Online Companion for "A Stochastic Approximation Method for Simulation-Based Quantile Optimization"

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A. Unbiasedness Conditions for GLR

(A.1) (Smoothness Conditions) $h(x;\theta)$ is twice continuously differentiable with respect to $x,\theta \in \Omega \times \Theta$, where Θ is an open neighborhood for the parameter of interest, and $f(x;\theta)$ is continuously differentiable with respect to $(x,\theta) \in \Re^l \times \Theta$ and goes to zero as x approaches infinity.

(A.2) (Uniform Convergence Conditions) Let $A^{\varepsilon}_{y,\theta} = \left\{x \in \Re^l : y-\varepsilon \le h(x;\theta) \le y+\varepsilon\right\}$. Then $\forall\ \theta \in \Theta$, $\forall\ y \in \Re$,

$$\lim_{\varepsilon\to 0}\sup_{y\in\Re} \mathfrak{v}\left(A^{\varepsilon}_{y,\theta}\right)=0, \quad \lim_{\varepsilon\to 0}\sup_{\theta\in\Theta} \mathfrak{v}\left(A^{\varepsilon}_{y,\theta}\right)=0\ ,$$

where $\mathfrak{v}$ denotes the Lebesgue measure on $\Re^l$. For the special case where h is invertible, i.e., there exits $i=1,2,\cdots,l$, s.t. $x_i = h^{-1}(y;x_{-i},\theta)$, where $x_{-i}=(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_l)$, then the condition holds if h^{-1} is globally Lipschitz continuous with respect to y and θ.

(A.3) (Integrability Conditions) $\forall\ x \in \Re^l$, there exit functions $v_\ell(\cdot)$, $\ell=1,2,\cdots,l$, s.t. $\left|\left(\partial h(x;\theta)/\partial x_i\right)^{-1}\right| \le \prod_{\ell=1}^{l} v_\ell(x_\ell;\theta)$,

$$\lim_{x_i\to\pm\infty} v_i(x_i;\theta) f_i(x_i;\theta)=0, \quad \int_{\Re} v_\ell(x_\ell;\theta) f_\ell(x_\ell;\theta)\,dx_\ell < \infty, \quad \ell \ne i,$$

and

$$\lim_{x_i \to \pm\infty} \int_{\Re^{l-1}} \left| \left(\frac{\partial h(x;\theta)}{\partial x_i} \right)^{-1} \frac{\partial h(x;\theta)}{\partial \theta} \right| f(x;\theta) \prod_{\ell \neq i} dx_\ell = 0 \ .$$

In addition,

$$\int_{x\in\Re^l} \sup_{\theta\in\Theta} |\Psi_{1,i}(x;\theta) f(x;\theta)| dx < \infty, \quad \int_{x\in\Re^l} |\Psi_{2,i}(x;\theta)| f(x;\theta) \, dx < \infty \ .$$

For the special case when h^{-1} is globally Lipschitz continuous with respect to y and θ, then the condition holds if $\mathbb{E}\left[\left| \frac{\partial^2 h(x;\theta)}{\partial x_i^2} \Big|_{x=X} \right| \right] < \infty$, and $\lim_{x_i\to\pm\infty} f_i(x_i;\theta) = 0$, $f(x;\theta) < \infty$.

B. Proof of Lemma 1

For notational convenience, denote $G_1(X;\theta_k,q_k)$ and $G_2(X;\theta_k,q_k)$ as $G_{1,k}$ and $G_{2,k}$, respectively. We write equation (9) in the form $D_{k+1} = (1-\alpha_k G_{2,k})D_k + \alpha_k G_{1,k}$. It follows that

$$\|D_{k+1}\|^2 \le (1-\alpha_k G_{2,k})^2 \|D_k\|^2 + 2\alpha_k |1-\alpha_k G_{2,k}| \|D_k\| \|G_{1,k}\| + \alpha_k^2 \|G_{1,k}\|^2.$$

Taking conditional expectations at both sides and using A2(b), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|D_{k+1}\|^2 \middle| \mathcal{F}_k\right] \\
&\quad \le \mathbb{E}\left[(1-\alpha_k G_{2,k})^2 \middle| \mathcal{F}_k\right] \|D_k\|^2 + 2\alpha_k \mathbb{E}\left[\|G_{1,k}\| |1-\alpha_k G_{2,k}| \middle| \mathcal{F}_k\right] \|D_k\| + \alpha_k^2 \mathbb{E}[\|G_{1,k}\|^2 | \mathcal{F}_k] \\
&\quad \le \mathbb{E}\left[(1-\alpha_k G_{2,k})^2 \middle| \mathcal{F}_k\right] \|D_k\|^2 + 2\alpha_k \sqrt{\mathbb{E}[\|G_{1,k}\|^2 | \mathcal{F}_k]} \sqrt{\mathbb{E}[(1-\alpha_k G_{2,k})^2 | \mathcal{F}_k]} \|D_k\| + \alpha_k^2 C_1 \\
&\quad \le \mathbb{E}\left[(1-\alpha_k G_{2,k})^2 \middle| \mathcal{F}_k\right] \|D_k\|^2 + 2\alpha_k \sqrt{C_1} \sqrt{\mathbb{E}[(1-\alpha_k G_{2,k})^2 | \mathcal{F}_k]} \|D_k\| + \alpha_k^2 C_1. \qquad \text{(A-1)}
\end{aligned}$$

We now derive a bound for the term $\mathbb{E}[(1-\alpha_k G_{2,k})^2 | \mathcal{F}_k]$. Note that by A2, $2\mathbb{E}[G_{2,k} | \mathcal{F}_k] - \alpha_k \mathbb{E}[G_{2,k}^2 | \mathcal{F}_k] \ge 2\epsilon - \alpha_k C_2$ w.p.1. Since $\alpha_k \to 0$, there exists $N_1 > 0$ such that w.p.1,

$$2\mathbb{E}[G_{2,k} | \mathcal{F}_k] - \alpha_k \mathbb{E}[G_{2,k}^2 | \mathcal{F}_k] \ge 2\epsilon - \alpha_k C_2 > \epsilon, \quad \forall k \ge N_1.$$

Therefore, we have for all $k \ge N_1$, $\mathbb{E}[(1-\alpha_k G_{2,k})^2 | \mathcal{F}_k] = 1 - \alpha_k (2\mathbb{E}[G_{2,k} | \mathcal{F}_k] - \alpha_k \mathbb{E}[G_{2,k}^2 | \mathcal{F}_k]) \le 1 - \alpha_k \epsilon$. Substituting this bound into (A-1), we have w.p.1 that

$$\mathbb{E}\left[\|D_{k+1}\|^2 \middle| \mathcal{F}_k\right] \le (1-\alpha_k\epsilon)\|D_k\|^2 + 2\alpha_k \sqrt{C_1}\sqrt{1-\alpha_k\epsilon}\|D_k\| + \alpha_k^2 C_1.$$

Next, taking expectations at both sides yields

$$\begin{aligned}
\mathbb{E}[\|D_{k+1}\|^2] &\le (1-\alpha_k\epsilon)\mathbb{E}[\|D_k\|^2] + 2\alpha_k\sqrt{C_1}\sqrt{1-\alpha_k\epsilon}\sqrt{\mathbb{E}[\|D_k\|^2]} + \alpha_k^2 C_1 \\
&= \left[\sqrt{1-\alpha_k\epsilon}\sqrt{\mathbb{E}[\|D_k\|^2]} + \alpha_k\sqrt{C_1}\right]^2 \\
&\le \left[\left(1-\frac{\alpha_k\epsilon}{2}\right)\sqrt{\mathbb{E}[\|D_k\|^2]} + \alpha_k\sqrt{C_1}\right]^2 \\
&= \left[\left(1-\frac{\alpha_k\epsilon}{2}\right)\sqrt{\mathbb{E}[\|D_k\|^2]} + \frac{\alpha_k\epsilon}{2}\frac{2\sqrt{C_1}}{\epsilon}\right]^2
\end{aligned}$$

$$\leq \Big(\max\Big\{\sqrt{\mathbb{E}[\|D_k\|^2]}, \frac{2\sqrt{C_1}}{\epsilon}\Big\}\Big)^2,$$

where in the second inequality above we have used the fact that $\sqrt{1-x} \leq 1 - x/2$ for $x \in [0,1]$. This shows inductively that $\mathbb{E}[\|D_k\|^2] \leq (\max\{\sqrt{\mathbb{E}[\|D_{N_1}\|^2]}, \frac{2\sqrt{C_1}}{\epsilon}\})^2$ for $k \geq N_1$, and because N_1 is finite, we have $\sup_k \mathbb{E}[\|D_k\|^2] < \infty$.

To show part (ii) of the Lemma, we write (9) as

$$\begin{aligned} D_{k+1} &= (1 - \alpha_k \mathbb{E}[G_{2,k}|\mathcal{F}_k])D_k + \alpha_k \mathbb{E}[G_{1,k}|\mathcal{F}_k] + \alpha_k U_{1,k} + \alpha_k U_{2,k} \\ &= (1 - \tilde{\alpha}_k)D_k + \tilde{\alpha}_k R_k + \alpha_k U_{1,k} + \alpha_k U_{2,k}, \end{aligned} \tag{A-2}$$

where $U_{1,k} = (\mathbb{E}[G_{2,k}|\mathcal{F}_k] - G_{2,k})D_k$, $U_{2,k} = G_{1,k} - \mathbb{E}[G_{1,k}|\mathcal{F}_k]$, and we have defined $\tilde{\alpha}_k = \alpha_k \mathbb{E}[G_{2,k}|\mathcal{F}_k]$ and $R_k = \mathbb{E}[G_{1,k}|\mathcal{F}_k]/\mathbb{E}[G_{2,k}|\mathcal{F}_k]$. From A2, we have that w.p.1 $\alpha_k \epsilon \leq \tilde{\alpha}_k \leq \alpha_k \sqrt{C_2}$, and because $\alpha_k \to 0$ (by A4(a)), $0 < \tilde{\alpha}_k < 1$ for all $k \geq N_2$ for some integer $N_2 > 0$ sufficiently large. Expanding (A-2) recursively starting from the term D_{N_2} and taking norms at both sides, we have that for all $k \geq N_2$

$$\begin{aligned} \|D_{k+1}\| \leq{} & \prod_{i=N_2}^{k} (1 - \tilde{\alpha}_i)\|D_{N_2}\| + \sum_{i=N_2}^{k} \prod_{j=i+1}^{k} (1 - \tilde{\alpha}_j)\tilde{\alpha}_i \|R_i\| \\ & + \Big\| \sum_{i=N_2}^{k} \prod_{j=i+1}^{k} (1 - \tilde{\alpha}_j)\alpha_i U_{1,i} \Big\| + \Big\| \sum_{i=N_2}^{k} \prod_{j=i+1}^{k} (1 - \tilde{\alpha}_j)\alpha_i U_{2,i} \Big\|. \end{aligned} \tag{A-3}$$

From A4(a), $\prod_{i=N_2}^{k}(1 - \tilde{\alpha}_i)\|D_{N_2}\| \leq e^{-\sum_{i=N_2}^{k} \tilde{\alpha}_i}\|D_{N_2}\| \leq e^{-\epsilon \sum_{i=N_2}^{k} \alpha_i}\|D_{N_2}\| \to 0$ as $k \to \infty$. Consider the second term on the right-hand side of (A-3). We have from A2 that $\|R_i\| \leq \mathbb{E}[\|G_{1,i}\||\mathcal{F}_i]/|\mathbb{E}[G_{2,i}|\mathcal{F}_i]| \leq \sqrt{C_1}/\epsilon$ w.p.1. This, together with the fact $\sum_{i=N_2}^{k} \prod_{j=i+1}^{k}(1 - \tilde{\alpha}_j)\tilde{\alpha}_i \leq 1$ (which can be easily verified using an inductive argument), indicates that $\sum_{i=N_2}^{k} \prod_{j=i+1}^{k}(1 - \tilde{\alpha}_j)\tilde{\alpha}_i \|R_i\| \leq \sqrt{C_1}/\epsilon$ for all $k \geq N_2$ w.p.1.

Next, notice that $\sum_{i=N_2}^{k} \alpha_i U_{1,i}$ is a martingale. In addition

$$\begin{aligned} \mathbb{E}\Big[\Big\| \sum_{i=N_2}^{k} \alpha_i U_{1,i} \Big\|^2\Big] &= \sum_{i=N_2}^{k} \alpha_i^2 \mathbb{E}[(\mathbb{E}[G_{2,i}|\mathcal{F}_i] - G_{2,i})^2 \|D_i\|^2] \\ &= \sum_{i=N_2}^{k} \alpha_i^2 \mathbb{E}\Big[\mathbb{E}\big[(\mathbb{E}[G_{2,i}|\mathcal{F}_i] - G_{2,i})^2|\mathcal{F}_i\big]\|D_i\|^2\Big] \\ &\leq \sum_{i=N_2}^{k} \alpha_i^2 \mathbb{E}\big[\mathbb{E}[G_{2,i}^2|\mathcal{F}_i]\|D_i\|^2\big] \\ &\leq C_2 \sum_{i=N_2}^{k} \alpha_i^2 \mathbb{E}[\|D_i\|^2] \\ &< \infty, \end{aligned}$$

where the second inequality follows from A2(b) and the last step is due to A4(a) and part (i) of the lemma. Consequently, the martingale is bounded in L^2 and converges to a finite random vector. Let $a_i := \big(\prod_{j=N_2}^{i}(1-\tilde{\alpha}_j)\big)^{-1}$. It follows from A4(a) that $0 < a_i \le a_{i+1}$ for all $i \ge N_2$ and $a_k \to \infty$ as $k \to \infty$. Therefore, regarding the third term on the right-hand side of (A-3), we have

$$\sum_{i=N_2}^{k}\prod_{j=i+1}^{k}(1-\tilde{\alpha}_j)\alpha_i U_{1,i} = \prod_{j=N_2}^{k}(1-\tilde{\alpha}_j)\sum_{i=N_2}^{k}\frac{1}{\prod_{j=N_2}^{i}(1-\tilde{\alpha}_j)}\alpha_i U_{1,i} = \frac{1}{a_k}\sum_{i=N_2}^{k} a_i\alpha_i U_{1,i},$$

which tends to zero as $k \to \infty$ by applying Kronecker's lemma (e.g., Shiryaev (1996)). The same argument can be used to show that the last term on the right-hand side of (A-3) also tends to zero as $k \to \infty$ w.p.1. Hence, we conclude that all four terms on the right-hand side of (A-3) are bounded. This completes the proof of part (ii) of the lemma. □

C. Proofs of Lemmas in Section 4.1

C.1. Proof of Lemma 2

Without loss of generality, we consider the interval $[0, T]$ for a fixed $T > 0$. Let $t \in [0, T]$ be given. Suppose there exists an integer d satisfying $t = t_{n+d} - t_n$, then it is easy to observe that the sum $J_\beta^n(t)$ coincides with the integral $\int_0^t J(\theta_\beta^n(s), q_\beta^n(s))\, ds$, and we have $|\rho_\beta^n(t)| = 0$. Otherwise, we can find a unique integer d such that $t_{n+d} < t_n + t < t_{n+d+1}$. Thus, for almost all sample paths generated by the algorithm, we have

$$\begin{aligned}
|\rho_\beta^n(t)| &= \Big|J_\beta^n(t) - \int_0^t J(\theta_\beta^n(s), q_\beta^n(s))\, ds\Big| \\
&= \Big|\int_{t_{n+d}}^{t} J(\theta_\beta^n(s), q_\beta^n(s))\, ds\Big| \\
&\le \int_{t_{n+d}}^{t} \big|J(\theta_\beta^n(s), q_\beta^n(s))\big|\, ds \\
&\le \beta_{n+d},
\end{aligned}$$

where the last step holds because $|J|$ is bounded by one and $t - t_{n+d} < \beta_{n+d}$. Because this holds for all $t \in [0, T]$, we have $\sup_{t\in[0,T]} |\rho_\beta^n(t)| \to 0$ w.p.1 as $n \to \infty$. □

C.2. Proof of Lemma 3

Define $M_n = \sum_{i=0}^{n-1}\beta_i\delta_i$. It is clear that $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$. Thus, by martingale inequality, for any given $\varepsilon > 0$,

$$P\big(\sup_{n\le j\le m}|M_j - M_n| \ge \varepsilon\big) \le \frac{\mathbb{E}|\sum_{i=n}^{m-1}\beta_i\delta_i|^2}{\epsilon^2} = \frac{\sum_{i=n}^{m-1}\beta_i^2\mathbb{E}[\delta_i^2]}{\epsilon^2},$$

where the equality holds because the cross terms $\mathbb{E}[\delta_i\delta_j] = \mathbb{E}[\mathbb{E}[\delta_j|\mathcal{F}_j]\delta_i] = 0$ for all $i < j$. Using the fact that $|\delta_i| \le 1$, it can be seen that the last term above is bounded by $\sum_{i=n}^{\infty}\beta_i^2/\varepsilon^2$. Thus, by A4(b), we have

$$\lim_{n\to\infty} P\big(\sup_{n\le j}|M_j - M_n| \ge \varepsilon\big) = 0. \tag{A-4}$$

Now, given $T > 0$ and any $t \in [0, T]$, we have

$$|\Delta_\beta^n(t)| = \Big|\sum_{i=n}^{m(t_n+t)-1} \beta_i \delta_i\Big| = |M_{m(t_n+t)} - M_n| \\ \le \sup_{n \le j} |M_j - M_n|,$$

which tends to 0 w.p.1 as $n \to \infty$ as a result of (A-4). $\square$

C.3. Proof of Lemma 4

Because $H_1 < \min_\theta q(\theta;\alpha) \le \max_\theta q(\theta;\alpha) < H_2$, there exist constants $0 < \delta < \min\{\alpha, 1-\alpha\}$ and $\varepsilon > 0$ such that $H_1 < \min_\theta q(\theta;\alpha-\delta) - 3\varepsilon \le \max_\theta q(\theta;\alpha+\delta) + 3\varepsilon < H_2$. Let $\mathcal{B}_1 := \min_\theta q(\theta;\alpha-\delta)$ and $\mathcal{B}_2 := \max_\theta q(\theta;\alpha+\delta)$ and define intervals $\mathcal{A}_i = [\mathcal{B}_1 - i\varepsilon, \mathcal{B}_2 + i\varepsilon]$, $i = 1, 2, 3$.

Let Ω_1 be the set of sample paths on which Lemma 3 holds. Clearly, $P(\Omega_1) = 1$. For a fixed $\omega \in \Omega_1$, let $N(\omega) > 0$ be a sufficiently large integer such that $\sup_{m \ge k} |\sum_{j=k}^{m-1} \beta_j \delta_j| \le \varepsilon$ and $\beta_k \le \varepsilon$ for all $k \ge N(\omega)$. Suppose there is a time $n \ge N(\omega)$ such that $q_n \notin \mathcal{A}_2$. We expand (15) starting from term q_n as

$$q_{k+1} = q_n + \sum_{i=n}^{k} \beta_i J(\theta_i, q_i) + \sum_{i=n}^{k} \beta_i \delta_i + \sum_{i=n}^{k} \beta_i E_i$$

and consider two cases: (i) $q_n \in [H_1, \mathcal{B}_1 - 2\varepsilon)$. Note that for every q_i, $i \ge n+1$ that stays in this interval, we have $J(\theta_i, q_i) = \alpha - P(Y \le q_i|\mathcal{F}_i) \ge \alpha - P(Y \le \mathcal{B}_1) \ge \alpha - (\alpha - \delta) = \delta$ and $E_{i-1} \ge 0$. Consequently,

$$q_{k+1} \ge H_1 + \delta \sum_{i=n}^{k} \beta_i - \Big|\sum_{i=n}^{k} \beta_i \delta_i\Big| \ge H_1 + \delta \sum_{i=n}^{k} \beta_i - \varepsilon.$$

(ii) When $q_n \in (\mathcal{B}_2 + 2\varepsilon, H_2]$, we have $J(\theta_i, q_i) \le -\delta$ and $E_{i-1} \le 0$ for every q_i, $i \ge n+1$ in the interval. Thus,

$$q_{k+1} \le H_2 - \delta \sum_{i=n}^{k} \beta_i + \Big|\sum_{i=n}^{k} \beta_i \delta_i\Big| \le H_2 - \delta \sum_{i=n}^{k} \beta_i + \varepsilon.$$

Since $\sum_{j=n}^{\infty} \beta_j = \infty$ by A4(b), the term $\delta \sum_{i=n}^{k} \beta_i$ can be made arbitrarily large by increasing k. Thus, in either case (i) or (ii) the sequence $\{q_k\}$ will enter $\mathcal{A}_2$ in finite time, i.e., there must be an $n' \ge N(\omega)$ such that $q_{n'} \in \mathcal{A}_2$.

Now for any $k \ge n' \ge N(\omega)$, we have

$$q_{k+1} = q_{n'} + \sum_{i=n'}^{k} \beta_i J(\theta_i, q_i) + \sum_{i=n'}^{k} \beta_i \delta_i + \sum_{i=n'}^{k} \beta_i E_i.$$

Consider the recursion $\tilde{q}_{k+1} = \tilde{q}_k + \beta_k J(\theta_k, q_k)$ for $k \geq n'$ with $\tilde{q}_{n'} = q_{n'}$. By applying a simple inductive argument, it can be verified that

$$q_{k+1} = \tilde{q}_{k+1} + \sum_{i=n'}^{k} \beta_i \delta_i + \sum_{i=n'}^{k} \beta_i E_i. \tag{A-5}$$

Next, we show that $\mathcal{A}_2$ is invariant with respect to $\{\tilde{q}_k\}$ for all $k \geq n'$. Because $|\tilde{q}_{k+1} - \tilde{q}_k| \leq \beta_k \leq \varepsilon$, each increment in $\tilde{q}_k$ is bounded in magnitude by ε. Therefore, in order for the sequence to exit $\mathcal{A}_2$, there must be a time $k' \geq n' \geq N(\omega)$ such that $\tilde{q}_{k'} \in \mathcal{A}_2 \backslash \mathcal{A}_1$ and $\tilde{q}_{k'+1} \notin \mathcal{A}_2$. Also note that since $\{\tilde{q}_k, k = n', \ldots, k'\} \subseteq \mathcal{A}_2 \subset \mathcal{A}_3 \subset \mathcal{H}$ and $\sup_{k \geq n'} |\sum_{i=n'}^{k} \beta_i \delta_i| \leq \varepsilon$, it follows inductively from (15) and (A-5) that $E_k = 0$ and $|\tilde{q}_k - q_k| \leq \varepsilon$ for all $k = n', \ldots, k'$. We have the following two cases:

(i) Suppose $\mathcal{B}_1 - 2\varepsilon \leq \tilde{q}_{k'} < \mathcal{B}_1 - \varepsilon$. Then using the fact that $|\tilde{q}_{k'} - q_{k'}| \leq \varepsilon$, we get $\mathcal{B}_1 - 3\varepsilon \leq q_{k'} < \mathcal{B}_1$. It thus follows from the definition of $\tilde{q}_{k+1}$ that

$$\tilde{q}_{k'+1} = \tilde{q}_{k'} + \beta_{k'} J(\theta_{k'}, q_{k'}) \geq \tilde{q}_{k'} + \beta_{k'} \delta > \tilde{q}_{k'} \geq \mathcal{B}_1 - 2\varepsilon,$$
$$\tilde{q}_{k'+1} = \tilde{q}_{k'} + \beta_{k'} J(\theta_{k'}, q_{k'}) \leq \tilde{q}_{k'} + \beta_{k'} \leq \tilde{q}_{k'} + \varepsilon < \mathcal{B}_1.$$

(ii) On the other hand, suppose $\mathcal{B}_2 + \varepsilon < \tilde{q}_{k'} \leq \mathcal{B}_2 + 2\varepsilon$, then $\mathcal{B}_2 < q_{k'} \leq \mathcal{B}_2 + 3\varepsilon$. This suggests that

$$\tilde{q}_{k'+1} = \tilde{q}_{k'} + \beta_{k'} J(\theta_{k'}, q_{k'}) \leq \tilde{q}_{k'} - \beta_{k'} \delta < \tilde{q}_{k'} \leq \mathcal{B}_2 + 2\varepsilon,$$
$$\tilde{q}_{k'+1} = \tilde{q}_{k'} + \beta_{k'} J(\theta_{k'}, q_{k'}) \geq \tilde{q}_{k'} - \beta_{k'} \geq \tilde{q}_{k'} - \varepsilon > \mathcal{B}_2.$$

In either case, we obtain that $\tilde{q}_{k'+1} \in \mathcal{A}_2$, which contradicts the hypothesized relationship that $\tilde{q}_{k'+1} \notin \mathcal{A}_2$. This shows that $\tilde{q}_k \in \mathcal{A}_2$ for all $k \geq n'$.

Finally, because every q_k lies within a distance of ε from $\tilde{q}_k$ whenever $\tilde{q}_k \in \mathcal{A}_2$, the sequence $\{q_k,\ k \geq n'\}$ must remain in $\mathcal{A}_3 \subset \mathcal{H}$. This implies that the projection term E_k in (15) is identically zero for all $k \geq n'$. Hence, we conclude that along each $\omega \in \Omega_1$ $\sup_{t \in \mathcal{I}} |E_\beta^n(t)| = \sup_{t \in \mathcal{I}} \left| \sum_{i=n}^{m(t_n+t)-1} \beta_i E_i \right| = 0$ when n becomes sufficiently large. $\square$

C.4. Proof of Lemma 5

For a given $T > 0$, let $t \in [0, T]$. From (14), the projection term satisfies $-Z_k^T(\theta - \theta_{k+1}) \leq 0$ for all $\theta \in \Theta$, which suggests that $-Z_k^T(\theta_k - \theta_{k+1}) \leq 0$. We thus obtain from (13) that $0 \geq -Z_k^T(\theta_k - \theta_{k+1}) = -\gamma_k Z_k^T(D_k - Z_k) = -\gamma_k Z_k^T D_k + \gamma_k \|Z_k\|^2$. Rearranging terms and applying the CauchyCSchwarz inequality $Z_k^T D_k \leq \|Z_k\| \|D_k\|$, we get $\|Z_k\| \leq \|D_k\|$. This leads to the following bound on $\|\phi_\beta^n(t)\|$:

$$\begin{aligned} \|\phi_\beta^n(t)\| = \left\| \sum_{i=n}^{m(t_n+t)-1} \beta_i \tilde{D}_i \right\| &\leq T \sup_{i \geq n} \|\tilde{D}_i\| \\ &= T \sup_{i \geq n} \frac{\gamma_i}{\beta_i} \|Z_i - D_i\| \\ &\leq 2T \sup_{i \geq n} \frac{\gamma_i}{\beta_i} \|D_i\|. \end{aligned}$$

From part (ii) of Lemma 1, we know that on almost every sample path ω generated by the algorithm, there exists a bound $B_D(\omega) < \infty$ such that $\|D_i(\omega)\| \leq B_D(\omega)$ for all i. This, when combined with the fact that $\gamma_i/\beta_i \to 0$ (Assumption A5), implies that $\sup_{t\in[0,T]} \|\phi_\beta^n(t)\| \to 0$ as $n \to \infty$ w.p.1. □

D. Proofs of Lemmas in Section 4.2

D.1. Proof of Lemma 6

Given $T > 0$ and consider an arbitrary time $\tau \in [0, T]$. It is easy to observe that if there exists an integer d such that $\tau = \tau_{n+d} - \tau_n$, then $\|\rho_\alpha^n(\tau)\| = 0$ and the result holds trivially. Otherwise, let d be the integer such that $\tau_{n+d} < \tau_n + \tau < \tau_{n+d+1}$. Note that by the definition of H, we have $\|H(D_k, \theta_k, q_k)\| \leq \|\nabla_\theta F(q_k; \theta)|_{\theta=\theta_k}\| + f(q_k; \theta_k)\|D_k\|$. Conditions A2(b) and A3 suggest that there are positive constants $\bar{C}_1,\ \bar{C}_2 < \infty$ such that $\|H(D_k, \theta_k, q_k)\| \leq \bar{C}_1 + \bar{C}_2\|D_k\|$ w.p.1. In addition, part (ii) of Lemma 1 implies that on almost every sample path ω generated by the algorithm, there exists a finite $B_D(\omega)$ such that $\sup_k \|D_k(\omega)\| \leq B_D(\omega)$. Thus, by further noting that $D_\alpha^n(\cdot)$, $\theta_\alpha^n(\cdot)$, and $q_\alpha^n(\cdot)$ are piece-wise constant interpolations of D_k, θ_k, and q_k, we have that along the same path ω,

$$\begin{aligned}\|\rho_\alpha^n(\tau)\| &= \left\| H_\alpha^n(\tau) - \int_0^\tau H(D_\alpha^n(s), \theta_\alpha^n(s), q_\alpha^n(s))\ ds \right\| \\ &= \left\| \int_{\tau_{n+d}}^\tau H(D_\alpha^n(s), \theta_\alpha^n(s), q_\alpha^n(s))\ ds \right\| < \alpha_{n+d}\big(\bar{C}_1 + \bar{C}_2 B_D(\omega)\big).\end{aligned}$$

Because this holds for almost all trajectories generated by the algorithm, we conclude that $\sup_{\tau\in[0,T]} \|\rho_\alpha^n(\tau)\| \to 0$ w.p.1 as $n \to \infty$. □

D.2. Proof of Lemma 7

Let $\tau \in [0, T]$ for a given $T > 0$. From A3, for any $\varepsilon > 0$, there exists a path-dependent integer $N_3 > 0$ such that $\|b_{1,i}\| \leq \varepsilon$ for all $i \geq N_3$ w.p.1. Therefore, along (almost) every path, when $n \geq N_3$, we have

$$\|B_{1,\alpha}^n(\tau)\| = \left\| \sum_{i=n}^{m(\tau_n+\tau)-1} \alpha_i b_{1,i} \right\| \leq \sum_{i=n}^{m(\tau_n+\tau)-1} \alpha_i \|b_{1,i}\| \leq T\varepsilon.$$

Because ε is arbitrary and the bound holds for all $\tau \in [0, T]$, it follows that $\sup_{\tau\in[0,T]} \|B_{1,\alpha}^n(\tau)\| \to 0$ as $n \to \infty$. Similarly, using A3 and part (ii) of Lemma 1, it is easy to show that $\sup_{\tau\in[0,T]} \|B_{2,\alpha}^n(\tau)\| \to 0$ as $n \to \infty$ w.p.1. □

D.3. Proof of Lemma 9

Define $\mathscr{M}_n = \sum_{i=0}^{n-1} \alpha_i w_i$. It is clear that $\mathbb{E}[\mathscr{M}_n|\mathscr{F}_{n-1}] = \mathscr{M}_{n-1}$ and that $\mathbb{E}[\|\mathscr{M}_n\|^2] = \sum_{i=0}^{n-1} \alpha_i^2 \mathbb{E}[\|w_i\|^2]$. Note that from A2 and the definition of w_k,

$$\mathbb{E}[\|w_i\|^2] = \mathbb{E}\Big[\big(\mathbb{E}[G_2(X; \theta_i, q_i)|\mathcal{F}_i] - G_2(X; \theta_i, q_i)\big)^2 \|D_i\|^2\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\mathbb{E}\big[\big(\mathbb{E}[G_2(X;\theta_i,q_i)|\mathcal{F}_i] - G_2(X;\theta_i,q_i)\big)^2\big|\mathcal{F}_i\big]\,\|D_i\|^2\Big] \\
&\le \mathbb{E}\Big[\mathbb{E}\big[G_2^2(X;\theta_i,q_i)|\mathcal{F}_i\big]\,\|D_i\|^2\Big] \\
&\le C_2\mathbb{E}[\|D_i\|^2] \\
&< \infty,
\end{aligned}
$$

where the finiteness of the last term follows from part (i) of Lemma 1. The rest of the proof is identical to that of Lemma 3 and is thus omitted. □

D.4. Proof of Lemma 10

For a given $T > 0$, let $\tau \in [0,T]$. Using the same argument as in the proof of Lemma 5, it is easy to show that $|E_k| \le |\alpha - I\{Y \le q_k\}| \le 1$. Thus, $|\check{I}_i| \le 2\frac{\beta_i}{\alpha_i}$ for all i. Hence, we have w.p.1 that

$$
\begin{aligned}
|\psi_\alpha^n(\tau)| = \Big|\sum_{i=n}^{m(\tau_n+\tau)-1} \alpha_i \check{I}_i\Big| &\le T\sup_{i\ge n}|\check{I}_i| \\
&\le 2T\sup_{i\ge n}\frac{\beta_i}{\alpha_i},
\end{aligned}
$$

which tends to 0 as $n \to \infty$ by A5. □

E. A Toy Example in Section 5.1

E.1. Unbiasedness of GLR Estimators in Section 5.1

Proposition 1 *The GLR estimators in Section 5.1.1 satisfy conditions (A.1)-(A.3) of Section A.*

Proof. The input density $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is differentiable and $h(x;\theta) = (2\theta - \theta^2)\,x$ is twice differentiable on $\Re \times \Theta$, where $\frac{\partial^2 h(x;\theta)}{\partial x^2} = 0$ and $\Theta = (0,2)$. Note that h is a linear function of x and $2\theta - \theta^2 \ne 0$ when $\theta \in \Theta$, so h is invertible and $h^{-1}(y;\theta) = \frac{y}{2\theta-\theta^2}$.

Now we turn to the Lipschitz condition. Since

$$
\left|\left(\frac{\partial h(x;\theta)}{\partial x}\right)^{-1}\right| = \left|\frac{1}{2\theta-\theta^2}\right| \ge 1, \quad \left|\left(\frac{\partial h(x;\theta)}{\partial \theta}\right)^{-1}\right| = \left|\frac{1}{(2-2\theta)\,x}\right| > \left|\frac{1}{2x}\right|,
$$

then we have h^{-1} is globally Lipschitz continuous with respect to y and θ so the uniform convergence condition (A.2) holds for $\forall\theta \in \Theta$ and $\forall y \in \Re$.

Since $\mathbb{E}\left[\left|\frac{\partial^2 h(x;\theta)}{\partial x^2}\right|\right] = 0 < \infty$ and $\lim_{x\to\pm\infty} f_X(x) = 0$, $\sup_x f_X(x) = \frac{1}{\sqrt{2\pi}} < \infty$, the integrability condition (A.3) holds. □

Proposition 2 *The GLR estimators in Section 5.1.2 satisfy conditions (A.1)-(A.3).*

Proof. With the transformation $X' = \ln(X-2)$, the input density $f'_X(x') = \frac{2e^{x'}}{(e^{x'}+2)^2}$ is differentiable and $h(x';\theta) = (2\theta-\theta^2)\,x'$ is twice differentiable on $\Re \times \Theta$, where $\frac{\partial^2 h(x';\theta)}{\partial x'^2} = 0$ and $\Theta = (0,2)$.

Note that h is a linear function of x' and $2\theta-\theta^2\neq 0$ when $\theta\in\Theta$, so h is invertible and $h^{-1}(y;\theta)=\frac{y}{2\theta-\theta^2}$.

Now we turn to the Lipschitz condition. Since

$$\left|\left(\frac{\partial h(x';\theta)}{\partial x'}\right)^{-1}\right|=\left|\frac{1}{2\theta-\theta^2}\right|\geq 1,\quad \left|\left(\frac{\partial h(x';\theta)}{\partial \theta}\right)^{-1}\right|=\left|\frac{1}{(2-2\theta)x'}\right|>\left|\frac{1}{2x'}\right|,$$

then we have h^{-1} is globally Lipschitz continuous with respect to y and θ so the uniform convergence condition (A.2) holds for $\forall\theta\in\Theta$ and $\forall y\in\Re$.

Since $\mathbb{E}\left[\left|\frac{\partial^2 h(x';\theta)}{\partial x'^2}\right|\right]=0<\infty$ and $\lim_{x'\to\pm\infty} f_{X'}(x')=0$, $\sup_{x'} f_{X'}(x')=\frac{1}{4}<\infty$, the integrability condition (A.3) holds. $\square$

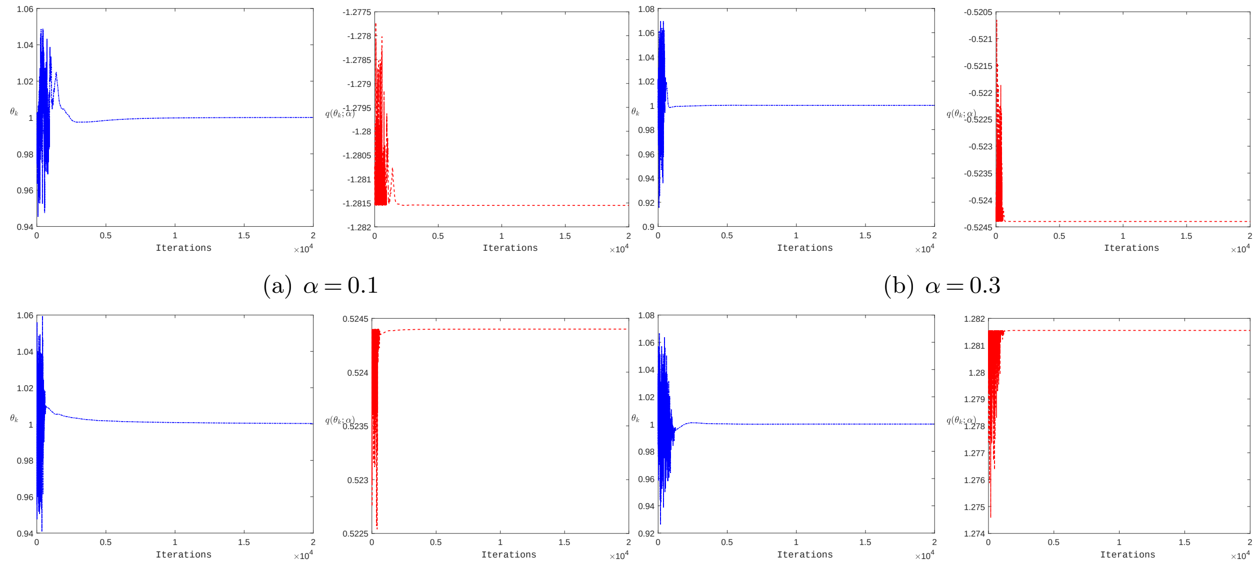
Proposition 3 *The GLR estimators in Section 5.1.2 have bounded variances.*

Proof. With $x_m>0$, $\alpha'>0$ and $f_{X'}(x')=\frac{\alpha' x_m^{\alpha'} e^{x'}}{(e^{x'}+x_m)^{\alpha'+1}}$, $x'\in\Re$, we have,

$$\begin{aligned}
\mathbb{E}[\|G_1(X';\theta_k,q_k)\|^2|\mathcal{F}_k] &= \left(\frac{2-2\theta_k}{\theta_k(2-\theta_k)}\right)^2\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left(1+x'-\frac{(\alpha'+1)x'e^{x'}}{e^{x'}+x_m}\right)^2\frac{\alpha' x_m^{\alpha'}e^{x'}}{(e^{x'}+x_m)^{\alpha'+1}}dx'\\
&=\left(\frac{2-2\theta_k}{\theta_k(2-\theta_k)}\right)^2\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left(1+\frac{x_m-\alpha'e^{x'}}{x_m+e^{x'}}x'\right)^2\frac{\alpha' x_m^{\alpha'}e^{x'}}{(e^{x'}+x_m)^{\alpha'+1}}dx'\\
&<\left(\frac{2-2\theta_k}{\theta_k(2-\theta_k)}\right)^2\alpha' x_m^{-1}\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left[1+\left(x_m-\alpha'e^{x'}\right)^2x'^2x_m^{-2}+2\left(x_m-\alpha'e^{x'}\right)x'x_m^{-1}\right]e^{x'}dx'\\
&<\left(\frac{2-2\theta_k}{\theta_k(2-\theta_k)}\right)^2\alpha' x_m^{-1}\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left[(x'+1)^2+\alpha'^2e^{2x'}x'^2x_m^{-2}\right]e^{x'}dx'\\
&=\left(\frac{2-2\theta_k}{\theta_k(2-\theta_k)}\right)^2\alpha' x_m^{-1}\left(\frac{\alpha'^2x_m^{-2}}{27}e^{\frac{3q_k}{2\theta_k-\theta_k^2}}\left(\frac{9q_k^2}{(2\theta_k-\theta_k^2)^2}-\frac{6q_k}{2\theta_k-\theta_k^2}+2\right)+e^{\frac{q_k}{2\theta_k-\theta_k^2}}\left(\frac{q_k^2}{(2\theta_k-\theta_k^2)^2}+1\right)\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[G_2^2(X';\theta_k,q_k)|\mathcal{F}_k] &= \frac{1}{\theta_k^2(2-\theta_k)^2}\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left(1-\frac{(\alpha'+1)e^{x'}}{e^{x'}+x_m}\right)^2\frac{\alpha' x_m^{\alpha'}e^{x'}}{(e^{x'}+x_m)^{\alpha'+1}}dx'\\
&<\frac{1}{\theta_k^2(2-\theta_k)^2}\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}\left[1+\frac{(\alpha'+1)^2e^{2x'}}{(e^{x'}+x_m)^2}\right]\frac{\alpha' x_m^{\alpha'}e^{x'}}{(e^{x'}+x_m)^{\alpha'+1}}dx'\\
&<\frac{1}{\theta_k^2(2-\theta_k)^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]\int_{-\infty}^{\frac{q_k}{2\theta_k-\theta_k^2}}e^{x'}dx'\\
&=\frac{1}{\theta_k^2(2-\theta_k)^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]e^{\frac{q_k}{2\theta_k-\theta_k^2}}.
\end{aligned}$$

Since both $\frac{2-2\theta_k}{\theta_k(2-\theta_k)}$ and $\frac{1}{\theta_k(2-\theta_k)}$ are continuous functions on the closed interval $\Theta=[0.01,1.99]$, we have $\frac{2-2\theta_k}{\theta_k(2-\theta_k)}$ and $\frac{1}{\theta_k(2-\theta_k)}$ are bounded. With $q_k\in\mathcal{H}=[-10^{10},10^{10}]$ and $\frac{1}{2\theta_k-\theta_k^2}$ is continuous on Θ, we have $\frac{q_k}{2\theta_k-\theta_k^2}$ is bounded. Therefore, there exists $C_1>0$ and $C_2>0$ such that $\sup_k\mathbb{E}[\|G_1(X';\theta_k,q_k)\|^2|\mathcal{F}_k]\leq C_1$ and $\sup_k\mathbb{E}[G_2^2(X';\theta_k,q_k)|\mathcal{F}_k]\leq C_2$ w.p.1, which completes the proof. $\square$



(a) $\alpha = 0.1$ (b) $\alpha = 0.3$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

Figure 1 **Performance of SQO on the example of Section 5.1.1: (a)** $\alpha = 0.1$**; (b)** $\alpha = 0.3$**; (c)** $\alpha = 0.7$**; (d)** $\alpha = 0.9$**.**

E.2. SQO with Different Quantile Levels in Section 5.1.1

To illustrate the effect of the quantile level α on the performance of SQO in Section 5.1.1, Figure 1 shows the performance of SQO (averaged over 1000 independent runs) under four different values of α: $0.1, 0.3, 0.7, 0.9$. All algorithm parameters are taken to be the same as described in Section 5.1.1. The optimal quantiles obtained by inverting the cumulative density function of $Y(\theta)$ are $q(\theta^*; 0.1) = -1.2816$, $q(\theta^*; 0.3) = -0.5244$, $q(\theta^*; 0.7) = 0.5244$ and $q(\theta^*; 0.9) = 1.2816$, respectively. Numerical results indicate the robust performance of SQO with respect to the choice of different quantile levels.

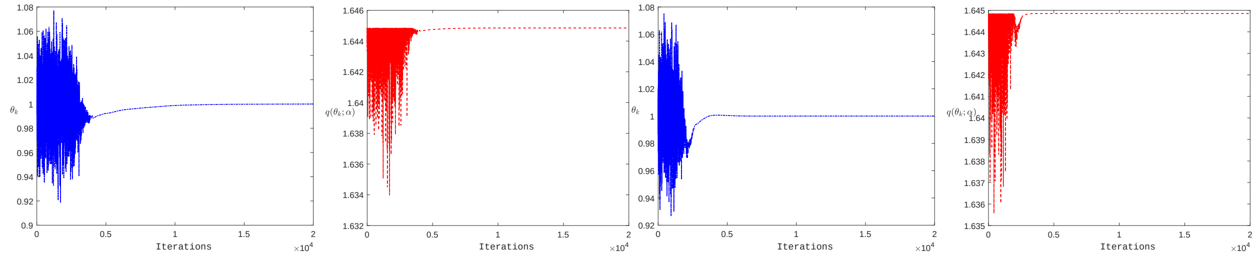
In our repeated simulations, both SQO and QG are able to achieve average relative errors of under 3.8×10^{-4}, 2.6×10^{-2}, 1.7×10^{-2} and 6.7×10^{-4}, respectively, within an average CPU time of 0.1 second. SG yields an average relative error of 1.0×10^{-1}, 1.5×10^{-1}, 1.8×10^{-1} and 9.2×10^{-2}, respectively, which are larger than those of the other two algorithms.

E.3. SQO with Different Step Sizes in Section 5.1.1

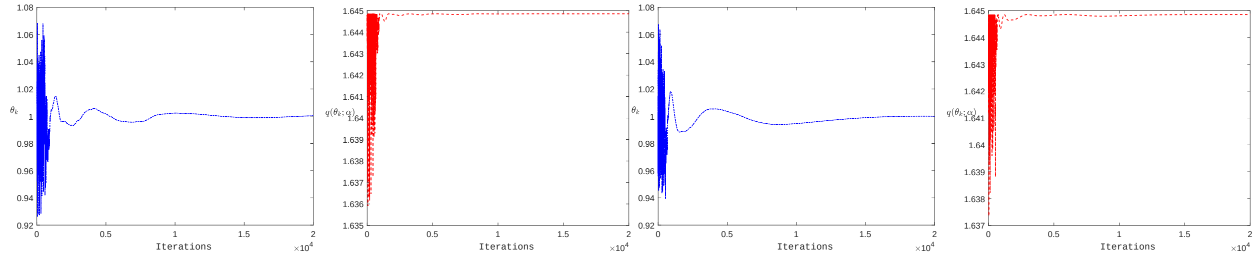
To illustrate the effect of different step sizes on the performance of SQO, tests are performed on the example in Section 5.1.1 (with $\alpha = 0.95$) using four different sets of step size sequences. Figure 2 shows the performance of SQO (averaged over 1000 independent runs) in each of the respective test cases.

E.4. SQO with Different Quantile Levels in Section 5.1.2

To illustrate the effect of parameter α on the performance of SQO algorithm in Section 5.1.2, we report in Figure 3 the performance of SQO (averaged over 1000 independent runs) under



(a) $\alpha_k = 1/k^{0.501}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k$ (b) $\alpha_k = 1/k^{0.55}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k^{0.9}$



(c) $\alpha_k = 1/k^{0.65}$, $\beta_k = 1/k^{0.7}$, $\gamma_k = 1/k^{0.8}$ (d) $\alpha_k = 1/k^{0.7}$, $\beta_k = 1/k^{0.8}$, $\gamma_k = 1/k^{0.9}$

Figure 2 **Performance of SQO on the example of Section 5.1.1 under different step sizes: (a)** $\alpha_k = 1/k^{0.501}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k$; **(b)** $\alpha_k = 1/k^{0.55}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k^{0.9}$; **(c)** $\alpha_k = 1/k^{0.65}$, $\beta_k = 1/k^{0.7}$, $\gamma_k = 1/k^{0.8}$; **(d)** $\alpha_k = 1/k^{0.7}$, $\beta_k = 1/k^{0.8}$, $\gamma_k = 1/k^{0.9}$.

four different α values: $0.05, 0.3, 0.7, 0.9$. The optimal quantiles obtained by inverting the cumulative density function of $Y(\theta)$ are $q(\theta^*; 0.05) = 2.1053$, $q(\theta^*; 0.3) = 2.8571$, $q(\theta^*; 0.7) = 6.6667$ and $q(\theta^*; 0.9) = 20$, respectively.

Both SQO and QG are able to achieve average relative errors of under 8.3×10^{-3}, 1.8×10^{-3}, 5.1×10^{-5} and 7.7×10^{-4}, respectively, in less than 0.15 second. Compared to SQO and QG, SG yields higher average relative errors of 4.3×10^{-2}, 4.4×10^{-1}, 6.0×10^{-2} and 5.0×10^{-2}, respectively.

E.5. SQO with Different Step Sizes in Section 5.1.2

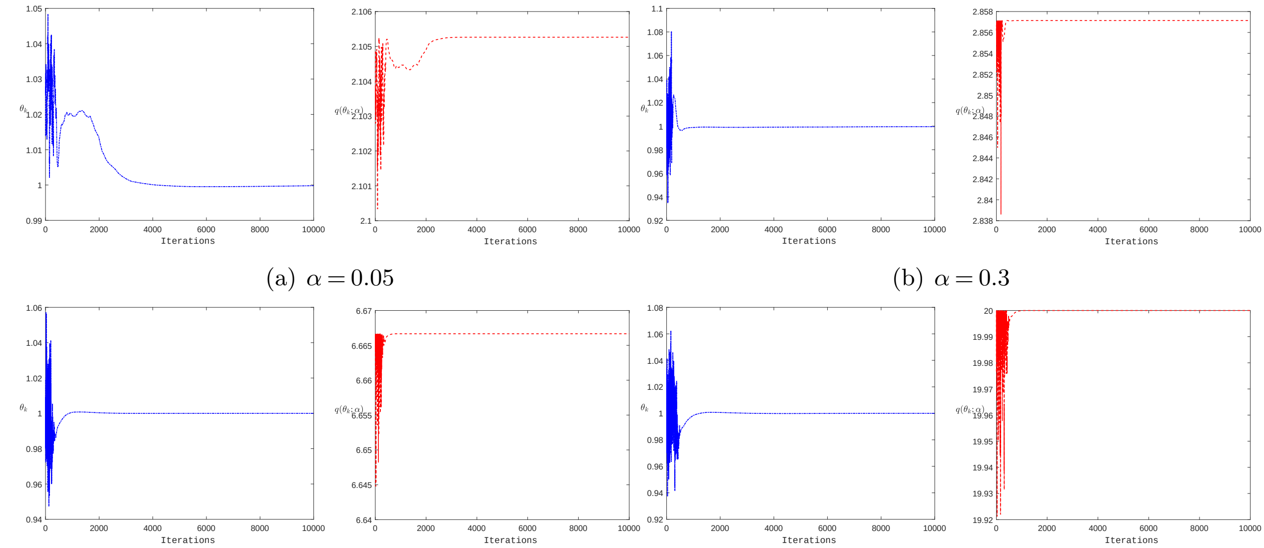
To illustrate the effect of different step sizes on the performance of SQO in Section 5.1.2, Figure 4 reports the performance of SQO (averaged over 1000 independent runs) under different choices of the step size sequences with $\alpha = 0.95$.

F. Newsvendor Problem

F.1. Unbiasedness of GLR Estimators in Section 5.2

Proposition 4 *The GLR estimators in Section 5.2 satisfy conditions (A.1)-(A.3).*

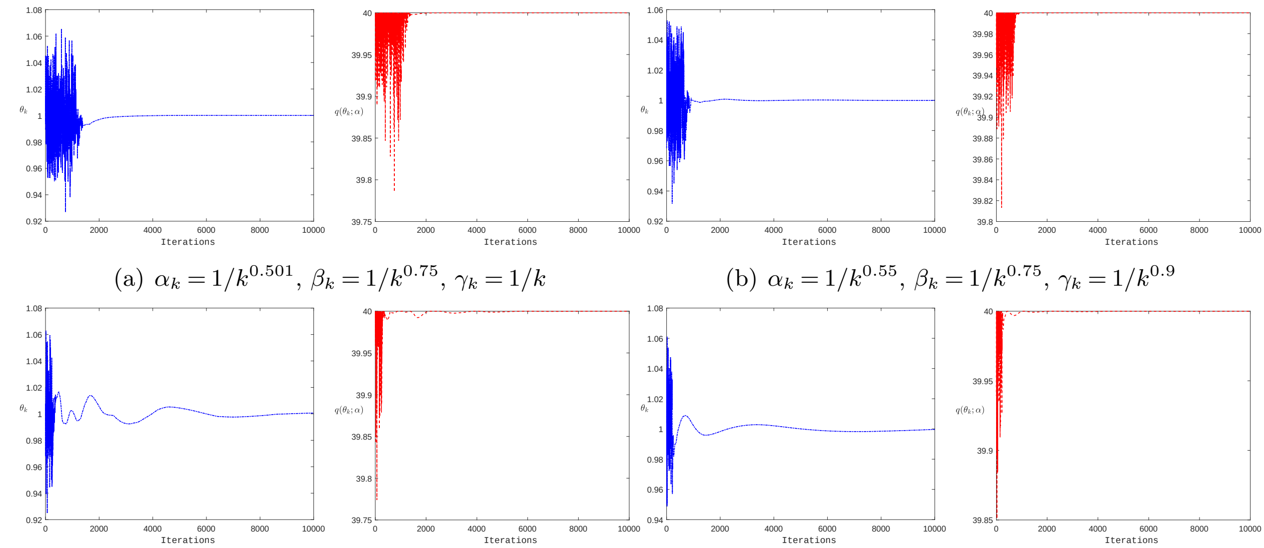
Proof. With the transformation $X' = \ln(X - 1)$, the input density $f'_X(x') = \frac{e^{x'}}{(e^{x'}+1)^2}$ is differentiable and $h(x'; \theta) = px' - c\theta$ is twice differentiable on $\Re \times \Theta$, where $\frac{\partial^2 h(x';\theta)}{\partial x'^2} = 0$ and $\Theta = [20, 80]$. Note that h is a linear function of x' and $p \neq 0$, so h is invertible and $h^{-1}(y; \theta) = \frac{y + c\theta}{p}$.



(a) $\alpha = 0.05$ (b) $\alpha = 0.3$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

Figure 3 **Trajectories of SQO with (a)** $\alpha = 0.05$**; (b)** $\alpha = 0.3$**; (c)** $\alpha = 0.7$**; (d)** $\alpha = 0.9$ **on the example of Section ??.**



(a) $\alpha_k = 1/k^{0.501}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k$ (b) $\alpha_k = 1/k^{0.55}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k^{0.9}$

(c) $\alpha_k = 1/k^{0.65}$, $\beta_k = 1/k^{0.7}$, $\gamma_k = 1/k^{0.8}$ (d) $\alpha_k = 1/k^{0.7}$, $\beta_k = 1/k^{0.8}$, $\gamma_k = 1/k^{0.9}$

Figure 4 **Performance of SQO on the example of Section 5.1.2 with different step sizes (a)** $\alpha_k = 1/k^{0.501}$**,** $\beta_k = 1/k^{0.75}$**,** $\gamma_k = 1/k$**; (b)** $\alpha_k = 1/k^{0.55}$**,** $\beta_k = 1/k^{0.75}$**,** $\gamma_k = 1/k^{0.9}$**; (c)** $\alpha_k = 1/k^{0.65}$**,** $\beta_k = 1/k^{0.7}$**,** $\gamma_k = 1/k^{0.8}$**; (d)** $\alpha_k = 1/k^{0.7}$**,** $\beta_k = 1/k^{0.8}$**,** $\gamma_k = 1/k^{0.9}$**.**

Now we turn to the Lipschitz condition. Since

$$\left| \left(\frac{\partial h\,(x';\theta)}{\partial x'} \right)^{-1} \right| = \left| \frac{1}{p} \right| > 0, \quad \left| \left(\frac{\partial h\,(x';\theta)}{\partial \theta} \right)^{-1} \right| = \left| \frac{1}{c} \right| > 0,$$

then we have h^{-1} is globally Lipschitz continuous with respect to y and θ so the uniform convergence condition (A.2) holds for $\forall \theta \in \Theta$ and $\forall y \in \Re$.

Since $\mathbb{E}\left[\left|\frac{\partial^2 h(x';\theta)}{\partial x'^2}\right|\right]=0<\infty$ and $\lim_{x'\to\pm\infty} f_{X'}(x')=0$, $\sup_{x'} f_{X'}(x')=\frac{1}{4}<\infty$, the integrability condition (A.3) holds. □

Proposition 5 *The GLR estimators in Section 5.2 have bounded variances.*

Proof. With $x_m>0$, $\alpha'>0$, $p>c>0$ and $f_{X'}(x')=\frac{\alpha' x_m^{\alpha'} e^{x'}}{\left(e^{x'}+x_m\right)^{\alpha'+1}}$, $x'\in\Re$, we have,

$$\begin{aligned}
\mathbb{E}[\|G_1(X';\theta_k,q_k)\|^2|\mathcal{F}_k] &= \frac{c^2}{p^2}\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}\left(1-\frac{(\alpha'+1)\,e^{x'}}{e^{x'}+x_m}\right)^2\frac{\alpha' x_m^{\alpha'}e^{x'}}{\left(e^{x'}+x_m\right)^{\alpha'+1}}dx'\\
&< \frac{c^2}{p^2}\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}\left[1+\frac{(\alpha'+1)^2e^{2x'}}{\left(e^{x'}+x_m\right)^2}\right]\frac{\alpha' x_m^{\alpha'}e^{x'}}{\left(e^{x'}+x_m\right)^{\alpha'+1}}dx'\\
&< \frac{c^2}{p^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}e^{x'}dx'\\
&= \frac{c^2}{p^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]e^{\frac{q_k+c\theta_k}{p}},
\end{aligned}$$

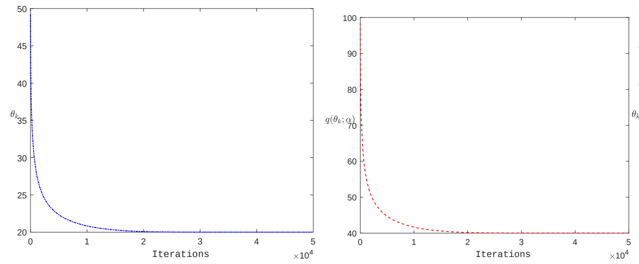
$$\begin{aligned}
\mathbb{E}[G_2^2(X';\theta_k,q_k)|\mathcal{F}_k] &= \frac{1}{p^2}\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}\left(1-\frac{(\alpha'+1)\,e^{x'}}{e^{x'}+x_m}\right)^2\frac{\alpha' x_m^{\alpha'}e^{x'}}{\left(e^{x'}+x_m\right)^{\alpha'+1}}dx'\\
&< \frac{1}{p^2}\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}\left[1+\frac{(\alpha'+1)^2e^{2x'}}{\left(e^{x'}+x_m\right)^2}\right]\frac{\alpha' x_m^{\alpha'}e^{x'}}{\left(e^{x'}+x_m\right)^{\alpha'+1}}dx'\\
&< \frac{1}{p^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]\int_{-\infty}^{\frac{q_k+c\theta_k}{p}}e^{x'}dx'\\
&= \frac{1}{p^2}\alpha' x_m^{-1}\left[1+(\alpha'+1)^2\right]e^{\frac{q_k+c\theta_k}{p}}.
\end{aligned}$$

Since both $\theta_k\in\Theta=[20,80]$ and $q_k\in\mathcal{H}=[-10^{10},10^{10}]$ are bounded, we have $\frac{q_k+c\theta_k}{p}$ is bounded. Therefore, there exist $C_1>0$ and $C_2>0$ such that $\sup_k\mathbb{E}[\|G_1(X';\theta_k,q_k)\|^2|\mathcal{F}_k]\le C_1$ and $\sup_k\mathbb{E}[G_2^2(X';\theta_k,q_k)|\mathcal{F}_k]\le C_2$ w.p.1, which completes the proof. □

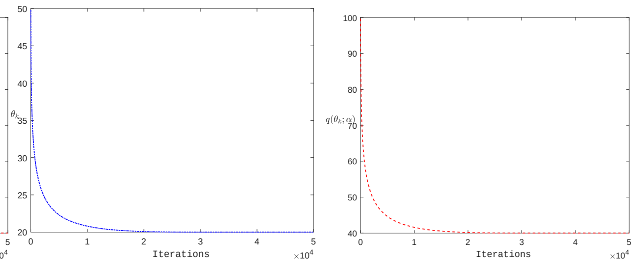
F.2. SQO with Different Quantile Levels in Section 5.2

To illustrate the effect of the quantile level α on the performance of SQO in Section 5.2, Figure 5 reports the performance of SQO averaged over 1000 independent runs under four different values of α: $0.05, 0.3, 0.7, 0.9$. All algorithm parameters are taken to be the same as in Section 5.2.

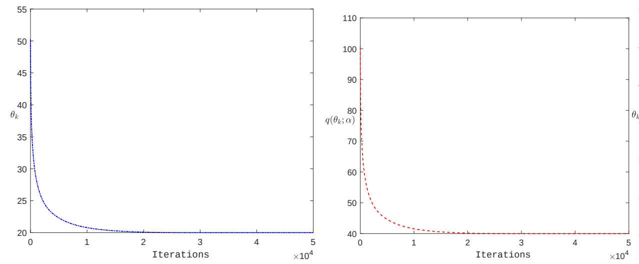
Both QG and SG are implemented with the same algorithm parameters as described in Section 5.2. Table 1 records the average relative errors and averaged CPU times (in seconds) over 1000 independent runs for all algorithms under different quantile levels.



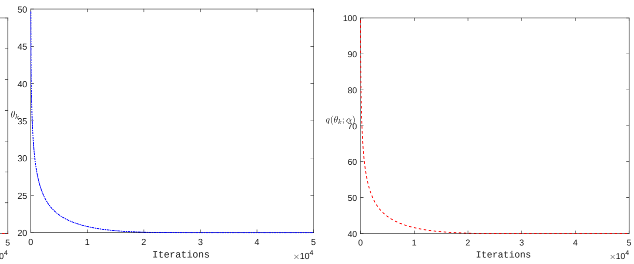
(a) $\alpha = 0.05$, SQO



(b) $\alpha = 0.3$, SQO



(c) $\alpha = 0.7$, SQO



(d) $\alpha = 0.9$, SQO

Figure 5 **Performance of SQO (averaged over 1000 runs) (a)** $\alpha = 0.05$**, SQO; (b)** $\alpha = 0.3$**, SQO; (c)** $\alpha = 0.7$**, SQO; (d)** $\alpha = 0.9$**, SQO on the example of Section 5.2.**

Table 1 **Performance of SQO, QG and SG on the example of Section 5.2 under different quantile levels**

	SQO		QG		SG	
Settings	Relative Error	CPU Time	Relative Error	CPU Time	Relative Error	CPU Time
$\alpha = 0.05$	0	0.43s	1.05×10^{-4}	3.76s	3.67×10^{-3}	0.94s
$\alpha = 0.3$	0	0.43s	1.05×10^{-4}	3.75s	9.91×10^{-3}	1.00s
$\alpha = 0.7$	0	0.43s	1.05×10^{-4}	3.76s	2.69×10^{-2}	0.99s
$\alpha = 0.9$	0	0.49s	1.05×10^{-4}	3.76s	1.15×10^{-1}	1.03s

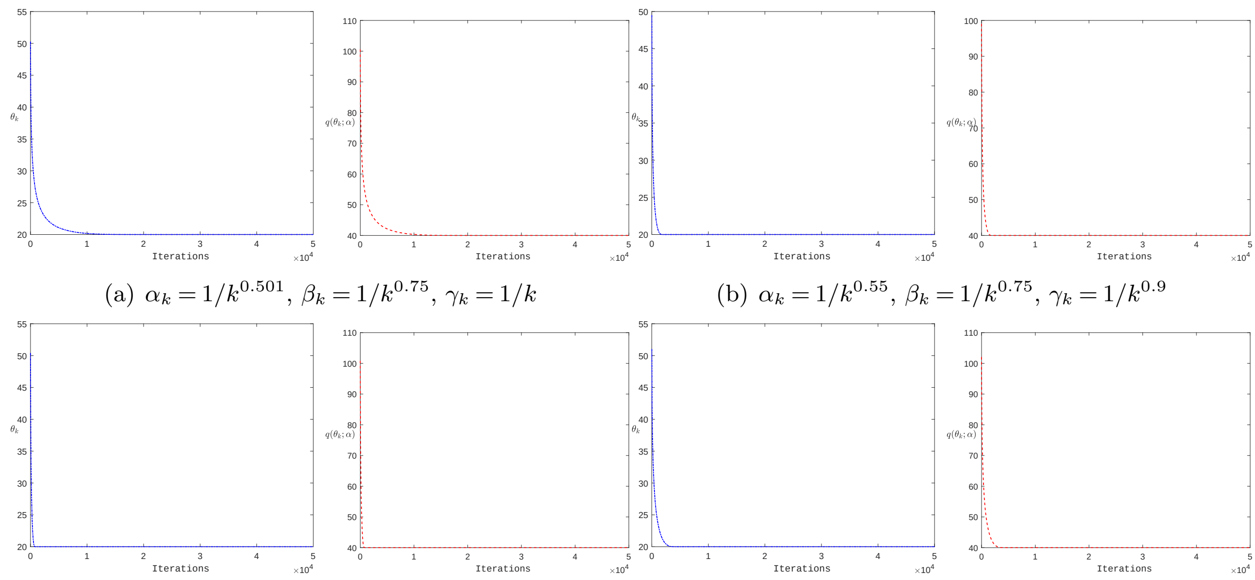
F.3. SQO with Different Step Sizes in Section 5.2

To illustrate the effect of different step sizes on the performance of SQO, tests are performed on the example in Section 5.2 (with $\alpha = 0.95$) using four different sets of step size sequences. Figure 6 shows the performance of SQO (averaged over 1000 independent runs) in each of the respective test cases.

G. Portfolio Optimization

G.1. Alternative GLR Estimators for the Example in Section 5.3.1

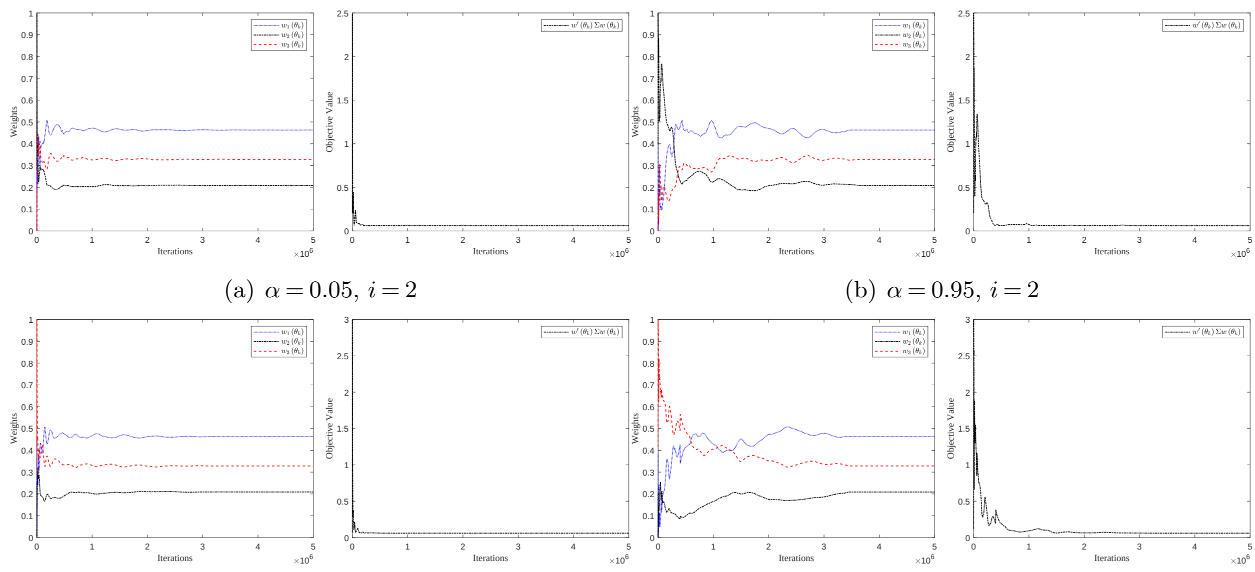
For the portfolio optimization problem with zero-mean asset returns ($\mu = 0$), we have also tested the performance of SQO based on alternative GLR estimators with $i = 2$ and $i = 3$. The parameter settings are the same as in Section 5.3.1. In both cases, the performance of SQO is similar to the $i = 1$ case presented in the main body of the paper. The performance of the algorithm, averaged over 1000 independent runs, is shown in Figure 7.



(a) $\alpha_k = 1/k^{0.501}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k$ (b) $\alpha_k = 1/k^{0.55}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k^{0.9}$

(c) $\alpha_k = 1/k^{0.65}$, $\beta_k = 1/k^{0.7}$, $\gamma_k = 1/k^{0.8}$ (d) $\alpha_k = 1/k^{0.7}$, $\beta_k = 1/k^{0.8}$, $\gamma_k = 1/k^{0.9}$

Figure 6 **Performance of SQO on the example of Section 5.2 with different step sizes (a)** $\alpha_k = 1/k^{0.501}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k$; **(b)** $\alpha_k = 1/k^{0.55}$, $\beta_k = 1/k^{0.75}$, $\gamma_k = 1/k^{0.9}$; **(c)** $\alpha_k = 1/k^{0.65}$, $\beta_k = 1/k^{0.7}$, $\gamma_k = 1/k^{0.8}$; **(d)** $\alpha_k = 1/k^{0.7}$, $\beta_k = 1/k^{0.8}$, $\gamma_k = 1/k^{0.9}$.



(a) $\alpha = 0.05$, $i = 2$ (b) $\alpha = 0.95$, $i = 2$

(c) $\alpha = 0.05$, $i = 3$ (d) $\alpha = 0.95$, $i = 3$

Figure 7 **Performance of SQO with different GLR estimators (a)** $\alpha = 0.05$, $i = 2$; **(b)** $\alpha = 0.95$, $i = 2$; **(c)** $\alpha = 0.05$, $i = 3$; **(d)** $\alpha = 0.95$, $i = 3$ **on the example of Section 5.3.1.**

G.2. Unbiasedness of GLR Estimators in Section 5.3.2

Proposition 6 *The GLR estimators in Section 5.3.2 satisfy conditions (A.1)-(A.3).*

Proof. The input density $f(X) = (2\pi)^{-l/2}|\Sigma|^{-1/2}\exp\left\{-\frac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)\right\}$ is differentiable and $h(X;\theta) = X^T w(\theta)$ is twice differentiable on $\Re^l \times \Theta$, where $\frac{\partial^2 h(X;\theta)}{\partial x_1^2} = 0$ and $\Theta = \left(0, \frac{\pi}{2}\right)$. Note that h is a linear function of X and for $\theta \in \Theta$, $w_i(\theta) \neq 0$, $i = 1,2,3$, so h is invertible and $h^{-1}(y;\theta) = \frac{y - \sum_{i=2}^3 x_i w_i(\theta)}{w_1(\theta)}$.

Now we turn to the Lipschitz condition. Note

$$\left|\left(\frac{\partial h(x;\theta)}{\partial \vartheta_1}\right)^{-1}\right| = \left|\frac{1}{\sin 2\theta_1\left(-x_1 + \cos^2\vartheta_2 x_2 + \sin^2\theta_2 x_3\right)}\right| .$$

For $x_2 > x_3$, we have

$$\left|\left(\frac{\partial h(x;\theta)}{\partial \vartheta_1}\right)^{-1}\right| > \left|\frac{1}{x_2 - x_1}\right| ,$$

and for $x_2 < x_3$,

$$\left|\left(\frac{\partial h(x;\theta)}{\partial \vartheta_1}\right)^{-1}\right| > \left|\frac{1}{x_3 - x_1}\right| .$$

In addition, we have

$$\left|\left(\frac{\partial h(x;\theta)}{\partial \vartheta_2}\right)^{-1}\right| = \left|\frac{1}{\sin^2\theta_1 \sin 2\theta_2 (x_3 - x_2)}\right| > \left|\frac{1}{x_3 - x_2}\right| ,$$

and

$$\left|\left(\frac{\partial h(x;\theta)}{\partial x_1}\right)^{-1}\right| = \left|\frac{1}{\cos^2\theta_1}\right| > 1 .$$

Therefore, h^{-1} is globally Lipschitz continuous with respect to y and θ, so the uniform convergence condition (A.2) holds for $\forall\theta \in \Theta$ and $\forall y \in \Re$.

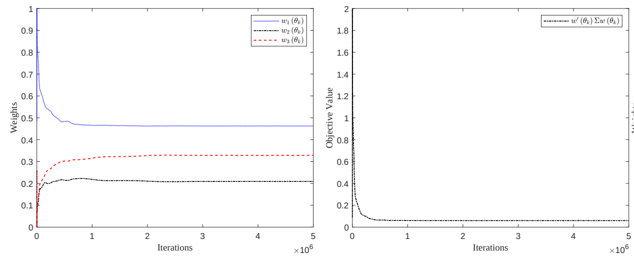
Since $\mathbb{E}\left[\left|\frac{\partial^2 h(x;\theta)}{\partial x_1^2}\right|\right] = 0 < \infty$ and $\lim_{x_1 \to \pm\infty} f_1(x_1) = 0$, $\sup_x f_X(x) < \infty$, the integrability condition (A.3) holds. □

Similarly, one can show the GLR estimators in Section 5.3.1 satisfy conditions (A.1)-(A.3).

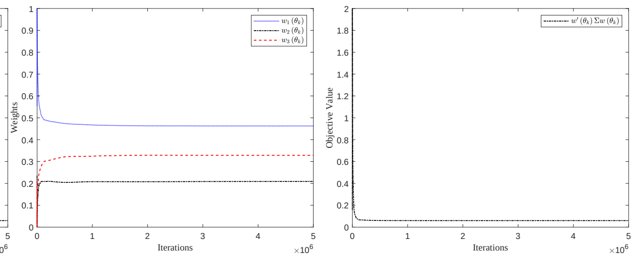
G.3. SQO with Different Quantile Levels in Section 5.3.1

To illustrate the effect of the quantile level α on the performance of SQO in Section 5.3.1, Figure 8 reports the performance of the algorithm, averaged over 1000 independent runs, under four different α values: $0.1, 0.3, 0.7, 0.9$. The algorithm is implemented using a GLR estimator with $i = 1$ and the step sizes $\alpha_k = 1/k^{\frac{3}{5}}$, $\beta_k = 1/k^{\frac{4}{5}}$, $\gamma_k = 1/k$. The initial estimates are the same as in Section 5.3.1.

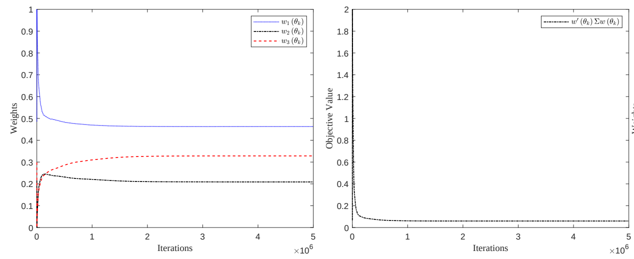
The QG algorithm is implemented with the step sizes $\rho_0 = \delta_0 = n_0 = 1$ and $b_1 = 1$, $b_2 = 0.501$, $b_3 = 2.003$. The initial estimates are taken to be the same as described in Section 5.3.1. Table 2 records the average relative errors in the objective value of (26) and the averaged CPU times (in seconds) for both algorithms under different quantile levels.



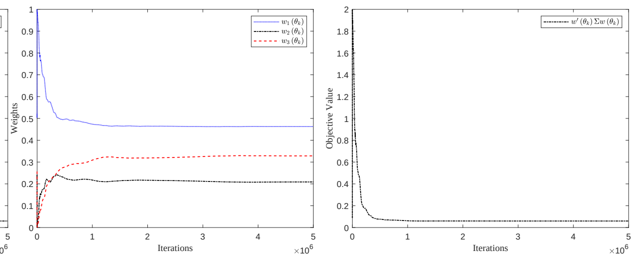
(a) $\alpha = 0.1$



(b) $\alpha = 0.3$



(c) $\alpha = 0.7$



(d) $\alpha = 0.9$

Figure 8 **Performance of SQO on the example of Section 5.3.1 under different α values: (a) $\alpha = 0.1$; (b) $\alpha = 0.3$; (c) $\alpha = 0.7$; (d) $\alpha = 0.9$.**

Table 2 **Performance of SQO and QG on the example of Section 5.3.1 under different quantile levels**

	SQO		QG	
Settings	Relative Error	CPU Time	Relative Error	CPU Time
$\alpha = 0.1$	3.53×10^{-5}	78.87s	1.04×10^{-3}	20155.82s
$\alpha = 0.3$	7.24×10^{-5}	75.11s	1.52×10^{-2}	20068.88s
$\alpha = 0.7$	8.54×10^{-5}	74.75s	3.85×10^{-3}	20759.95s
$\alpha = 0.9$	8.18×10^{-5}	72.53s	3.49×10^{-4}	20810.31s

G.4. SQO with Different Quantile Levels in Section 5.3.2

To illustrate the effect of α on the performance of SQO in Section 5.3.2, Table 3 shows the weight vectors $w(\theta)$, the objective values (26), and the true quantiles $q(\theta;\alpha)$ obtained by SQO (each averaged over 1000 runs) under different α values: $0.1, 0.2, 0.3, 0.9$. SQO is implemented using a GLR estimator with $i = 1$ and the step sizes $\alpha_k = 1/k^{\frac{3}{5}}$, $\beta_k = 1/k^{\frac{4}{5}}$, $\gamma_k = 1/k$. The initial estimates are taken to be the same as that in Section 5.3.2.

Table 3 **Results obtained by SQO on the example of Section 5.3.2 under different quantile levels**

Settings	Weights	Objective Value	Quantile
$\alpha = 0.1$	$(0.4324, 0.2046, 0.3630)^T$	-1.8615	1.7534
$\alpha = 0.2$	$(0.4110, 0.2012, 0.3878)^T$	-1.8896	1.8299
$\alpha = 0.3$	$(0.3023, 0.1856, 0.5121)^T$	-1.8857	1.8698
$\alpha = 0.9$	$(0.4931, 0.2143, 0.2926)^T$	-1.7300	1.9776

H. Asset Returns Following A Jump-Diffusion Model

H.1. Unbiasedness of GLR Estimators in Section 5.4

Proposition 7 *The GLR estimators in Section 5.4 satisfy conditions (A.1)-(A.3).*

Proof. The input density $f_X(X) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{W_1^2}{2t^2}} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{W_2^2}{2t^2}}$ is differentiable and $h(X;\theta) = \sum_{i=1}^{2} w_i(\theta) r_i(\theta)$ is twice differentiable on $\Re^l \times \Theta$, where $\frac{\partial^2 h(X;\theta)}{\partial W_1^2} = 0$ and $\Theta = \left(0, \frac{\pi}{2}\right)$. Note that h is a linear function of W_1 and $w_1(\theta)\sigma_1 + w_2(\theta)\sigma_2\sqrt{\rho_{12}} \neq 0$ for $\theta \in \Theta$, so h is invertible and

$$h^{-1}(y;\theta) = \frac{y - \left(w_1\sigma_1\sqrt{\rho_{12}} + w_2\sigma_2\right) W_2 - w_1(\mu_1 + J_1) - w_2(\mu_2 + J_2)}{w_1\sigma_1 + w_2\sigma_2\sqrt{\rho_{12}}} .$$

Now we turn to the Lipschitz condition. Note that

$$\left|\left(\frac{\partial h(X;\theta)}{\partial W_1}\right)^{-1}\right| = \left|\frac{1}{\cos^2\theta\sigma_1 + \sin^2\theta\sqrt{\rho_{12}}}\right| .$$

For $\sqrt{\rho_{12}} > \sigma_1$, we have

$$\left|\left(\frac{\partial h(X;\theta)}{\partial W_1}\right)^{-1}\right| > \frac{1}{\sqrt{\rho_{12}}} ,$$

and for $\sqrt{\rho_{12}} < \sigma_1$,

$$\left|\left(\frac{\partial h(X;\theta)}{\partial W_1}\right)^{-1}\right| > \frac{1}{\sigma_1} .$$

In addition,

$$\left|\left(\frac{\partial h(X;\theta)}{\partial \theta}\right)^{-1}\right| = \left|\frac{1}{\sin(2\theta)(r_2 - r_1)}\right| \geq \left|\frac{1}{r_2 - r_1}\right| .$$

Therefore, we have h^{-1} is globally Lipschitz continuous with respect to y and θ, so the uniform convergence condition (A.2) holds for $\forall\theta \in \Theta$, $\forall y \in \Re$.

Since $\mathbb{E}\left[\left|\frac{\partial^2 h(X;\theta)}{\partial W_1^2}\right|\right] = 0 < \infty$ and $\lim_{W_1 \to \pm\infty} f_1(W_1) = 0$, $\sup_X f_X(X) < \infty$, the integrability condition (A.3) holds. □

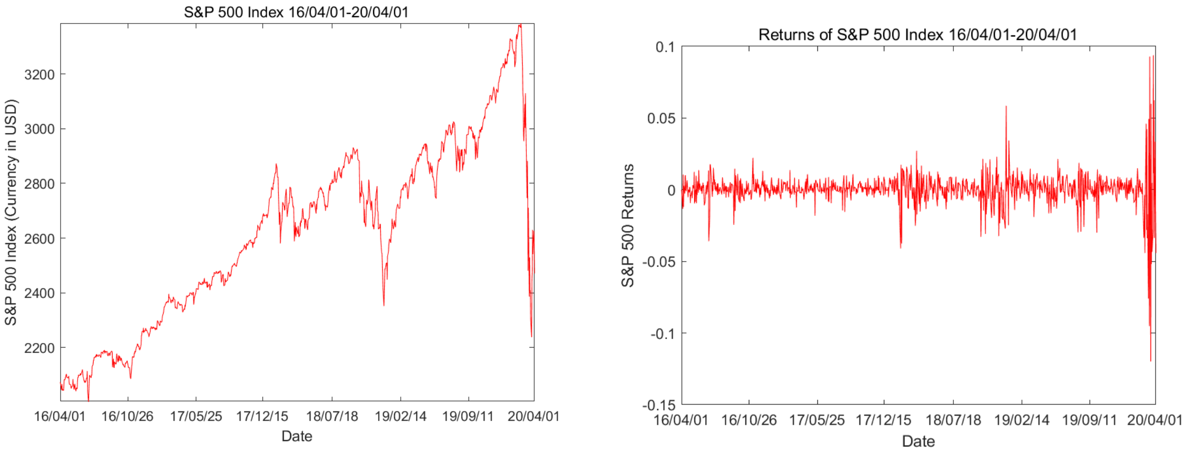
H.2. A Real Data Experiment

We calibrate the jump-diffusion model by using S&P 500 Index and Hang Seng Index (HSI) daily adjusted closing price data from April 1st 2016 to April 1st 2020. The data is collected from Yahoo Finance, and there are 963 observations for each index during the period (deleting the missing values). Figures 9 and 10 present the prices and returns of the S&P 500 Index and HSI, respectively.

We first calibrate a traditional diffusion model without jump terms, i.e.,

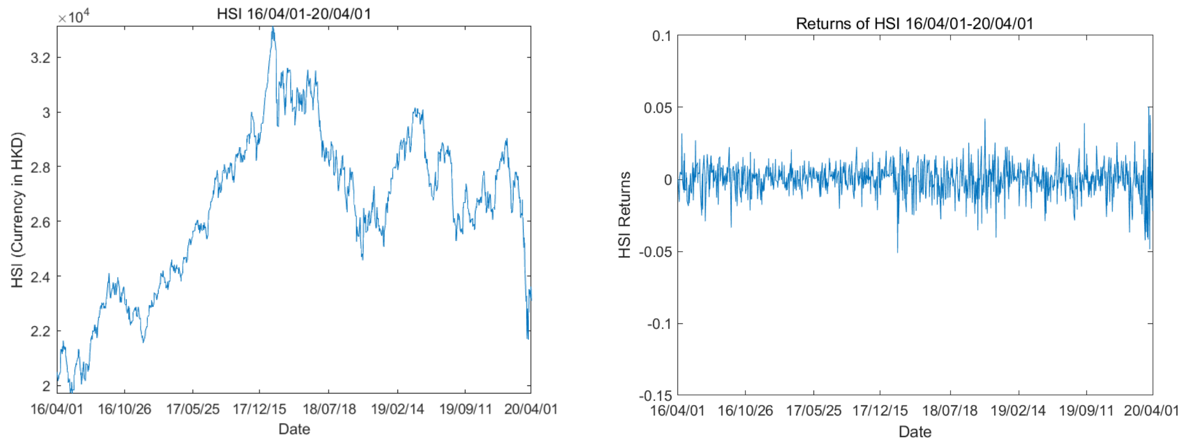
$$r_1(t) = \mu_1 t + \sigma_1 W_1(t) + \sigma_1\sqrt{\rho_{12}} W_2(t),$$

$$r_2(t) = \mu_2 t + \sigma_2\sqrt{\rho_{12}} W_1(t) + \sigma_2 W_2(t).$$



(a) (b)

Figure 9 **(a) Prices and (b) Returns of S&P 500 Index during 04/01/16-04/01/20.**



(a) (b)

Figure 10 **(a) Prices and (b) Returns of HSI during 04/01/16-04/01/20.**

We assume that for all $0 = t_0 < t_1 < \cdots < t_{963}$, the increments $\Delta_j r_i = r_i(t_j) - r_i(t_{j-1})$, $j = 1, \ldots, 963$, are independent, $i = 1, 2$. Set $\Delta t = t_j - t_{j-1} = 1$, then the increments $\Delta_j r_i$ follow the normal distributions $N(\mu_i, \sigma_i^2(1+\rho_{12}))$, $j = 1, 2, \cdots, 963$, and the correlation between the returns of two assets is $2\sqrt{\rho_{12}}/(1+\rho_{12})$. The MLE estimates of μ, σ and ρ_{12} using the data from the S&P 500 Index and HSI are

$$\widehat{\mu} = \left(-4.5886 \times 10^{-5}, -2.2799 \times 10^{-5}\right)^T, \quad \widehat{\sigma} = \left(1.7148 \times 10^{-2}, 1.4129 \times 10^{-2}\right)^T$$

and $\widehat{\rho}_{12} = 0.2539$, respectively.

We then calibrate the jump diffusion model in Section 5.4. Let λ be the jump intensity of Poisson counting process $\{N(t)\}_{t=0}^{\infty}$, and suppose the jump sizes follow an exponential distribution with mean $1/b$, i.e., $D_j^i \sim Exp(b)$, $j=1,2,\cdots,963$, $i=1,2$. Since the Poisson jump processes are assumed to be independent, the correlation between the returns of two assets is the same as that in the diffusion process, i.e., $2\sqrt{\rho_{12}}/(1+\rho_{12})$. An estimate of the correlation parameter from samples is $\widehat{\rho}_{12}=0.1572$. Then the parameters μ_i, σ_i, λ_i and b_i are estimated by MLE. The log-likelihood of $\Delta_j r_i$, $j=1,\ldots,963$, is given by

$$\sum_{j=1}^{963} \log f(\Delta_j r_i;\theta),$$

where f is the density of $\Delta_j r_i$, and θ is a generic parameter containing μ_i, σ_i, λ_i and b_i. We use a gradient-based optimization approach to find the MLE, and the derivative of the log-likelihood is

$$\sum_{j=1}^{963} \frac{\partial f(\Delta_j r_i;\theta)}{\partial\theta} \bigg/ f(\Delta_j r_i;\theta).$$

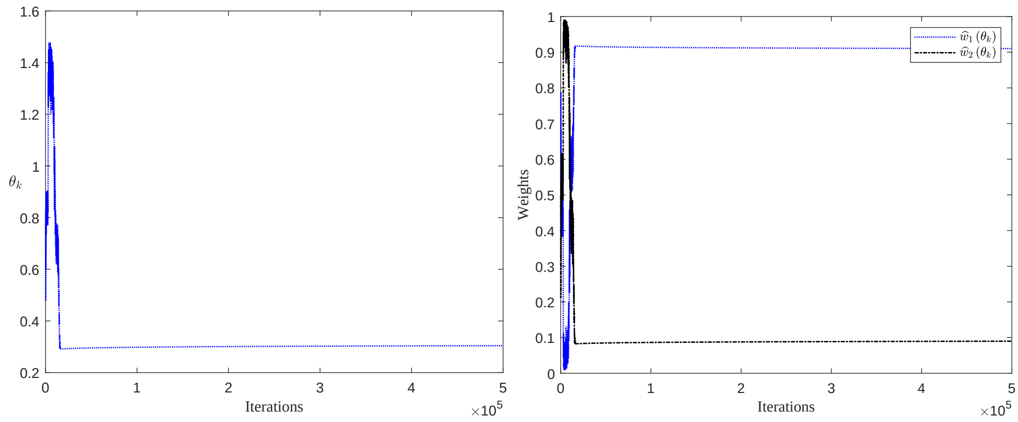
Unlike in the diffusion model, f and $\partial f/\partial\theta$ do not have analytical forms in the jump-diffusion model and we estimate them by simulation using the GLR estimators in Peng et al. (2019). We can use the SQO algorithm to avoid the ratio bias. The estimates of the parameters using the real data of the S&P 500 Index and HSI are

$$\begin{aligned}
&\widehat{\mu}=\left(6.2905\times 10^{-5}, 1.9545\times 10^{-4}\right)^T, \quad \widehat{\sigma}=\left(5.4937\times 10^{-3}, 2.0877\times 10^{-3}\right)^T,\\
&\widehat{\lambda}=(0.9959, 0.9992)^T, \quad \widehat{b}=(100000.1509, 100000.0005)^T.
\end{aligned}$$

For model selection between the diffusion model and jump-diffusion model, we use the Bayesian information criterion (BIC) defined as $p\log(T)-2\mathcal{L}(\widehat{\theta},\mathcal{M})$, where p is the number of parameters to be estimated in the model, T is the number of observations, and $\mathcal{L}(\widehat{\theta},\mathcal{M})$ is the value of the maximum log-likelihood of the model $\mathcal{M}$ with parameter $\widehat{\theta}$ estimated by MLE. The model with the lower BIC is preferred, which implies either fewer explanatory variables, better fit, or both. Denote Δ_{BIC} as the difference in BICs between the two models. In the literature, the strength of the evidence against higher BIC value can be summarized as follows: if $\Delta_{BIC}\in[0,2)$, the evidence against the other model is insignificant; $\Delta_{BIC}\in[2,6)$, positive; $\Delta_{BIC}\in[6,10)$, strong; $\Delta_{BIC}\geq 10$, very strong. In our problem, we have $T=963$, $p_{\textit{diffusion}}=2$ and $p_{\textit{jump-diffusion}}=4$ given correlation ρ_{12}. For S&P 500 returns model, we have estimates $BIC_{\textit{diffussion}}=-6926$ and $BIC_{\textit{jump-diffusion}}=-7492$. Similarly, we have for HSI returns model $BIC_{\textit{diffusion}}=-7299$ and $BIC_{\textit{jump-diffusion}}=-7932$. The difference in BIC values indicates that the strength of the evidence favoring the jump-diffusion model over the diffusion model is very strong.

With the estimated parameters for jump-diffusion model, we then optimize the 0.05-quantile of the parameterized stochastic model $Y(\theta)=w_1(\theta)r_1(t)+w_2(\theta)r_2(t)$ using SQO with step sizes

$\alpha_k = 1/k^{\frac{3}{5}}$, $\beta_k = 1/k^{\frac{4}{5}}$, $\gamma_k = 1/k$. We run the algorithm for 5×10^5 iterations. The average θ value (averaged over 1000 independent runs) obtained by the algorithm is $\theta = 0.3159$ and the corresponding weight vectors is given by $\widehat{w} = (0.9035, 0.0965)^T$. The performance of SQO on this example is shown in Figure 11.



(a) θ_k (b) $\widehat{w}(\theta_k)$

Figure 11 **(a) θ_k and the corresponding weights (b) $\widehat{w}(\theta_k)$ found by SQO on the real-data example.**

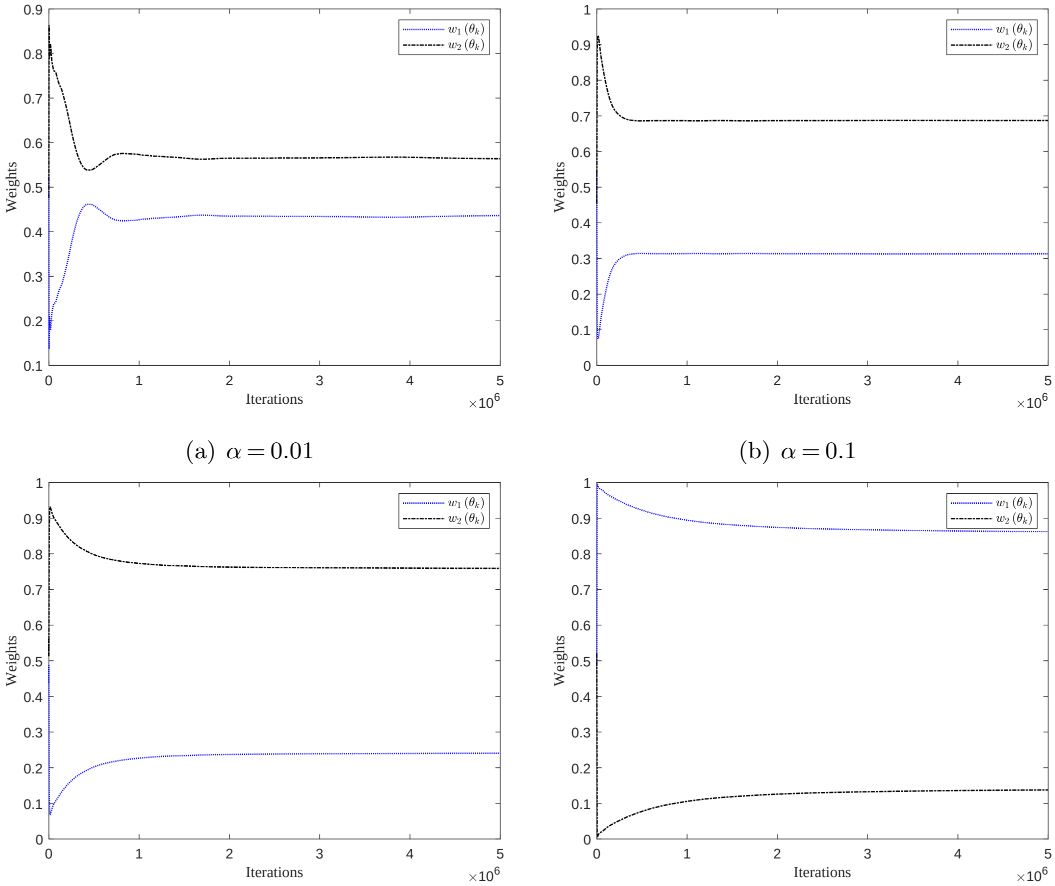
H.3. SQO with Different Quantile Levels in Section 5.4

To illustrate the effect of the quantile level α on the performance of SQO, we have also tested the algorithm on the example in Section 5.4 using four different α values: $0.01, 0.1, 0.15, 0.9$. The results are based on using the step sizes $\alpha_k = 1/k^{\frac{3}{5}}$, $\beta_k = 1/k^{\frac{4}{5}}$, $\gamma_k = 1/k$. The performance of SQO (averaged over 1000 runs) is plotted in Figure 12.

The QG algorithm is implemented with the step sizes $\rho_0 = \delta_0 = n_0 = 1$, $b_1 = 1$, $b_2 = 0.501$, and $b_3 = 2.003$. The initial estimates are taken to be the same as in Section 5.4. The θ values, the corresponding weight vectors $w(\theta)$ and the CPU times (in seconds) (all averaged over 1000 and 10 independent runs, respectively) obtained by both algorithms under different quantile levels are provided in Table 4.

Table 4 **Results obtained by SQO and QG on the example of Section 5.4 under different α values**

	SQO			QG		
	θ	Weights	CPU Time	θ	Weights	CPU Time
$\alpha = 0.01$	0.8494	$(0.4362, 0.5638)^T$	160.57s	0.8370	$(0.4485, 0.5515)^T$	55778.67s
$\alpha = 0.1$	0.9770	$(0.3130, 0.6870)^T$	153.86s	0.9631	$(0.3260, 0.6740)^T$	55829.16s
$\alpha = 0.15$	1.0580	$(0.2407, 0.7593)^T$	153.53s	1.0653	$(0.2345, 0.7655)^T$	56119.16s
$\alpha = 0.9$	0.3801	$(0.8624, 0.1376)^T$	163.96s	0.3859	$(0.8583, 0.1417)^T$	56279.50S



(a) $\alpha = 0.01$ (b) $\alpha = 0.1$

(c) $\alpha = 0.15$ (d) $\alpha = 0.9$

Figure 12 **Performance of SQO with (a)** $\alpha = 0.01$**; (b)** $\alpha = 0.1$**; (c)** $\alpha = 0.15$**; (d)** $\alpha = 0.9$ **on the example of Section 5.4.**

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