Online supplement

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A Reformulations

A.1 Standard formulation

$$
\begin{aligned}
\text{(STD)} \quad \max \ & \sum_{k\in K}\sum_{a\in A_1} \eta^k T_a x_a^k \\
\text{s.t.} \ & \sum_{a\in A_1^+(i)} x_a^k + \sum_{a\in A_2^+(i)} y_a^k - \sum_{a\in A_1^-(i)} x_a^k - \sum_{a\in A_2^-(i)} y_a^k = b_i^k, && k\in K, i\in V, \\
& \lambda_i^k - \lambda_j^k \le c_a + T_a, && k\in K, a\equiv(i,j)\in A_1, \\
& \lambda_i^k - \lambda_j^k \le c_a, && k\in K, a\equiv(i,j)\in A_2, \\
& \sum_{a\in A_1}(c_a+T_a)x_a^k + \sum_{a\in A_2} c_a y_a^k = \lambda_{o^k}^k - \lambda_{d^k}^k, && k\in K, \\
& x_a^k \ge 0, && k\in K, a\in A_1, \\
& y_a^k \ge 0, && k\in K, a\in A_2, \\
& T_a \ge 0, && a\in A_1.
\end{aligned}
$$

A.2 Value function formulation

$$
\begin{aligned}
\text{(VF)} \quad \max \ & \sum_{k\in K}\sum_{a\in A_1} \eta^k T_a x_a^k \\
\text{s.t.} \ & \sum_{a\in A_1^+(i)} x_a^k + \sum_{a\in A_2^+(i)} y_a^k - \sum_{a\in A_1^-(i)} x_a^k - \sum_{a\in A_2^-(i)} y_a^k = b_i^k, && k\in K, i\in V, \\
& L^k \le \sum_{a\in A} \delta_a^p c_a + \sum_{a\in A_1} \delta_a^p T_a, && k\in K, p\in P^k, \\
& (c_a+T_a)x_a^k + \sum_{a\in A_2} c_a y_a^k = L^k, && k\in K, \\
& x_a^k \ge 0, && k\in K, a\in A_1, \\
& y_a^k \ge 0, && k\in K, a\in A_2, \\
& T_a \ge 0, && a\in A_1.
\end{aligned}
$$

A.3 Path complementary slackness formulation

$$
\begin{aligned}
\text{(PCS)} \qquad & \max \sum_{k\in K}\sum_{a\in A_1}\sum_{p\in P^k} \eta^k \delta_a^p T_a z_p^k \\
& \text{s.t.} \sum_{p\in P^k} z_p^k = 1, && k\in K,\\
& L^k \le \sum_{a\in A} \delta_a^p c_a + \sum_{a\in A_1} \delta_a^p T_a, && k\in K, p\in P^k,\\
& \left(\sum_{a\in A} \delta_a^p c_a + \sum_{a\in A_1} \delta_a^p T_a - L^k\right) z_p^k = 0, && k\in K, p\in P^k,\\
& z_p^k \ge 0, && k\in K, p\in P^k,\\
& T_a \ge 0, && a\in A_1.
\end{aligned}
$$

A.4 Hybrid model

Suppose we divide the set of commodities K into 3 parts based on the number of paths: K_1 for commodities having only 1 path, K_2 for commodities with less than N paths, and K_3 for commodities with more than N paths. If we assign (PCS2) to K_2 and (STD) to K_3, then we get the following hybrid model:

$$
\begin{aligned}
\max \; & \sum_{k\in K_2} \eta^k \tau^k + \sum_{k\in K_3}\sum_{a\in A_1} \eta^k t_a^k \\
\text{s.t.} \; & (4), (5), && k\in K_2,\\
& L^k \ge \sum_{a\in A} \delta_a^p c_a + \sum_{a\in A_1} \delta_a^p T_a - S_p^k(1 - z_p^k), && k\in K_2, p\in P^k,\\
& \sum_{p\in P^k}\sum_{a\in A} \delta_a^p c_a z_p^k + \tau^k = L^k, && k\in K_2,\\
& z_p^k \in \{0,1\}, && k\in K_2, p\in P^k,\\
& (1), (2), (3), (19), (20), && k\in K_3,\\
& \sum_{a\in A_1} (c_a x_a^k + t_a^k) + \sum_{a\in A_2} c_a y_a^k = \lambda_{o^k}^k - \lambda_{d^k}^k, && k\in K_3,\\
& x_a^k \in \{0,1\}, && k\in K_3, a\in A_1.
\end{aligned}
$$

B Proof of lemma 4

If the problem $(R(q^i), S(q^i))$ is infeasible, then the above statement is trivially true. Suppose it has a feasible solution $\hat{q}^i$ which is a simple path (path with no loops). We will construct another path (not necessarily a solution of $(R(q^i), S(q^i))$) which dominates $\hat{q}^i$ by replacing a segment in $\hat{q}^i$ by a shorter segment in $p^{(j)}$. First, we need to sort the arcs in $S(q^i)$ by the order of appearance in $p^{(j)}$:

$$\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots, \hat{a}_{n-1}, \hat{a}_n$$

where $n = |S(q^i)|$. We also define $\hat{a}_0$ as a virtual arc whose target is o^k (its source is not relevant). Next, for $0 \le i < n$, we will try to replace the segment of $\hat{q}^i$ between $\hat{a}_i$ and $\hat{a}_{i+1}$ with the same segment of $p^{(j)}$ if $\hat{a}_{i+1}$ comes after $\hat{a}_i$ in $\hat{q}^i$ and if the latter segment has smaller cost. If we are able to make such a replacement, due to the definition of $S(q^i)$, the segment connecting $\hat{a}_i$ and $\hat{a}_{i+1}$ in $p^{(j)}$ is toll-free; hence, we only remove tolled arcs while building a better path, and thus the resulting path dominates $\hat{q}^i$. If we cannot replace any segment, this means that either all segments in $\hat{q}^i$ are identical to those in $p^{(j)}$ or there is a segment in $\hat{q}^i$ with smaller cost.

Consider the first case: all segments in $\hat{q}^i$ are identical to those in $p^{(j)}$. In this case, the parts of $\hat{q}^i$ and $p^{(j)}$ from o^k to $s(q^i)$ are identical. The other part of $\hat{q}^i$ must be a path from $s(q^i)$ to d^k. Because we assume that $\hat{q}^i$ is a simple path, we must exclude all nodes preceding $s(q^i)$. However, since we also assume that q^i does not exist, this implies $s(q^i)$ and d^k to be disconnected if we exclude those nodes. This is a contradiction.

Consider the second case: there is a segment in $\hat{q}^i$ with smaller cost. Let $\hat{a}_m$ be the first arc such that the segment of $\hat{q}^i$ from $\hat{a}_m$ to $\hat{a}_{m+1}$ is cheaper than its counterpart of $p^{(j)}$. All segments preceding $\hat{a}_m$ in both $\hat{q}^i$ and $p^{(j)}$ must be identical since we assume that we cannot replace any segment. Consequently, the part from o^k to $\hat{a}_m$ of both paths are the same. By Lemma 3, the segment of $p^{(j)}$ from $\hat{a}_m$ to $\hat{a}_{m+1}$ is the optimal path while excluding all preceding nodes. However, the same segment of $\hat{q}^i$ is better, which means it must violate that constraint and must repeat some node preceding $\hat{a}_m$. This is again a contradiction as we assumed that $\hat{q}^i$ is a simple path.

Since both cases result in a contradiction, we can always replace some segment in $\hat{q}^i$ with a better segment in $p^{(j)}$. Therefore, $\hat{q}^i$ cannot be bilevel feasible. $\square$

C Example of path enumeration producing redundant paths

Consider the graph in Figure 1. The number of each arc represents the initial cost c_a. The first shortest path is $o^k - u - v - d^k$ with the cost of 3. By Algorithm 1, three subproblems are generated, each producing a candidate path. They are the second shortest path $o^k - u - d^k$ (cost 4), the third shortest path $o^k - u - v - w - d^k$ (cost 6), and the toll-free path $o^k - d^k$ (cost 10). All 4 paths will be returned, however, the path $o^k - u - d^k$ dominates $o^k - u - v - w - d^k$. The final set of bilevel feasible paths only has 3 paths: $o^k - u - v - d^k$, $o^k - u - d^k$, and $o^k - d^k$.

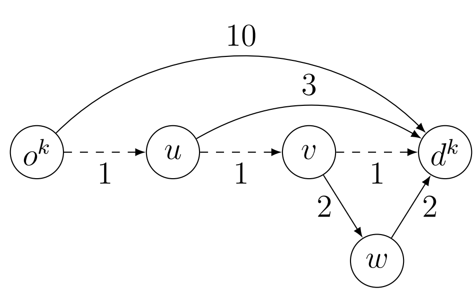

Figure 1: Graph with redundant paths (dashed arcs are tolled arcs).

D Instance generation by Brotcorne et al. (2000)

For a given graph and a set of O-D pairs, first, the shortest path of each commodity is found. Next, we count the number of paths passing through each arc and sort them in the descending order according to that count. Following that order, each arc is converted into a tolled arc until 2/3 of the desired number of tolled arcs is reached. The last 1/3 is selected randomly among all remaining arcs. An arc is converted only if it does not remove the last toll-free path for all commodities. To make the generated data more realistic, we also enforce the properties of the arc to be symmetrical, which means that any arc and its reversed arc will have the same cost, and both must be either tolled or toll-free. The cost of 80% of all arcs will be distributed uniformly from 5 to 35, while the remaining 20% will have the maximum cost of 35. The cost of tolled arcs are halved after the conversion. The proportion of tolled arcs is 20%.

References

Brotcorne L, Labbé M, Marcotte P, Savard G (2000) A bilevel model and solution algorithm for a freight tariff-setting problem. *Transportation Science* 34(3):289–302.