Online Supplement: Detailed Computational Results

Table 1 Summary of the notation used in the following numerical analysis.

Abbreviation	Meaning
b	Right hand side constraint value
n	Size of the coefficient vector a
r_1	Optimal radius (bypass point) on the positive real axis
r_2	Optimal radius (bypass point) on the positive imaginary axis
r^*	Optimal radius (bypass point) on the positive real axis
sol	Computed (approximate) number of feasible solutions
$\mathrm{sol}_{\mathrm{SP}}^{N}$	Computed (approximate) number of feasible solutions using a shortest path with parameters $(N, 0.001, r^*)$
$\mathrm{sol}_{\mathrm{LattE}}$	Computed (approximate) number of feasible solutions using the LattE software
t_{circ}	Total computation time for the circular parameterization
t_{ellip}	Total computation time for the elliptic parameterization
t_{SP}^{N}	Total computation time for the shortest path parameterization with parameters $(N, 0.001, r^*)$
t_{SPint}^{N}	Integration time only for the shortest path parameterization with parameters $(N, 0.001, r^*)$
t_{LattE}	Total computation time using the software LattE
c_{circ}	Condition number for the circular parameterization
c_{ellip}	Condition number for the elliptical parameterization
c_{SP}^{N}	Condition number for the shortest path parameterization with parameters $(N, 0.001, r^*)$
$\mathrm{adj}_{\mathrm{SPint}}^{N}$	Number of radial adjustments for the shortest path with parameters $(N, 0.001, r^*)$

Table 2 Overview of the path parameters for the simple geometries, the circle and the ellipse, as well as the two shortest paths on the instances taken from Pisinger (2005). While r_1 parameterizes the circle and the starting point of the shortest paths, both r_1 and r_2 are necessary to parameterize the ellipse. The elliptic path presents significantly smaller condition numbers than the circle, while the shortest paths achieve even smaller condition numbers. Comparing the two shortest paths, the more fine-grained parameterization presents an additional improvement in the condition number, presenting the minimal condition number of all paths presented here as expected.

	r_1	r_2	c_{circ}	c_{ellip}	$\text{adj}_{\text{SP}}^{36}$	c_{SP}^{36}	$\text{adj}_{\text{SP}}^{500}$	c_{SP}^{500}
p1	0.98962	0.99887	678655	612697	40	608288	432	600547
p2	0.99131	0.9963	49176	36205	46	41433	366	155863
p3	0.99096	0.99759	271239	257825	44	252396	258	252434
p4	0.99074	0.99822	22999	10675	48	197617	552	642928
p5	0.99303	0.99745	207614	183162	24	173175	208	172550
p6	0.99112	0.99821	779029257	778130512	32	777281397	240	777281340
p7	0.99348	0.99809	2168347	1683157	30	1718130	246	1618354
p8	0.99356	0.99861	68189975	66345913	28	66499849	192	66548927
p9	0.99401	0.99733	6135125	5417261	18	5206232	86	5205048
p10	0.99513	0.99773	56384073	55023923	16	54470791	64	54468836
p11	0.99365	0.99781	45228184538	45203883128	34	45212704840	206	45212672445
p12	0.9961	0.9985	17917502	14771042	8	14037288	16	14036941
p13	0.99501	0.99837	14296550740	14257710276	10	14250577663	18	14250575775
p14	0.99551	0.99899	60901637061	60843896655	12	60832767794	64	60832758832
p15	0.9961	0.99894	2125025756	2087465101	8	2081446425	8	2081445184
p16	0.99606	0.9985	2344359185	2283324821	8	2269475079	12	2269472159
p17	0.99571	0.99812	339691087150	339414418877	28	339366178574	100	339362115922
p18	0.99594	0.99852	-	-	8	3074645993635	8	3074645967170
p19	0.99553	0.99859	-	-	12	348618396597779	24	348618396575776
p20	0.99666	0.99877	-	-	8	150376189492629	28	150376189489663
p21	0.99599	0.99869	-	-	-	-	-	-
p22	0.99616	0.99762	-	-	10	697432025032480	10	697432024320699
p23	0.99637	0.99797	-	-	8	156126260246894	8	156126260225020
p24	0.99624	0.99854	-	-	-	-	-	-

Table 3 Overview of the computational results for the simple geometries, the circle and the ellipse, as well as the two shortest paths on the instances taken from Pisinger (2005). The ellipse requires longer computation times for smaller instances, but performs faster on the larger instances when compared to the circle. The shortest paths require longer computation times than the simple geometries on the smaller instances, but outperform the circle and ellipse on the larger instances. Several instances that cannot be solved by the simple geometries can be solved by the shortest paths. While the shortest path with $N = 500$ angular nodes requires longer computation times for all instances as compared to the parameterization with $N = 36$ angular nodes, it is able to achieve shorter integration times for some instances, for example $p1$. The integration introduces an error in the larger instances, which is evident in the discrepancies in the approximate solution counts. The computation time for the LattE software exceeds the 300 second time limit for these instances, underlining that these instances are indeed hard to compute.

	$\mathrm{sol}_{\mathrm{circ}}$	t_{circ}	$\mathrm{sol}_{\mathrm{ellip}}$	t_{ellip}	$\mathrm{sol}^{36}_{\mathrm{SP}}$	t^{36}_{SP}	t^{36}_{SPint}	$\mathrm{sol}^{500}_{\mathrm{SP}}$	t^{500}_{SP}	t^{500}_{SPint}	$\mathrm{sol}_{\mathrm{LattE}}$	t_{LattE}
p1	544429	1.49	544429	1.71	544429	5.24	4.68	544429	11.65	3.75	-	>300
p2	28060	1.62	28060	1.90	28060	3.40	2.95	28060	10.10	3.33	-	>300
p3	220560	1.34	220560	1.26	220560	3.17	2.60	220560	8.53	2.41	-	>300
p4	5502	1.34	5502	1.38	5501	4.62	4.15	5502	11.05	4.38	-	>300
p5	152417	1.50	152417	2.70	152416	2.94	2.61	152417	6.97	2.47	-	>300
p6	715681393	2.00	715681393	2.06	715681393	3.06	2.64	715681393	8.58	2.74	-	>300
p7	1439298	2.80	1439298	2.53	1439298	4.15	3.83	1439298	8.24	3.74	-	>300
p8	60979453	2.66	60979453	2.83	60979453	4.77	4.46	60979453	8.54	4.04	-	>300
p9	4849303	3.04	4849303	4.26	4849303	3.00	2.74	4849303	6.55	2.74	-	>300
p10	50856154	5.51	50856154	3.43	50856154	3.59	3.37	50856154	6.58	3.44	-	>300
p11	42417243268	3.84	42417243268	3.09	42417243269	4.83	4.51	42417243268	9.17	4.66	-	>300
p12	13244852	5.59	13244852	5.00	13244852	3.83	3.66	13244852	6.17	3.69	-	>300
p13	13466832804	5.50	13466832804	4.45	13466832804	3.48	3.26	13466832803	6.50	3.36	-	>300
p14	57612978960	7.37	57612978960	5.37	57612978960	5.29	5.07	57612978959	8.18	5.02	-	>300
p15	1981736210	5.81	1981736210	5.13	1981736210	3.90	3.72	1981736210	6.57	4.08	-	>300
p16	2169533751	19.88	2169533751	11.41	2169533751	3.81	3.64	2169533751	6.38	3.89	-	>300
p17	324480407213	9.84	324480407213	8.21	324480407213	5.55	5.33	324480407213	9.53	6.36	-	>300
p18	-	>300	-	>300	2938495443637	3.83	3.65	2938495443638	6.38	3.91	-	>300
p19	-	>300	-	>300	333806322641172	5.32	5.10	333806322641188	8.01	4.87	-	>300
p20	-	>300	-	>300	144048903115529	7.17	6.99	144048903115544	9.20	6.73	-	>300
p21	-	>300	-	>300	-	>300	>300	-	>300	>300	-	>300
p22	-	>300	-	>300	672271093810708	4.47	4.30	672271093810763	7.19	4.71	-	>300
p23	-	>300	-	>300	150556678598701	5.01	4.84	150556678598714	7.52	5.05	-	>300
p24	-	>300	-	>300	-	>300	>300	-	>300	>300	-	>300

Table 4 Overview of the path parameters for the simple geometries, the circle and the ellipse, as well as the two shortest paths on a set of instances with no solutions. Choosing a smaller b value will diminish the relative order of the pole at the origin. Dominant poles on the unit circle provide the shortest path with more opportunities to reduce the condition number, which is reflected in the number of adjustments.

	a	b	r_1	r_2	c_{circ}	c_{ellip}	$\text{adj}_{\text{SP}}^{36}$	c_{SP}^{36}	$\text{adj}_{\text{SP}}^{500}$	c_{SP}^{500}
q1	$[3, 5, 6, 9, 10]$	1	0.53882	0.66762	2	2	960	2	1330	2
q2	$[11, 13, 17, 19, 23]$	31	0.92388	0.98272	13	7	418	4	656	4
q3	$[500, 501, ...550]$	551	0.99259	0.99805	61	22	32	12	-	-

Table 5 Overview of the computational results for the simple geometries, the circle and the ellipse, as well as the two shortest paths on a set of instances with no solutions. Making more adjustments increases the computation time of the shortest path, exceeding the time limit for the final instance, which demonstrates an intricate pole structure on the unit circle due to the large number of factors in the coefficient vector. The results for the LattE software are included for reference.

	sol_{circ}	t_{circ}	$\text{sol}_{\text{ellip}}$	t_{ellip}	$\text{sol}_{\text{SP}}^{36}$	t_{SP}^{36}	t_{SPint}^{36}	$\text{sol}_{\text{SP}}^{500}$	t_{SP}^{500}	t_{SPint}^{500}	$\text{sol}_{\text{LattE}}$	t_{LattE}
q1	0.00001	0.04	0.00002	0.01	0.00000	5.09	1.64	0.00000	48.34	2.50	0	0.01
q2	0.00002	0.02	0.00009	0.04	0.09948	1.12	0.64	-0.08572	8.28	1.61	0	0.01
q3	0.00000	1.00	0.00006	1.17	0.19306	0.85	0.51	-	>300	>300	-	>300

Table 6 Path parameterizations and condition numbers for the shortest paths on instances with increasing b values. As expected, the condition number for the parameterization with $N = 500$ is considerably smaller for all instances, while the number of adjustments in the path are significantly higher.

b	$\text{adj}_{\text{SP}}^{36}$	c_{SP}^{36}	$\text{adj}_{\text{SP}}^{500}$	c_{SP}^{500}
50	416	634	1962	604
100	298	38661	1420	38193
150	332	953080	986	949750
200	272	13733403	860	13623868
250	228	134828544	670	134737635
300	204	1014519018	572	1014318588
350	180	6389699415	488	6187498785
400	166	32910844190	462	31900185106
450	148	152339950309	376	143249092647
500	114	606976426541	370	573273368341
550	104	2189928633123	336	2078739683042
600	100	7444688718918	382	6924267384509
650	94	22833010488655	284	21419757867733
700	98	66952634340672	272	62063790182628
750	96	181032537420752	258	169725296202847
800	88	609365125506520	240	440793071785804
850	86	1338341315912620	208	1092689749563940
900	78	2955780219429130	194	2596004038988620
950	70	6463996048212330	166	5937510299177790
1000	76	13875597544999900	214	13108782372348000
1050	-	-	-	-

Table 7 **Increasing the b value increases the size of the pole at the origin. This moves the starting radius further to the boundary, and decreases the number of nodes in the shortest path problem, decreasing the computation time. The two parameterizations produce the exact same solutions for the smaller instances, and begin producing small relative discrepancies for the last few instances. The error pattern is similar to what is observed in Table 2 and results from the error in the numerical integration scheme.**

b	$\text{sol}_{\text{SP}}^{36}$	t_{SP}^{36}	t_{SPint}^{36}	$\text{sol}_{\text{SP}}^{500}$	t_{SP}^{500}	t_{SPint}^{500}	$\text{sol}_{\text{LattE}}$	t_{LattE}
50	510	3.64	0.94	510	35.90	4.94	510	32.45
100	34434	2.92	0.88	34434	27.06	4.00	34434	243.22
150	875553	2.37	0.97	875553	22.53	3.08	-	>300
200	12706352	2.08	0.95	12706352	20.32	2.94	-	>300
250	126508562	1.89	0.91	126508562	15.50	2.47	-	>300
300	956593952	1.82	0.97	956593952	12.39	1.85	-	>300
350	5852912692	1.54	0.83	5852912692	11.20	1.76	-	>300
400	30242661625	1.76	1.02	30242661625	10.17	1.69	-	>300
450	136049809507	1.39	0.81	136049809507	10.94	1.82	-	>300
500	545027910355	1.96	1.18	545027910355	11.56	1.98	-	>300
550	1978312132399	1.21	0.76	1978312132399	8.77	1.58	-	>300
600	6595122641660	1.67	1.12	6595122641660	9.95	2.14	-	>300
650	20413951150269	1.69	1.11	20413951150269	9.28	1.92	-	>300
700	59191214892887	1.77	1.23	59191214892891	10.22	2.06	-	>300
750	161955797166090	1.42	1.00	161955797166113	7.78	1.95	-	>300
800	420735861015145	1.31	0.95	420735861015155	6.57	1.51	-	>300
850	1043161193458690	1.23	0.90	1043161193458690	6.01	1.41	-	>300
900	2479412642859080	1.19	0.88	2479412642859090	6.10	1.41	-	>300
950	5671026462624020	1.13	0.81	5671026462624090	5.60	1.46	-	>300
1000	12523653711398100	1.35	1.05	12523653711398000	5.79	1.63	-	>300
1050	-	>300	>300	-	>300	>300	-	>300

Table 8 **The b value is maintained at $b = 200$ while the number of coefficients in the a vector is increased. The new coefficients are randomly generated in $[1, 50]$. This causes the poles on the boundary of the unit disk to increase in size, while the pole at the origin remains constant. Again, the shortest path with more degrees of freedom achieves smaller condition numbers.**

n	$\text{adj}_{\text{SP}}^{36}$	c_{SP}^{36}	$\text{adj}_{\text{SP}}^{500}$	c_{SP}^{500}
20	272	13733403	860	13623868
30	400	3416220179	982	3414894481
40	672	2018576130445	1330	2018339150639
50	684	117363732058470	1406	117358635936184
60	740	259247694103850	1470	259238025981771
70	754	834172365560111	1724	834139231395788
80	760	3420974571790020	1878	3420857654483900
90	718	10889817400463400	1946	10889564880473200
100	-	-	1984	457795568727533000
110	-	-	2206	756858595535446000
120	-	-	-	-

Table 9 **The shortest path problem becomes more costly to compute for all parameterizations as the size of the poles on the boundary of the unit disk increases. Lower condition numbers can be achieved by carefully circumventing the poles. While avoiding the poles decreases the condition number significantly, the computation time increases when increasing the degrees of freedom in the shortest path parameterization. Nevertheless, the lower condition numbers ultimately permits additional instances to converge in the numerical integration. The LattE code does not solve the instances within the time limit, but the time for the first instance is included for reference.**

n	$\text{sol}_{\text{SP}}^{36}$	t_{SP}^{36}	t_{SPint}^{36}	$\text{sol}_{\text{SP}}^{500}$	t_{SP}^{500}	t_{SPint}^{500}	$\text{sol}_{\text{LattE}}$	t_{LattE}
20	12706352	1.99	0.92	12706352	15.37	2.22	12706352	(767.20)
30	3231494637	3.18	1.38	3231494637	26.8	2.63	-	>300
40	1926080459195	4.60	1.75	1926080459195	42.05	3.53	-	>300
50	112741441577786	5.95	1.92	112741441577787	59.68	4.16	-	>300
60	249378103539524	7.46	2.27	249378103539526	72.86	4.41	-	>300
70	802703537456477	8.24	2.33	802703537456480	85.42	5.19	-	>300
80	3293394167012470	9.45	2.48	3293394167012480	100.74	5.94	-	>300
90	10497306954818300	10.79	2.62	10497306954818300	117.89	6.43	-	>300
100	-	>300	>300	442500538243987000	142.45	7.17	-	>300
110	-	>300	>300	731645471006155000	168.69	9.03	-	>300
120	-	>300	>300	-	>300	>300	-	>300