.99	71	0.99	18	0.99	14	1	348	0.99	17	1	10	1	5
15-6	4497.2	1	7	0.99	40	0.99	15	0.99	15	1	79	0.99	64	1	5	1	6
18-3	8383	1	4	0.99	164	0.99	41	0.99	32	1	13	0.99	180	1	1	1	2
18-4	6884.3	1	23	0.98	144	0.98	29	0.98	24	1	119	0.98	169	1	12	1	25
18-6	5957.5	1	248	0.99	123	0.99	47	0.99	39	1	4021	0.99	135	1	239	1	395
18-8	4956.1	1	4762	0.99	125	0.99	43	0.99	40	0.76	10800	0.99	129	1	3748	1	6587
20-3	16245.3	1	20	0.97	533	0.97	103	0.97	55	1	75	0.97	561	1	25	1	5
20-4	13942.2	1	492	0.98	459	0.98	53	0.98	25	1	1701	0.98	502	1	170	1	37
20-6	9919.2	1	4867	0.99	224	0.99	42	0.99	23	0.51	10800	0.99	270	1	9593	1	2166
20-8	8697.8	0.75	10800	0.98	156	0.98	76	0.98	31	0.53	10800	0.98	250	0.71	10800	0.76	10800
20-10	4878.3	0.88	10800	0.99	194	0.99	87	0.99	34	0.64	10800	0.99	280	0.74	10800	0.81	10800
22-3	44517.4	1	444	0.96	1092	0.96	34	0.96	21	1	1632	0.96	1432	1	328	1	183
22-4	25071.6	1	4806	0.96	589	0.96	89	0.96	36	1	9336	0.96	768	1	3453	1	2033
22-6	18872.5	0.79	10800	0.97	312	0.97	89	0.97	31	0.55	10800	0.97	391	0.76	10800	0.70	10800
22-8	7587.9	0.82	10800	0.97	322	0.97	109	0.97	59	0.62	10800	0.97	511	0.75	10800	0.80	10800
22-10	6121.4	0.84	10800	0.99	336	0.99	84	0.99	107	0.65	10800	0.99	432	0.72	10800	0.78	10800
22-12	4566.7	0.88	10800	0.99	353	0.99	111	0.99	126	0.62	10800	0.99	478	0.71	10800	0.90	10800
25-3	44875.1	1	206	0.96	3207	0.96	516	0.96	176	1	918	0.96	4427	1	181	1	196
25-4	14811.7	1	83	0.95	1256	0.95	122	0.95	66	1	567	0.95	1298	1	184	1	153
25-6	9887.9	0.79	10800	0.96	2361	0.96	376	0.96	195	0.47	10800	0.96	2629	0.74	10800	0.79	10800
25-8	8999.1	0.73	10800	0.98	2051	0.98	673	0.98	402	0.53	10800	0.98	1492	0.64	10800	0.67	10800
25-10	6509.8	0.74	10800	0.97	2698	0.97	367	0.97	543	0.39	10800	0.97	1822	0.55	10800	0.63	10800
25-12	4921.6	0.47	10800	0.96	1130	0.96	1024	0.96	916	0.35	10800	0.96	1712	0.45	10800	0.57	10800
30-3	53502.1	1	203	0.93	8255	0.94	412	0.94	79	1	2902	0.94	1863	1	361	1	208
30-4	21896.3	1	1359	0.93	10800	0.93	4229	0.93	4350	1	8366	0.93	9636	1	1101	1	2049
30-6	14736.8	0.73	10800	0.95	10800	0.96	2324	0.96	3722	0.43	10800	0.96	10800	0.65	10800	0.64	10800
30-8	10807.5	0.40	10800	0.95	8560	0.95	4120	0.95	2720	0.24	10800	0.95	8120	0.44	10800	0.47	10800
30-14	6978.9	0.36	10800	0.96	7894	0.96	1063	0.96	10800	0.26	10800	0.96	10800	0.33	10800	0.39	10800

Table 4 QSAP – 10% out-of-star quadratic matrix density – Comparing different methods – None of the methods stops within the time limit

Instance (n-h)	BFS	BQP	CG+BQPPricing	CG+Heuristic BQPPricing	CG+Heuristic SLTPricing	SLT	CG+SLTPricing	RLT	Outer Approximation
		LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS
40-3	51693.8	0.75	0.83	0.68	0.87	0.63	0.83	0.68	0.72
40-4	41684.2	0.52	0.78	0.71	0.71	0.22	0.78	0.40	0.48
40-6	19765.3	0.31	0.76	0.18	0.18	0.17	0.68	0.24	0.27
40-8	18865.9	0.21	0.43	0.35	0.35	0.17	0.43	0.23	0.26
40-14	11586.1	0.25	0.51	0.51	0.38	0.21	0.51	0.25	0.28
50-3	60996.3	0.83	0.87	0.88	0.82	0.58	0.79	0.73	0.65
50-4	62141.8	0.46	0.79	0.81	0.76	0.26	0.73	0.44	0.53
50-6	51281.2	0.27	0.78	0.79	0.27	0.16	0.50	0.18	0.23
50-8	44208.2	0.18	0.77	0.79	0.33	0.15	0.45	0.15	0.19
50-14	14426.3	0.22	0.37	0.25	0.30	0.20	0.29	0.17	0.23

Appendix C: Primal bounding

As mentioned before, primal bounding methodologies are beyond the scope of this project. However, to show the strength of the proposed reformulation framework and CG for the adjacent-only class of BQPs, we obtain the primal bounds for all of the methods. In the next sections we first discuss the obtained UB for the adjacent-only problems and then we point out some discussions on the QSAP as an example of general BQPs.

C.1. Upper bound for adjacent-only problems

Although we use a very trivial heuristic to find the feasible solutions after the CG methods terminate, in a relatively large fraction of instances in adjacent-only problems, the BFS is obtained through the CG, meaning that the reformulation and CG outperform other formulations and methods in terms of both primal bound and dual bound for these problems.

To find the BFS for each instance, we compare the primal bounds obtained by different methods. In Figure 9, we present the fraction of instances where the BFS is found through each method. This figure proves the capabilities of the CG methods to find the best upper bounds in both the AQSAP and the MOT problem. According to Figure 9a, the CG+HeuristicBQPPricing obtains the BFS for the largest fraction of instances (93%) among all of the methods. For MOT, Figure 9b indicates that the CG+RLTPricing is the best method, finding the BFS for 70% of instances.

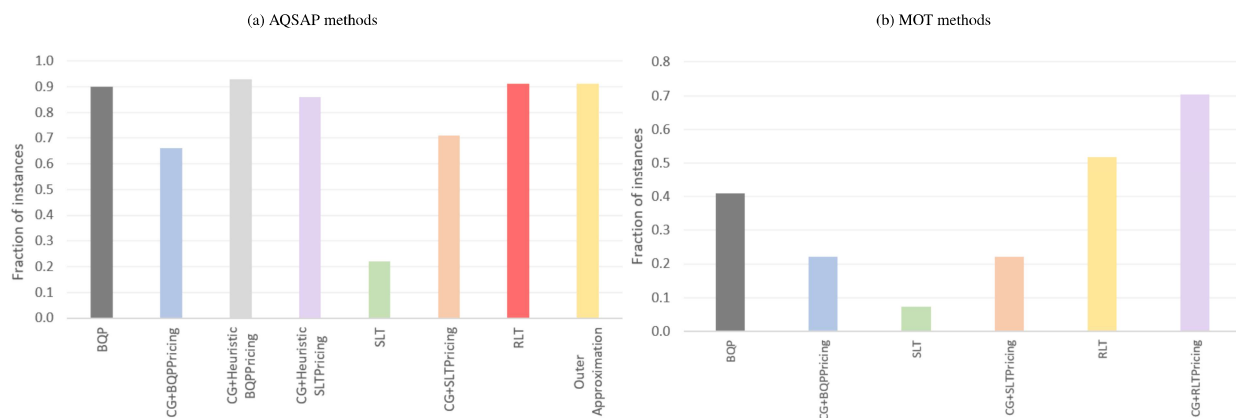
Table 5 QAP – 15% out-of-star quadratic matrix density – Comparing different methods – At least one of the methods stops within the time limit

Instance (n-h)	BFS	BQP		CG+BQP		CG+HeuristicBQP		CG+HeuristicSLTPricing		SLT		CG+SLTPricing		RLT		OuterApproximation	
		LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time
10-3	3172.5	1	0	0.97	4	0.97	4	0.97	4	1	2	0.97	1	1	0	1	0
10-4	2881.7	1	1	0.99	5	0.99	5	0.99	4	1	4	0.99	1	1	0	1	0
15-3	33324.7	1	2	0.97	54	0.97	14	0.97	10800	1	8	0.97	18	1	1	1	1
15-4	22318.5	1	6	0.96	63	0.96	26	0.96	20	1	65	0.96	65	1	5	1	4
15-6	4568.5	1	4	0.98	41	0.98	16	0.98	21	1	38	0.98	56	1	3	1	4
18-3	8669.3	1	2	0.96	182	0.96	24	0.96	22	1	6	0.96	212	1	0	1	1
18-4	6978.7	1	10	0.97	194	0.97	29	0.97	26	1	38	0.97	193	1	4	1	13
18-6	6126.4	1	130	0.96	122	0.96	62	0.96	59	1	1544	0.96	137	1	75	1	136
18-8	5034.8	1	419	0.98	97	0.98	57	0.98	52	1	7584	0.98	122	1	500	1	840
20-3	16434.4	1	9	0.96	600	0.96	71	0.96	45	1	27	0.96	550	1	6	1	2
20-4	14529.7	1	97	0.94	349	0.94	32	0.94	19	1	788	0.94	444	1	76	1	19
20-6	10257.6	1	1707	0.95	206	0.95	57	0.95	25	1	10800	0.95	283	1	1724	1	391
20-8	9062.6	1	9703	0.95	227	0.95	84	0.95	36	1	10800	0.95	289	1	10800	1	10800
20-10	4965.2	1	4169	0.97	196	0.97	110	0.97	50	1	1037	0.97	315	1	10800	0.85	6248
22-3	45955.1	1	222	0.93	897	0.93	52	0.93	37	1	4682	0.93	1525	1	151	1	88
22-4	26366.2	1	899	0.91	464	0.91	121	0.91	44	1	10800	0.91	654	1	868	1	899
22-6	19512	1	8792	0.94	256	0.94	73	0.94	29	1	10800	0.94	433	1	10800	1	7364
22-8	8214.5	1	4946	0.9	525	0.9	106	0.9	78	1	10800	0.9	492	1	6424	1	8687
22-10	6349.8	0.85	10800	0.95	259	0.95	168	0.95	237	1	10800	0.95	368	1	10800	1	9966
22-12	4821.5	0.70	10800	0.93	433	0.93	160	0.93	131	1	10800	0.93	490	1	10800	1	8706
25-3	48713.8	1	117	0.89	4418	0.89	287	0.89	202	1	377	0.89	4186	1	117	1	129
25-4	16521.4	1	46	0.86	1685	0.86	121	0.86	75	1	241	0.86	1631	1	81	1	122
25-6	10827.9	1	7729	0.88	1932	0.88	384	0.88	191	1	10800	0.88	1895	1	7087	1	8188
25-8	9745.2	0.61	10800	0.9	4584	0.9	952	0.9	511	1	10800	0.9	2642	1	10800	0.76	10800
25-10	7063.3	0.59	10800	0.89	3336	0.89	942	0.89	840	1	10800	0.89	2130	1	10800	0.68	10800
25-12	4738.4	0.44	10800	0.85	1328	0.85	10156	0.85	10283	1	10800	0.85	10800	1	10800	0.58	10800
30-3	57082.6	1	93	0.88	10800	0.88	1751	0.88	264	1	1025	0.88	10800	1	103	1	118
30-4	24213	1	109	0.84	10800	0.85	2085	0.85	1132	1	2207	0.85	10800	1	398	1	322
30-6	18614.7	0.68	10800	0.77	10800	0.77	809	0.77	2739	1	10800	0.77	10800	1	10800	0.64	10800
30-8	14294	0.38	10800	0.78	5906	0.78	5366	0.78	3479	1	10800	0.78	8978	1	10800	0.44	10800
30-14	10284	0.31	10800	0.80	9593	0.80	1371	0.80	1365	1	10800	0.80	10800	1	10800	0.34	10800
40-14	17160.5	0.17	10800	0.36	10800	0.22	1540	0.22	10800	1	10800	0.37	10800	1	10800	0.2	10800

Table 6 QSAP – 15% out-of-star quadratic matrix density – Comparing different methods – None of the methods stops within the time limit

Instance (n-h)	BFS	BQP	CG+BQPPricing	CG+Heuristic BQPPricing	CG+Heuristic SLTPricing	SLT	CG+SLTPricing	RLT	Outer Approximation
		LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS
40-3	55376	0.85	0.75	0.62	0.82	0.65	0.76	0.78	0.76
40-4	47234.8	0.68	0.70	0.64	0.64	0.32	0.69	0.41	0.50
40-6	23518.1	0.29	0.62	0.22	0.22	0.15	0.60	0.25	0.27
40-8	27886.8	0.16	0.29	0.17	0.17	0.13	0.29	0.17	0.20
50-3	65385.1	0.91	0.81	0.82	0.77	0.66	0.76	0.83	0.80
50-4	70572.9	0.57	0.69	0.70	0.66	0.29	0.63	0.54	0.58
50-6	62755.8	0.24	0.63	0.65	0.23	0.14	0.51	0.17	0.25
50-8	53024.2	0.16	0.63	0.66	0.32	0.13	0.35	0.14	0.18
50-14	26288.9	0.12	0.17	0.03	0.16	0.12	0.16	0.10	0.14

Figure 9 Comparing methods in terms of obtaining BFS



C.2. Upper bound for QSAP

Akin to the heuristic in AQSAP and MOT, we use all of the entered columns in the QSAP-RMP to build our IP model to obtain a valid UB for QSAP. However, instead of solving the IP version of the last RMP directly, we solve an RMP with a slightly different objective function. In this model, we build a new quadratic cost function to reflect the interaction between each pair of stars (columns) of the RMP as below:

$$\begin{cases} Q_{s,s'} = 0, & \text{if } s \text{ and } s' \text{ have } i \text{ or } j \text{ in common} \\ Q_{s,s'} = \sum_{e \in s, f \in s'} q_{ef}, & \text{otherwise} \end{cases}$$

Therefore, the alternative model to solve is:

$$\min \sum_{s \in S} C_s \lambda_s + \sum_{s \in S} \sum_{s' > s \in S} Q_{s,s'} \lambda_s \lambda_{s'} \quad \text{s.t. } (30 - 32) \quad (58)$$

Through the new definition of the quadratic costs, we discard many impossible interactions among the stars from the beginning. This leads to reaching the same feasible solutions as with the previously mentioned heuristic in a much shorter time. Nonetheless, even with this reformulation the obtained feasible solutions are not promising. We do not report the details on the UB of the QSAP, but we would like to provide some remarks to direct future research on this topic:

Table 7 **QAP – 20% out-of-star quadratic matrix density – Comparing different methods – At least one of the methods stops within the time limit**

Instance (n-h)	BFS		BQP		CG+BQPPricing		CG+HeuristicBQP Pricing		CG+HeuristicSLTPricing		SLT		CG+SLTPricing		RLT		OuterApproximation	
	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time
10-3	3213.4	1	0	0	0.96	5	3	3	0.96	3	1	1	0.96	1	1	0	1	0
10-4	3001.4	1	1	1	0.96	9	5	4	0.96	4	1	1	0.96	2	1	0	1	0
15-3	34726.1	1	2	2	0.93	60	11	10	0.93	10	1	6	0.93	21	1	1	1	1
15-4	23852.4	1	7	7	0.90	72	19	19	0.90	19	1	54	0.90	75	1	4	1	3
15-6	5132.6	1	5	5	0.87	52	21	21	0.87	21	1	33	0.87	66	1	2	1	4
18-3	9895.4	1	2	2	0.84	182	31	26	0.84	26	1	5	0.84	227	1	1	1	1
18-4	7357.1	1	6	6	0.92	174	44	35	0.92	35	1	19	0.92	174	1	2	1	5
18-6	6379.6	1	43	43	0.92	138	54	48	0.92	48	1	374	0.92	138	1	21	1	63
18-8	5446	1	228	228	0.90	116	55	48	0.90	48	1	3503	0.90	134	1	201	1	571
20-3	18140.5	1	8	8	0.87	533	189	13	0.87	13	1	39	0.87	627	1	4	1	1
20-4	15836	1	73	73	0.86	394	28	20	0.86	20	1	553	0.86	461	1	76	1	14
20-6	11201.2	1	1681	1681	0.87	197	51	26	0.87	26	1	6065	0.87	286	1	1108	1	315
20-8	10102.4	1	2891	2891	0.85	169	124	40	0.85	40	0.62	10800	0.85	219	1	6039	1	5047
20-10	5464.7	1	4997	4997	0.88	242	167	71	0.88	71	0.69	10800	0.88	337	1	4995	1	5199
22-3	49327.2	1	248	248	0.87	1066	64	28	0.87	28	1	692	0.87	1693	1	153	1	81
22-4	28349	1	705	705	0.85	524	113	45	0.85	45	1	4661	0.85	659	1	832	1	510
22-6	21754.1	1	7202	7202	0.84	214	115	46	0.84	46	0.61	10800	0.84	431	1	10800	1	6822
22-8	9714.9	0.71	10800	10800	0.76	342	132	87	0.76	87	0.63	10800	0.76	470	1	4536	1	4872
22-10	7733.6	0.74	10800	10800	0.78	322	243	263	0.78	263	0.61	10800	0.78	438	1	10800	1	7526
22-12	5884.6	0.66	10800	10800	0.77	685	461	687	0.77	687	0.54	10800	0.77	818	0.78	10800	1	10800
25-3	51647	1	88	88	0.84	4549	218	149	0.84	149	1	321	0.84	4396	1	135	1	110
25-4	17712.6	1	21	21	0.80	1669	112	69	0.80	69	1	74	0.80	1812	1	51	1	71
25-6	12832.8	1	2871	2871	0.74	2300	241	196	0.74	196	0.77	10800	0.74	2235	1	4243	1	6120
25-8	13192.7	0.55	10800	10800	0.67	5744	542	384	0.67	384	0.41	10800	0.67	2463	0.68	10800	0.63	10800
25-10	8560.9	0.52	10800	10800	0.74	2446	1284	957	0.74	957	0.31	10800	0.74	1967	0.57	10800	0.62	10800
25-12	7883.8	0.35	10800	10800	0.60	1181	366	295	0.60	295	0.27	10800	0.60	1173	0.36	10800	0.44	10800
30-3	61853.4	1	48	48	0.81	10800	1105	237	0.81	237	1	257	0.81	10800	1	69	1	71
30-4	31188.4	1	222	222	0.65	10800	10800	1557	0.66	1557	1	2317	0.66	10800	1	546	1	711
30-6	25956.1	0.52	10800	10800	0.55	10800	2629	1965	0.55	1965	0.34	10800	0.55	10800	0.59	10800	0.59	10800
30-8	19918.5	0.33	10800	10800	0.52	10800	4722	2499	0.52	2499	0.18	10800	0.52	7811	0.30	10800	0.38	10800
30-14	14062.6	0.19	10800	10800	0.48	3044	1185	1191	0.48	1191	0.15	10800	0.48	1461	0.17	10800	0.22	10800
50-3	71284.3	1	4427	4427	0.75	10800	10800	10800	0.69	10800	1	5015	0.37	10800	1	7429	0.86	10800

Table 8 QSAP – 20% out-of-star quadratic matrix density – Comparing different methods – None of the methods stops within the time limit

Instance (n-h)	BFS	BQP	CG+BQPPricing	CG+Heuristic BQPPricing	CG+Heuristic SLTPricing	SLT	CG+SLTPricing	RLT	Outer Approximation
		LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS
40-3	59514.1	0.85	0.71	0.65	0.76	0.74	0.71	0.82	0.80
40-4	53056.6	0.70	0.62	0.50	0.50	0.27	0.61	0.42	0.50
40-6	28760	0.27	0.53	0.29	0.29	0.14	0.50	0.24	0.26
40-8	39915	0.14	0.19	0.11	0.11	0.10	0.21	0.13	0.16
40-14	28785.3	0.10	0.18	0.17	0.10	0.10	0.19	0.10	0.12
50-4	77901.2	0.74	0.63	0.64	0.60	0.36	0.56	0.68	0.65
50-6	74931.2	0.29	0.53	0.55	0.22	0.14	0.33	0.18	0.23
50-8	70230.7	0.15	0.48	0.49	0.23	0.11	0.27	0.13	0.15
50-14	27759.7	0.12	0.17	0.01	0	0.12	0.15	0.11	0.14

REMARK 6. GUROBI methods, and in particular BQP, attain the BFS in the vast majority of the cases in the QSAP data sets. However, as the tables in Appendix B suggest, it could not find the best LB and close the optimality gap for many instances of the problems, while CG methods can prove optimality in many instances of adjacent-only problems.

REMARK 7. Interestingly, we observe that in the cases where CG methods terminate within the time limit, the obtained UB is fairly close to the BFS provided by GUROBI. In the contrast, when the CG methods reach the three-hour time limit, they suggest a weak UB, while their associated LB is still better than GUROBI in a relevant subset of instances.

Table 9 QSAF – 25% out-of-star quadratic matrix density – Comparing different methods – At least one of the methods stops within the time limit

Instance (n-h)	BFS		BQP		CG+BQPPricing		CG+HeuristicBQPPricing		CG+HeuristicSLTPricing		SLT		CG+SLTPricing		RLT		OuterApproximation		
	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	LB/BFS	Time	
10-3	1	3456.3	1	1	0.89	7	0.89	5	0.89	4	1	1	0.89	1	1	1	0	1	0
10-4	1	3161	1	1	0.91	6	0.91	8	0.91	6	1	0	0.91	2	1	1	0	1	0
15-3	1	37682.9	1	2	0.86	66	0.86	11	0.86	8	1	6	0.86	20	1	1	1	1	1
15-4	1	25935.9	1	6	0.82	77	0.82	26	0.82	22	1	53	0.82	89	1	1	3	1	3
15-6	1	6285.4	1	3	0.71	52	0.71	24	0.71	23	1	50	0.71	65	1	1	2	1	4
18-3	1	10703.9	1	2	0.77	168	0.77	37	0.77	38	1	5	0.77	221	1	1	0	1	1
18-4	1	9346.3	1	7	0.72	203	0.72	32	0.72	29	1	23	0.72	222	1	1	3	1	10
18-6	1	7765.1	1	76	0.76	124	0.76	51	0.76	41	1	722	0.76	142	1	1	27	1	100
18-8	1	6158.9	1	187	0.80	116	0.80	80	0.80	71	1	2245	0.80	131	1	1	156	1	551
20-3	1	20576.4	1	10	0.77	519	0.77	52	0.77	16	1	33	0.77	704	1	1	4	1	2
20-4	1	16772.2	1	34	0.82	462	0.82	51	0.82	22	1	154	0.82	712	1	1	21	1	10
20-6	1	12269.6	1	960	0.80	239	0.80	77	0.80	27	1	4011	0.80	285	1	1	725	1	165
20-8	1	11915.7	0.75	10800	0.72	175	0.72	140	0.72	43	0.61	10800	0.72	264	1	1	6101	1	2875
20-10	1	6446.6	1	10800	0.75	238	0.75	140	0.75	68	0.66	10800	0.75	346	1	1	3108	1	8047
22-3	1	50813.7	1	126	0.84	1037	0.84	42	0.84	42	1	307	0.84	1644	1	1	79	1	60
22-4	1	30416.8	1	621	0.79	376	0.79	95	0.79	54	1	1807	0.79	729	1	1	545	1	542
22-6	1	24468.6	1	9098	0.75	249	0.75	98	0.75	59	0.70	10800	0.75	360	1	1	6559	1	10800
22-8	1	11778.1	0.73	10800	0.63	336	0.63	125	0.63	90	0.66	10800	0.63	443	1	1	4523	1	5447
22-10	1	10706.9	0.54	10800	0.56	334	0.56	257	0.56	164	0.50	10800	0.56	491	0.67	10800	0.77	10800	
22-12	1	7346.2	0.59	10800	0.61	590	0.61	812	0.61	723	0.46	10800	0.61	973	0.76	10800	1	10800	
25-3	1	54414.9	1	59	0.80	3991	0.80	368	0.80	283	1	150	0.80	4858	1	102	1	74	
25-4	1	19097.4	1	22	0.75	2513	0.75	175	0.75	75	1	45	0.75	2209	1	39	1	59	
25-6	1	15827.4	0.79	10800	0.60	3070	0.60	378	0.60	131	0.78	10800	0.60	1609	1	5183	1	3281	
25-8	1	16652.1	0.47	10800	0.53	3526	0.53	969	0.53	475	0.37	10800	0.53	1659	0.60	10800	0.58	10800	
25-10	1	13344.2	0.40	10800	0.47	1363	0.47	389	0.47	271	0.22	10800	0.47	1273	0.42	10800	0.46	10800	
25-12	1	12815	0.21	10800	0.37	8340	0.37	251	0.37	195	0.19	10800	0.37	1505	0.24	10800	0.31	10800	
30-3	1	65491.3	1	28	0.77	10800	0.77	904	0.77	272	1	190	0.77	10800	1	18	1	27	
30-4	1	35820.9	1	95	0.57	10800	0.57	6451	0.57	6451	1	1277	0.57	10800	1	212	1	349	
30-6	1	33203.3	0.51	10800	0.43	10800	0.43	7491	0.43	3041	0.29	10800	0.43	10800	0.60	10800	0.54	10800	
30-8	1	27684.8	0.27	10800	0.37	8469	0.37	1600	0.37	2229	0.15	10800	0.37	6692	0.29	10800	0.31	10800	
30-14	1	18797.5	0.15	10800	0.36	9217	0.36	1482	0.36	1422	0.12	10800	0.36	3186	0.14	10800	0.19	10800	
40-14	1	43918.3	0.07	10800	0.11	10800	0.11	1540	0.11	10800	0.06	10800	0.11	10800	0.07	10800	0.09	10800	
50-3	1	74787.6	1	236	0.71	10800	0.71	10800	0.71	10800	1	2195	0.66	10800	1	2001	0.94	10800	
50-4	1	78766.4	1	6021	0.62	10800	0.62	10800	0.62	10800	0.44	10800	0.55	10800	0.79	10800	0.83	10800	

Table 10 QSAP – 25% out-of-star quadratic matrix density – Comparing different methods – None of the methods stops within the time limit

Instance (n-h)	BFS	BQP	CG+BQPPricing	CG+Heuristic BQPPricing	CG+Heuristic SLTPricing	SLT	CG+SLTPricing	RLT	Outer Approximation
		LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS	LB/BFS
40-3	65784.3	0.96	0.66	0.58	0.69	0.86	0.64	0.85	0.82
40-4	59743.9	0.72	0.56	0.42	0.42	0.34	0.55	0.45	0.50
40-6	32330.7	0.3	0.47	0.26	0.26	0.15	0.44	0.25	0.31
40-8	47935.6	0.13	0.19	0.15	0.33	0.09	0.17	0.11	0.16
50-6	77643.1	0.3	0.51	0.52	0.23	0.17	0.39	0.22	0.26
50-8	74816.2	0.14	0.46	0.44	0.24	0.11	0.25	0.11	0.16
50-14	27759.7	0.13	0.16	0.01	0	0.13	0.13	0.12	0.16

Table 11 Data association on the MOT16-09 data set – Comparing GUROBI and CG – At least one of the methods stops within the time limit

Instance (J-h-d)	BFS		BQP		CG+BQPPricing		SLT		CG+SLTPricing		RLT		CG+RLTPricing					
	LB	LB/BFS	LB	LB/BFS	LB	LB/BFS	Time	LB	LB/BFS	LB	LB/BFS	Time	LB	LB/BFS	Time			
3-20-2	-266.4	1.51	-401.7	1.51	-266.8	1	75	-266.4	1	-266.8	1	64	-266.4	1	509	-266.8	1	12
4-25-2	-486.2	2.19	-1064.7	2.19	-486.2	1	72	-486.2	1	-486.2	1	6969	-486.2	1	367	-486.2	1	70
4-25-3	-516.7	2.66	-1372.8	2.66	-516.7	1	507	-551.9	1.07	-538.2	1.04	10800	-516.7	1	5498	-516.7	1	78
5-25-2	-636.1	2.66	-1693.5	2.66	-636.1	1	200	-665.3	1.05	-636.1	1	10800	-636.1	1	5939	-636.1	1	466
5-25-3	-740.1	2.4	-1774.6	2.4	-740.1	1	6522	-1477.4	2	-740.1	1	6347	-754.8	1.02	10800	-740.1	1	555
5-25-4	-783.2	2.58	-2024.3	2.58	-783.2	1	7291	-2177.8	2.78	-783.2	1	5676	-806.7	1.03	10800	-783.2	1	400
6-25-3	-910.1	2.59	-2359	2.59	-1365.8	1.5	10800	-2754.3	3.03	-945.1	1.04	10800	-937.6	1.03	10800	-913.1	1	733
6-25-4	-1014.5	2.78	-2817	2.78	-1898.7	1.87	10800	-3883	3.83	-1070.2	1.05	10800	-1064.4	1.05	10800	-1014.5	1	822
6-25-5	-1110.1	2.78	-3087.7	2.78	-2260.7	2.04	10800	-3493	3.15	-2288.8	2.06	10800	-1137.2	1.02	10800	-1110.6	1	821
7-30-3	-1040.5	2.82	-2932.4	2.82	-2509.7	2.41	10800	-3915.9	3.76	-1135.1	1.09	10800	-1077.3	1.04	10800	-1040.8	1	1304
7-30-4	-1172.7	3.02	-3538.3	3.02	-3038	2.59	10800	-4499	3.84	-1267	1.08	10800	-1236.6	1.05	10800	-1172.8	1	1374
7-30-5	-1336.9	2.81	-3753	2.81	-3496.2	2.62	10800	-5062.4	3.79	-1456.3	1.09	10800	-1417.3	1.06	10800	-1336.9	1	1718
8-30-3	-1229.1	2.75	-3379.1	2.75	-2261.9	1.84	10800	-4567.4	3.72	-1741.7	1.42	10800	-1303.5	1.06	10800	-1229.7	1	4293
8-30-4	-1497.8	3.19	-4778.2	3.19	-3806.2	2.54	10800	-5646.2	3.77	-2002	1.34	10800	-1638.2	1.09	10800	-1497.8	1	4252
8-30-5	-1619.4	3.22	-5207.5	3.22	-4373.9	2.7	10800	-6429.1	3.97	-2902	1.79	10800	-1892.7	1.17	10800	-1632.4	1.01	5088

Table 12 Data association on the MOT16-09 data set – Comparing GUROBI and CG – None of the methods stops within the time limit

Instance (T-h-d)	BFS	BQP		CG+BQPPricing		SLT		CG+SLTPricing		RLT		CG+RLTPricing	
		LB	LB/BFS	LB	LB/BFS	LB	LB/BFS	LB	LB/BFS	LB	LB/BFS	LB	LB/BFS
9-35-4	-1790.8	-5601.5	3.13	-5377.9	3	-6569.6	3.67	-4230.2	2.36	-2026.3	1.13	-1792.1	1.00
9-35-5	-1974.7	-6624.1	3.35	NA	NA	-7740.1	3.92	-5085.4	2.58	-2358.4	1.19	-1988.9	1.01
9-35-6	-2065.4	-7647.6	3.7	NA	NA	-8519.1	4.12	-5134.7	2.49	-2535.4	1.23	-2150.4	1.04
10-35-4	-1971.7	-6472	3.28	NA	NA	-7357.6	3.73	-5469.1	2.77	-2288.2	1.16	-2073.7	1.05
10-35-5	-2150.6	-8116.3	3.77	NA	NA	-8663.3	4.03	-5700.6	2.65	-2665.1	1.24	-2346.1	1.09
10-35-6	-2405.1	-9273.5	3.86	NA	NA	-9792.8	4.07	-6824.3	2.84	-2988.5	1.24	-2621.2	1.09
10-35-7	-2569.7	-10134.5	3.94	NA	NA	-10681.3	4.16	-7544.9	2.94	-3252.3	1.27	-2833.3	1.10
11-35-4	-2167.5	-7494.8	3.46	-6969.1	3.22	-8306.2	3.83	-5996.4	2.77	-2650.4	1.22	-2412.8	1.11
11-35-6	-2687	-10773.8	4.01	-8849.4	3.29	-11098.4	4.13	-8436	3.14	-3440.9	1.28	-3074.5	1.14
11-35-9	-3141.3	-12752.7	4.06	-9512.7	3.03	-13306.3	4.24	-9496.8	3.02	-4107.9	1.31	-3540.1	1.13
12-40-4	-2467.1	-8257	3.35	-8606.6	3.49	-9107.1	3.69	-8168.3	3.31	-2915.7	1.18	-2862.4	1.16
12-40-6	-3044.7	-11709.1	3.85	-10676.2	3.51	-12016.3	3.95	-10281.8	3.38	-3762.6	1.24	-3520.2	1.16