Online Supplementary Material for

Robust Direct Aperture Optimization for Radiation Therapy Treatment Planning

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Appendix A: Robust Counterpart of (RFMO) for Polyhedral Uncertainty

Polyhedral uncertainty set (2) leads to infinite possible realizations of $\tilde{\mathbf{p}}$ when $p_i - \underline{p}_i < p_i + \bar{p}_i$ for at least one phase i, and therefore, constraints (1b) are intractable. Chan et al. (2006) show that the equivalent robust counterpart of these constraints is both tractable and linear, at the expense of introducing a set of $|\mathcal{V}_T| \times (|\mathcal{I}|+1)$ new dual variables ($y_{i,v}$) and $|\mathcal{V}_T| \times |\mathcal{I}|$ new constraints as follows.

$$\min \sum_{s\in\{\mathcal{T},\mathcal{H}\}} \frac{c_s}{|\mathcal{V}_s|} \sum_{v\in\mathcal{V}_s} \sum_{b\in\mathcal{B}} \sum_{i\in\mathcal{I}} p_i D_{v,b,i}\omega_b$$

$$\text{s.t.} \sum_{i\in\mathcal{I}} \left[\underline{p}_i y_{0,v} - (\underline{p}_i + \bar{p}_i) y_{i,v} + (p_i - \underline{p}_i) \sum_{b\in\mathcal{B}} D_{b,v,i}\omega_b\right] \geq L_v \qquad \forall v \in \mathcal{V}_T, \tag{1a}$$

$$\sum_{b\in\mathcal{B}} D_{b,v,i}\omega_b - y_{0,v} + y_{i,v} \geq 0 \qquad \forall i \in \mathcal{I}, v \in \mathcal{V}_T, \tag{1b}$$

$$\omega_b \geq 0 \qquad \forall b \in \mathcal{B}, \tag{1c}$$

$$y_{i,v} \geq 0 \qquad \forall i \in \mathcal{I}, v \in \mathcal{V}_T, \tag{1d}$$

$$y_{0,v} \text{ URS} \qquad \forall v \in \mathcal{V}_T. \tag{1e}$$

The above robust counterpart to the (**RFMO**) model finds the worst-case realization of $\tilde{\mathbf{p}}$ for each voxel and optimizes over this realization, thereby immunizing the problem against the worst-case *feasible* realization of the uncertainty set for any chosen intensities.

Appendix B: Complete Robust Direct Aperture Model

The following is the complete (**RDAO**) formulation, with all relevant Section 2.2 deliverability constraints. The shaded lines are the base model, which assuming $|\mathcal{P}| = 1$ (i.e., no uncertainty) is the FMO model. When the shaded region is run and there is uncertainty ($|\mathcal{P}| > 1$), it is denoted the RFMO model. The remaining unannotated constraints are the classical deliverability constraints. When they are included in the model and certain, the DAO model is being run. If uncertain, the RDAO model is being run. Finally, when the boxed vertical and horizontal continuity constraints are included with the classical deliverability constraints on top of a certain base model, the model is denoted DAO-C, and when uncertain, the model is RDAO-C.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{s\in\{\mathcal{T},\mathcal{H}\}} \frac{c_s}{|\mathcal{V}_s|} \sum_{v\in\mathcal{V}_s}\sum_{b\in\mathcal{B}}\sum_{i\in\mathcal{I}}\sum_{a\in\mathcal{A}} p_i D_{v,b,i} w_{b,a} & \\
\text{s.t.} & \displaystyle\sum_{b\in\mathcal{B}}\sum_{i\in\mathcal{I}} \tilde{p}_i D_{b,v,i}\omega_b \geq L_v & \forall v\in\mathcal{V}_T, \tilde{\mathbf{p}}\in\mathcal{P}, \\
& w_{b,a} \leq M x_{b,a} & \forall b\in\mathcal{B}, a\in\mathcal{A}, \\
& w_{b,a} \leq f_a + M(1-x_{b,a}) & \forall b\in\mathcal{B}, a\in\mathcal{A}, \\
& w_{b,a} \geq f_a - M(1-x_{b,a}) & \forall b\in\mathcal{B}, a\in\mathcal{A}, \\
& \displaystyle\sum_{b\in\mathcal{B}_\theta} x_{b,a} \leq |\mathcal{B}_\theta| u_{a,\theta} & \forall a\in\mathcal{A}, \theta\in\Theta, \\
& \displaystyle\sum_{\theta\in\Theta} u_{a,\theta} = 1 & \forall a\in\mathcal{A}, \\
& l_{q,k+1,\theta,a} \geq l_{q,k,\theta,a} & \forall k\in\mathcal{K}', q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
& r_{q,k,\theta,a} \geq r_{q,k+1,\theta,a} & \forall k\in\mathcal{K}', q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
& x_{q,k,\theta,a} = -1 + l_{q,k,\theta,a} + r_{q,k,\theta,a} & \forall k\in\mathcal{K}, q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
\hline
\text{Continuity Constraints (“-C”)} & j_{q,\theta,a} = -1 + \bar{j}_{q,\theta,a} + \underline{j}_{q,\theta,a} & \forall q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
& j_{q,\theta,a} \leq \displaystyle\sum_{k\in\mathcal{K}} x_{q,k,\theta,a} & \forall q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
& |\mathcal{K}| \times j_{q,a,s} \geq \displaystyle\sum_{k\in\mathcal{K}} x_{q,k,\theta,a} & \forall q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
& \bar{j}_{q,\theta,a} \leq \bar{j}_{q+1,\theta,a} & \forall q\in\mathcal{Q}', a\in\mathcal{A}, s\in S, \\
& \underline{j}_{q+1,\theta,a} \leq \underline{j}_{q,\theta,a} & \forall q\in\mathcal{Q}', a\in\mathcal{A}, s\in S, \\
& j_{q,\theta,a} + j_{q-1,\theta,a} - \displaystyle\sum_{\delta=k+1}^{|\mathcal{K}|} x_{q,\delta,\theta,a} \leq 1 + \sum_{\delta=1}^{k} x_{q-1,\delta,\theta,a} & \forall k\in\mathcal{K}, q\in\mathcal{Q}'', \theta\in\Theta, a\in\mathcal{A}, \\
& j_{q,\theta,a} + j_{q-1,\theta,a} - \displaystyle\sum_{\delta=1}^{|\mathcal{K}|-k} x_{q,\delta,\theta,a} \leq 1 + \sum_{\delta=|\mathcal{K}|-k+1}^{|\mathcal{K}|} x_{q-1,\delta,\theta,a} & \forall k\in\mathcal{K}, q\in\mathcal{Q}'', a\in\mathcal{A}, s\in S, \\
& j_{q,\theta,a}, \bar{j}_{q,\theta,a}, \underline{j}_{q,\theta,a} \in \{0,1\} & \forall q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}, \\
\hline
& w_{b,a} \geq 0 & \forall b\in\mathcal{B}, a\in\mathcal{A}, \\
& f_a \geq 0 & \forall a\in\mathcal{A}, \\
& x_{b,a} \in \{0,1\} & \forall b\in\mathcal{B}, a\in\mathcal{A}, \\
& u_{a,\theta} \in \{0,1\} & \forall a\in\mathcal{A}, \theta\in\Theta, \\
& l_{q,k,\theta,a}, r_{q,k,\theta,a} \in \{0,1\} & \forall k\in\mathcal{K}, q\in\mathcal{Q}, \theta\in\Theta, a\in\mathcal{A}.
\end{array}
$$

Appendix C: CPG-S α-Parameter Selection

The value of the (**CPG-S**) parameter α was determined by running the problem at different α values in increments of 0.2 and choosing the best plan in terms of objective function value gap. The variation in objective function value as a function of α is depicted in Figure 1. The plot starts at $\alpha = 0.2$, since the problem collapses back into a fluence map optimization as α approaches 0, generating an optimality gap of 100% across all cases. Note that the bounds changed very little for $0.2 \leq \alpha \leq 0.8$, so in general, choosing any value in this range should lead to a high-quality warm start.

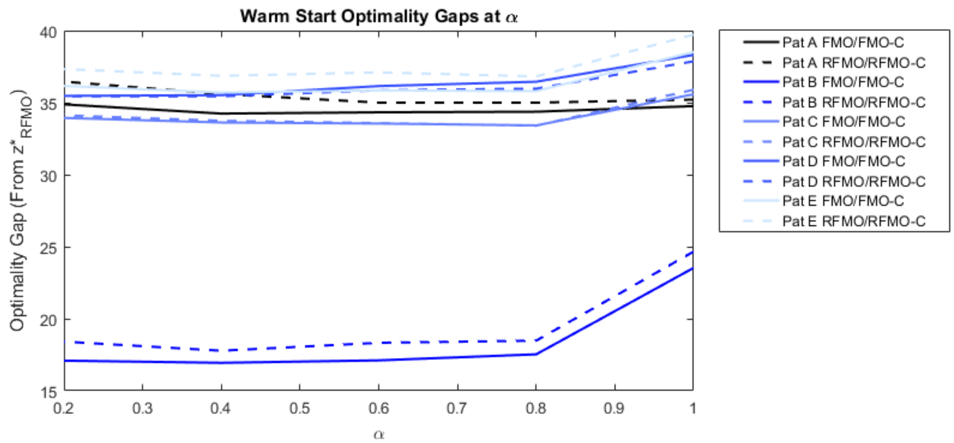

Figure 1 **The objective function gap at various α values between 0.2 and 1.**

Appendix D: A Note on RDAO Uncertainty Levels

The RDAO algorithm provides immunization against the worst-case realization of an uncertainty set, which is chosen to be the polyhedral set (2) in this paper, with $\underline{\mathbf{p}} = \bar{\mathbf{p}} = \mathbf{0.1}$. To give some insight into the impact of the magnitude of this uncertainty set, $\underline{\mathbf{p}}$ and $\bar{\mathbf{p}}$ are varied in Table 1. The objective function values of the CPG-proposed RDAO plan z^{CPG} for each of four different uncertainty conditions, around the nominal $\mathbf{p} = [0.125, 0.125, 0.125, 0.125, 0.5]$ are reported. Note that the DAO model (i.e., $\underline{\mathbf{p}} = \bar{\mathbf{p}} = 0$) and previously studied $\underline{\mathbf{p}} = \bar{\mathbf{p}} = 0.1$ results are included with the new smaller $\underline{\mathbf{p}} = \bar{\mathbf{p}} = 0.05$ and larger $\underline{\mathbf{p}} = \bar{\mathbf{p}} = 0.125$ uncertainty sets for reference.

	Patient				
$\underline{\mathbf{p}} = \bar{\mathbf{p}}$	A	B	C	D	E
0	55.3	37.7	51.5	50.6	51.7
0.05	56.8	38.1	52.2	51.0	52.7
0.1	56.8	38.2	51.8	51.3	53.0
0.125	58.1	38.6	52.3	51.6	53.7

Table 1 **The impact robust uncertainty set size on z^{CPG}.**

The overall trend is that the objective function increases monotonically as the uncertainty set grows larger. This is intuitive, seeing as a plan may have to compensate for larger proportions of time in the worst case phase-realization, thereby giving more dose to the target and healthy organs in order to ensure full dose to target. This trend is not true 100% of the time in the CPG case (as in Patient C, 0.05 exceeds 0.1) since the CPG heuristic finds non-optimal solutions, however it would be the case at optimality.

References

Chan TC, Bortfeld T, Tsitsiklis JN (2006) A robust approach to IMRT optimization. *Physics in Medicine & Biology* 51(10):2567.