Online Supplement
Optimal Power Flow in Distribution Networks under $N-1$ Disruptions: A Multi-stage Stochastic Programming Approach

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Online Supplement A. Test Case Construction

Due to the limited space of this paper, we present all data of Case 13 and the component subject to disruptions in Case 123. Case 33 and Case 123 have the same data structure and all detailed data files can be found in the GitHub repository (https://github.com/haoxiangyang89/Distribution_Disruption_Supplement). We include network topology, generator locations and costs, battery locations and their features, distribution line specifications, components subject to disruptions, and a load profile. Each case has one generator with large capacity and small costs, which serves as the approximation of the feeder between the transmission network and the distribution network.

Case 13:

- Network topology:

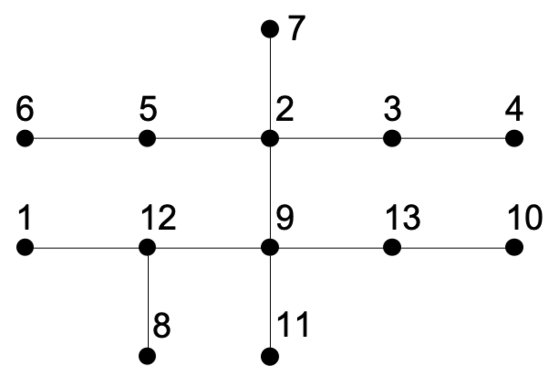

Figure 6 Distribution network of Case 13.

- The base voltage for all buses is 4.16KV. The squared voltage bounds are $\bar{V}_i = 1.05^2, \underline{V}_i = 0.95^2, \ \forall i \in \mathcal{N}$, in per unit (p.u.).
- Generator data: $c^l = 10000$
- Battery data: for all batteries $b \in \mathcal{B}$,

  — $|\mathcal{L}_b| = 4$

Table 8

Location	$\bar{s}_g^p$	$\bar{s}_g^q$	$\underline{s}_g^p$	$\underline{s}_g^p$	$c_{g,1}$	$c_{g,2}$	R_g^U	R_g^D
3	2.5	2.5	0	-2.5	5	10	6	-6
5	2.75	2.5	0	-2.5	10	20	6	-6
7	10	8	0	-8	2	1	20	-20
12	2.25	2.5	0	-2.5	20	40	6	-6
13	2.25	2.5	0	-2.5	20	40	6	-6

— $\alpha_b^1 = 2,\ \alpha_b^2 = 1.067,\ \alpha_b^3 = 0.938,\ \alpha_b^4 = 0.5$

— $\beta_b^1 = 0.7467,\ \beta_b^2 = 0,\ \beta_b^3 = 0,\ \beta_b^4 = 0.35$

— $\bar{I}_b = 1,\ \bar{u}_b = 1,\ c_b^B = 130$

- Line specifications:

Table 9

Line (i,j)	W_{ij} (MVA)	R_{ij} (p.u.)	X_{ij} (p.u.)
(1,12)	0.9568	0.00653	0.00662
(2,3)	4.2432	0.01226	0.01962
(2,5)	1.9136	0.02174	0.02216
(2,7)	9.1104	0.02241	0.06775
(2,9)	9.1104	0.02241	0.06775
(3,4)	9.1104	0.00011	0.00034
(5,6)	1.9136	0.01304	0.01329
(8,12)	1.2896	0.03520	0.01344
(9,11)	9.1104	0.01120	0.03387
(9,12)	1.9136	0.01304	0.01329
(9,13)	9.1104	0.00011	0.00034
(10,13)	4.1059	0.01303	0.00708

- Components subject to disruptions: all generators and lines
- Demand profile: the demand profiles for active and reactive power are generated from a normal distribution where the mean equals to the following base demand, $d_i^{p,0}$ and $d_i^{q,0}$, and the standard deviation is half of the base demand. If the randomly generated active demand is smaller than 0, we force it to take value 0. The unit is MW for the active demand and MVAr for the reactive demand.

— $d_1^{p,0} = 0.021,\ d_2^{p,0} = 0.12,\ d_4^{p,0} = 0.438,\ d_5^{p,0} = 0.631,\ d_6^{p,0} = 0.039,\ d_8^{p,0} = 0.12,$
$d_9^{p,0} = 3.002,\ d_{10}^{p,0} = 0.586,\ d_3^{p,0} = d_7^{p,0} = d_{11}^{p,0} = d_{12}^{p,0} = d_{13}^{p,0} = 0$

— $d_1^{q,0} = 0.013,\ d_2^{q,0} = 0.0744,\ d_4^{q,0} = 0.271,\ d_5^{q,0} = 0.391,\ d_6^{q,0} = 0.0242,\ d_8^{q,0} = 0.0892,$
$d_9^{q,0} = 1.861,\ d_{10}^{q,0} = 0.363,\ d_3^{q,0} = d_7^{q,0} = d_{11}^{q,0} = d_{12}^{q,0} = d_{13}^{q,0} = 0$

Case 123: Disruption scenarios

- $N-1$ setting:

— Generators at bus: 1, 25, 35, 60, 76, 97, 108, 116

— Lines: (7,8), (13,18), (25,26), (35,40), (67,97), (57,60), (86,87), (77,78), (105,108)

- Multiple component setting scenarios:
  1. Line (7,8), (13,18), Generator at bus 1
  2. Line (35,40), Generator at bus 35
  3. Line (57,60), Generator at bus 60
  4. Line (25,26), Generator at bus 25
  5. Line (77,78), Generator at bus 76
  6. Line (105,108), Generators at bus 97 and 108
  7. Line (67,97), Generators at bus 1 and 116

Online Supplement B. Dual Formulation of Model (2)

In this section we present the formulation of the dual problem of model (2), and then we explain the corresponding relationship between the dual variables and the primal constraints in model (2).

$$\begin{aligned}
\max \quad & \sum_{t=t_D}^{T}\Bigg[-\sum_{g\in\mathcal{G}}\left(\bar{s}_g^p B_{gt}^\omega\lambda_{gt}^{spu}+\underline{s}_g^p B_{gt}^\omega\lambda_{gt}^{spl}+\bar{s}_g^q B_{gt}^\omega\lambda_{gt}^{squ}+\underline{s}_g^q B_{gt}^\omega\lambda_{gt}^{sql}+R_g^U B_{gt}^\omega\lambda_{gt}^{RU}+R_g^D B_{gt}^\omega\lambda_{gt}^{RD}\right)- \\
& \sum_{i\in\mathcal{N}}\left(\bar{V}_i\lambda_{it}^{vu}+\underline{V}_i\lambda_{it}^{vl}+d_{it}^p\lambda_{it}^{pd}+d_{it}^q\lambda_{it}^{qd}\right)-\sum_{k\in\mathcal{E}}W_k B_{kt}^\omega\nu_{1,kt}-\sum_{b\in\mathcal{B}}\left(\bar{I}_b\lambda_b^{wu}+\sum_{\ell\in\mathcal{L}_b}\beta_b^\ell\lambda_{blt}^{\alpha\beta}\right)\Bigg]- \\
& \sum_{g\in\mathcal{G}}\hat{s}_{gt_D-1}^p\gamma_g-\sum_{b\in\mathcal{B}}\left(\bar{u}_b\lambda_b^{uu}+\hat{u}\zeta_b+\hat{w}_{bt_D-1}\eta_b\right)-\sum_{\omega\in\Omega}\sum_{t=t_D+\tau^\omega+1}^{T}\sum_{l\in\mathcal{C}_t^\omega}\lambda_{t\omega l}^c\Big(\hat{v}^{t\omega l}-\gamma^{t\omega l\top}\hat{s}_{\cdot t-1}^{p,t\omega l}- \\
& \eta^{t\omega l\top}\hat{w}_{\cdot t-1}^{t\omega l}-\zeta^{t\omega l\top}\hat{u}^{t\omega l}\Big)-\sum_{t=t_D}^{T}\left[\sum_{g\in\mathcal{G}}\left(\frac{c_{g1}^2-c_{g2}}{4c_{g2}}\mu_{31,gt}+\frac{c_{g1}}{2\sqrt{c_{g2}}}\mu_{32,gt}+\frac{c_{g1}^2+c_{g2}}{4c_{g2}}\nu_{3,gt}\right)\right] && \text{(1a)}
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \quad & \lambda_{gt}^{spu}+\lambda_{gt}^{spl}+\lambda_{i_gt}^{pd}+B_{gt}^\omega\lambda_{gt}^{RU}-B_{gt+1}^\omega\lambda_{gt+1}^{RU}+B_{gt}^\omega\lambda_{gt}^{RD}-B_{gt+1}^\omega\lambda_{gt+1}^{RD}-\sqrt{c_{g2}}\mu_{32,gt}- \\
& \quad \sum_{\omega\in\Omega}\sum_{\substack{l\in\mathcal{C}_{t+1}^\omega:\\ t\ge t_D+\tau^\omega}}\lambda_{t+1\omega l}^c\gamma_g^{t+1\omega l}=0 && \forall g\in\mathcal{G}, t=t_D,\dots,T-1 \quad \text{(1b)} \\
& \lambda_{gT}^{spu}+\lambda_{gT}^{spl}+\lambda_{i_gT}^{pd}+B_{gT}^\omega\lambda_{gT}^{RU}+B_{gT}^\omega\lambda_{gT}^{RD}-\sqrt{c_{g2}}\mu_{32,gT}=0 && \forall g\in\mathcal{G} \quad \text{(1c)} \\
& B_{gt_D}^\omega\lambda_{gt_D}^{RU}+B_{gt_D}^\omega\lambda_{gt_D}^{RD}+\gamma_g=0 && \forall g\in\mathcal{G} \quad \text{(1d)} \\
& \lambda_{gt}^{squ}+\lambda_{gt}^{sql}+\lambda_{i_gt}^{qd}=0 && \forall g\in\mathcal{G}, t=t_D,\dots,T \quad \text{(1e)} \\
& -\lambda_{it}^{pd}+R_kB_{kt}^\omega\lambda_{kt}^{pf}-B_{kt}^\omega\mu_{11,kt}=0 && \forall k=ij\in\mathcal{E}, t=t_D,\dots,T \quad \text{(1f)} \\
& -\lambda_{it}^{qd}+X_kB_{kt}^\omega\lambda_{kt}^{pf}-B_{kt}^\omega\mu_{12,kt}=0 && \forall k=ij\in\mathcal{E}, t=t_D,\dots,T \quad \text{(1g)} \\
& \lambda_{it}^{vu}+\lambda_{it}^{vl}+\sum_{k=ji\in\mathcal{E}}B_{kt}^\omega\lambda_{kt}^{pf}-\sum_{k=ij\in\mathcal{E}}B_{kt}^\omega\lambda_{kt}^{pf}=0 && \forall i\in\mathcal{N}, t=t_D,\dots,T \quad \text{(1h)} \\
& -c_{it}^L\le\lambda_{it}^{pd}\le c_{it}^L && \forall i\in\mathcal{N}, t=t_D,\dots,T \quad \text{(1i)} \\
& -c_{it}^L\le\lambda_{it}^{qd}\le c_{it}^L && \forall i\in\mathcal{N}, t=t_D,\dots,T \quad \text{(1j)} \\
& \lambda_{bt}^{wu}+\lambda_{bt}^I-\lambda_{bt+1}^I-\sum_{\omega\in\Omega}\sum_{\substack{l\in\mathcal{C}_{t+1}^\omega:\\ t\ge t_D+\tau^\omega}}\lambda_{t+1\omega l}^c\eta_b^{t+1\omega l}=0 && \forall b\in\mathcal{B}, t=t_D,\dots,T-1 \quad \text{(1k)} \\
& \lambda_{bT}^{wu}+\lambda_{bT}^I\ge 0 && \forall b\in\mathcal{B} \quad \text{(1l)} \\
& \eta_b-\lambda_{bt_D}^I\ge 0 && \forall b\in\mathcal{B} \quad \text{(1m)} \\
& \lambda_b^{uu}+\zeta_b-\sum_{t=t_D}^{T}\nu_{2,bt}-\sum_{\omega\in\Omega}\sum_{\substack{l\in\mathcal{C}_{t+1}^\omega:\\ t\ge t_D+\tau^\omega}}\lambda_{t+1\omega l}^c\zeta_b^{t+1\omega l}\ge 0 && \forall b\in\mathcal{B} \quad \text{(1n)} \\
& \lambda_{i_bt}^{pd}+\sum_{\ell\in\mathcal{L}_b}\lambda_{blt}^{\alpha\beta}-\mu_{21,bt}=0 && \forall b\in\mathcal{B}, t=t_D,\dots,T \quad \text{(1o)} \\
& \lambda_{i_bt}^{qd}-\mu_{22,bt}=0 && \forall b\in\mathcal{B}, t=t_D,\dots,T \quad \text{(1p)} \\
& \Delta t\lambda_{bt}^I-\sum_{\ell\in\mathcal{L}_b}\alpha_b^\ell\lambda_{blt}^{\alpha\beta}=0 && \forall b\in\mathcal{B}, t=t_D,\dots,T \quad \text{(1q)} \\
& \sum_{l\in\mathcal{C}_t^\omega}\lambda_{t\omega l}^c\ge -c_{t\omega}^\theta && \forall\omega\in\Omega, t=t_D+\tau^\omega+1,\dots,T \quad \text{(1r)}
\end{aligned}$$

$$\mu_{31,gt} + \nu_{3,gt} \le c^h_{gt} \qquad \forall g \in \mathcal{G}, t = t_D, \dots, T \tag{1s}$$

$$\|(\mu_{11,kt}, \mu_{12,kt})\| \le \nu_{1,kt} \qquad \forall k \in \mathcal{E}, t = t_D, \dots, T \tag{1t}$$

$$\|(\mu_{21,kt}, \mu_{22,kt})\| \le \nu_{2,kt} \qquad \forall k \in \mathcal{E}, t = t_D, \dots, T \tag{1u}$$

$$\|(\mu_{31,kt}, \mu_{32,kt})\| \le \nu_{3,kt} \qquad \forall k \in \mathcal{E}, t = t_D, \dots, T \tag{1v}$$

$$\lambda^{spu}, \lambda^{squ}, \lambda^{vu}, \lambda^{wu}, \lambda^{uu}, \lambda^{RU}, \lambda^{\alpha\beta}, \nu_1, \nu_2, \nu_3 \ge 0 \tag{1w}$$

$$\lambda^{spl}, \lambda^{sql}, \lambda^{vl}, \lambda^{RD} \le 0. \tag{1x}$$

As stated in Section 3, we denote the index set of cuts generated for a specific (t, ω) pair as $\mathcal{C}^\omega_t$. In this formulation, λ^{pd} and λ^{qd} are the dual variables of flow balance constraints (2b) and (2c). Power flow equations (2e) and (2f) can be reformulated as an equality just for the functional lines, with the corresponding dual variable λ^{pf}. We use λ^{RU} and λ^{RD} to represent the dual variables of the ramping constraints (2g) and (2h), $\lambda^{spu}, \lambda^{spl}, \lambda^{squ}$ and λ^{sql} to represent the dual variables corresponding to the upper and lower bounds for power generation (2i)-(2j), and λ^{vu} and λ^{vl} to represent the dual variables corresponding to the upper and lower bounds for squared voltage (2k). For constraints on batteries, we denote the dual variables for constraints (2l), (2n)-(2p) as $\lambda^I, \lambda^{\alpha\beta}, \lambda^{wu}$ and λ^{uu}, respectively. Dual variables γ, η and ζ correspond to the non-anticipativity constraints (2r)-(2t). As we generate cuts to approximate the value function f from below, the dual variables for those cuts are denoted as λ^c.

Finally, suppose we create an auxiliary variable h_{gt} to represent the generation cost at generator $g \in \mathcal{G}$ for time $t = t_D, \dots, T$. We can then introduce a set of second-order cone constraints as follows to characterize this quadratic cost function:

$$c_{g2} {s^p_{gt}}^2 + c_{g1} s^p_{gt} \le h_{gt} \quad \iff \quad \|(h_{gt} + \frac{c^2_{g1} - c_{g2}}{4c_{g2}}, \sqrt{c_{g2}} s^p_{gt} + \frac{c_{g1}}{2\sqrt{c_{g2}}})\| \le h_{gt} + \frac{c^2_{g1} + c_{g2}}{4c_{g2}}. \tag{2}$$

For the second-order cone constraints (2d), (2m) and (2), we employ three sets of dual variables as (μ_1, ν_1), (μ_2, ν_2) and (μ_3, ν_3). The dual SOCP constraints are formed as (1t)-(1v).

With the dual variables established, we build constraint (1b)-(1d) for the primal variable s^p (for time $t_D, \dots, T, T$ and $t_D - 1$), (1e) for s^q, (1h) for V, (1i) and (1j) for L^{p+}, L^{p-}, L^{q+} and L^{q-}, (1k)-(1m) for w (for time $t_D, \dots, T, T$ and $t_D - 1$), (1n) for u, (1o) and (1p) for z^p and z^q, (1q) for y, (1r) for the approximated value function term θ, and (1s) for the auxiliary variables h. Here for simplicity of representation, we define the coefficients in the primal objective function for L, θ and h terms as c^L, c^θ and c^h, which can be calculated from the probability parameters p^ω_t and the penalty coefficient c^l.

Online Supplement C. Cut Pre-generation

In the first few iterations of an SDDP-type algorithm, cuts (7) may be loose for earlier stages as they have coarse approximation of later-stage value functions in their objective functions. However, these cuts are tight approximations for any terminal stage value function from the beginning of the algorithm. The terminal stage value functions in our problem are $f_t^{\omega}, \forall t \geq T-\tau+1, \omega \in \Omega$. We can use this property to pre-generate cuts for terminal stage value functions so that the cuts for earlier stages are better from the beginning of the algorithm. We compare the run-time with and without the pre-generated cuts, as shown in Figure 7. For this test, we pre-generate 5 cuts in each iteration for 20 iterations before Algorithm 1. Although the pre-generation incurs an overhead and iterations with longer run-time towards the end, it helps achieve a better lower bound result if only a small sample budget is given.

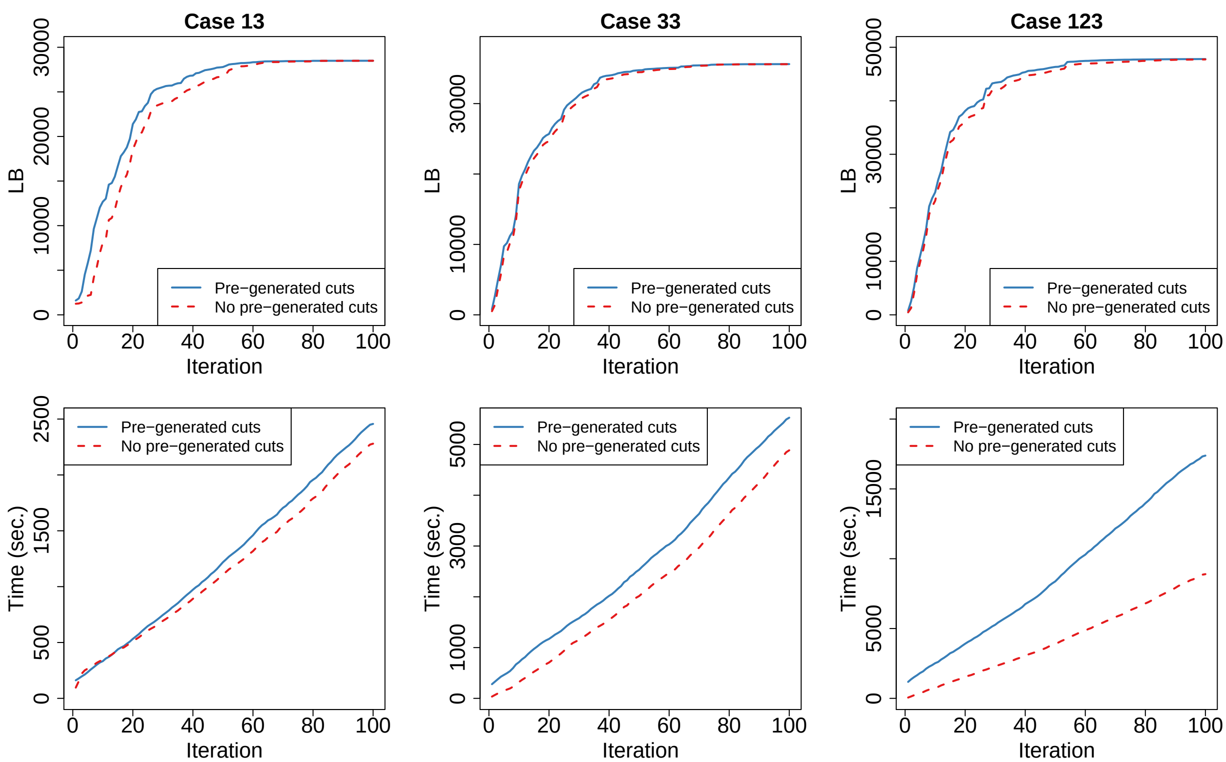

Figure 7 Lower bounds and run-times vs. the number of iterations with and without the terminal stage cut pre-generation. Five cuts are pre-generated in each iteration for 20 iterations.