Online Supplement of the Paper "Solving Large-Scale Fixed-Budget Ranking and Selection Problems"

L. Jeff Hong
School of Management and School of Data Science, Fudan University, Shanghai, 200433 China, hong_liu@fudan.edu.cn

Guangxin Jiang
School of Management, Harbin Institute of Technology, Harbin, 150001 China, gxjiang@hit.edu.cn

Ying Zhong*
School of Management and Economics, University of Electronic Science and Technology of China, Chengdu, 611731 China, yzhong4@uestc.edu.cn

Appendix A: Proof of Lemmas

A.1. Proof of Lemma 2

If $F=\prod_{r=1}^{\infty}(1-\exp(-\varrho r))$ for some constant $\varrho>0$, then we have

$$\log F = \sum_{r=1}^{\infty} \log(1-\exp(-\varrho r)). \tag{1}$$

Because the Taylor series for function $\log(1-x)$ is $-x-x^2/2-x^3/3-\cdots$, we can rewrite Equation (1) as,

$$\begin{aligned}
(1) &= -\sum_{r=1}^{\infty}\sum_{\gamma=1}^{\infty}\frac{\exp(-\varrho\gamma r)}{\gamma} \\
&= -\sum_{\gamma=1}^{\infty}\sum_{r=1}^{\infty}\frac{\exp(-\varrho\gamma r)}{\gamma} \\
&= -\sum_{\gamma=1}^{\infty}\frac{1}{\gamma}\frac{\exp(-\varrho\gamma)}{1-\exp(-\varrho\gamma)} \\
&\geq -\sum_{\gamma=1}^{\infty}\frac{1}{\gamma}\frac{1}{\varrho\gamma} = -\frac{1}{\varrho}\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^2} = -\frac{\pi^2}{6\varrho},
\end{aligned}$$

where the inequality holds because $\exp(-\varrho\gamma)/(1-\exp(-\varrho\gamma)) \leq 1/(\varrho\gamma)$ for any $\varrho\gamma>0$. It concludes the proof. □

* Corresponding Author

A.2. Proof of Lemma 3

By the Chernoff bound, for any positive constant $\kappa > 0$, we have

$$\mathbb{P}(Z \geq d) \leq \frac{\mathbb{E}[\exp(\kappa Z)]}{\exp(\kappa d)} = \exp\left(-\frac{1}{2}\kappa^2 - \kappa d\right). \tag{2}$$

Letting $\kappa = d$, Equation (2) yields,

$$\mathbb{P}(Z \geq d) \leq \exp\left(-\frac{d^2}{2}\right). \quad \square$$

Appendix B: Proof of Propositions

B.1. Proof of Proposition 1

We prove Proposition 1 in a similar way to that of Theorem 2. We show that if the sample allocation rule $N_r = \left\lfloor \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r N \right\rfloor$ is used and $\phi \geq 3$, for any positive constant $\eta_1' \geq 2\phi^2$ and IZ parameter $\delta > 0$, the PGS of the $\mathcal{FBKT}$ procedure is lower bounded for all $k \geq 2$. We first let $\hat{k} = 2^{\lceil \log_2 k \rceil}$. Notice that the procedure can select the best alternative in $\log_2 \hat{k}$ rounds. For $1 \leq r \leq \log_2 \hat{k}$, we let ζ_r denote the index of the alternative with the largest mean among the alternatives that are still in contention at the beginning of round r, i.e., $\mu_{\zeta_r} = \max_{i \in \mathcal{I}_r} \mu_i$, and ζ_r' denote the index of the alternative that competes with alternative ζ_r in round r. To ease the notation, we set $\zeta_r = \varphi$ for $r = \log_2 \hat{k} + 1$, where φ is the index of the alternative which is finally selected by the procedure. We also let,

$$\delta_r = \frac{\sqrt{2(\phi-1)} - \sqrt{\phi}}{\sqrt{\phi}} \left(\frac{\phi}{2(\phi-1)}\right)^{\frac{r}{2}} \delta,$$

for $r \geq 1$. It can be verified that if $\phi \geq 3$, $\delta_r > 0$ for $r \geq 1$ and $\sum_{r=1}^{\infty} \delta_r = \delta$. For $1 \leq r \leq \log_2 \hat{k}$, we let $\mathcal{Q}_r' = \{\mu_{\zeta_r} - \mu_{\zeta_r'} \leq \delta_r\} \cup \{\mu_{\zeta_r} - \mu_{\zeta_r'} > \delta_r \text{ and } \bar{X}_{\zeta_r}^r \geq \bar{X}_{\zeta_r'}^r\}$, and $\mathcal{Q}_r'^c$ denote the complement of event $\mathcal{Q}_r'$, i.e., $\mathcal{Q}_r'^c = \{\mu_{\zeta_r} - \mu_{\zeta_r'} > \delta_r \text{ and } \bar{X}_{\zeta_r}^r < \bar{X}_{\zeta_r'}^r\}$. Then, we can bound the PGS of the $\mathcal{FBKT}$ procedure as follows,

$$\begin{aligned}
\text{PGS} &\geq \mathbb{P}\left(\bigcap_{r=1}^{\log_2 \hat{k}} \left\{\mu_{\zeta_r} - \mu_{\zeta_{r+1}} \leq \delta_r\right\}\right) \\
&\geq \mathbb{P}\left(\bigcap_{r=1}^{\log_2 \hat{k}} \{\mathcal{Q}_r'\}\right) \\
&= \mathbb{P}(\mathcal{Q}_1') \prod_{r=2}^{\log_2 \hat{k}} \mathbb{P}\left(\mathcal{Q}_r' \middle| \mathcal{Q}_1', \ldots, \mathcal{Q}_{r-1}'\right) \\
&= \left(1 - \mathbb{P}\left(\mathcal{Q}_1'^c\right)\right) \prod_{r=2}^{\log_2 \hat{k}} \left(1 - \mathbb{P}\left(\mathcal{Q}_r'^c \middle| \mathcal{Q}_1', \ldots, \mathcal{Q}_{r-1}'\right)\right),
\end{aligned} \tag{3}$$

where the first inequality holds because if the mean difference between alternatives ζ_r and ζ_{r+1} is less than δ_r for $1 \leq r \leq \log_2 \hat{k}$, the mean difference between alternative φ and alternative k is less than $\sum_{r=1}^{\infty}\delta_r = \delta$, and the second inequality holds because event $\mathcal{Q}_r'$ implies event $\{\mu_{\zeta_r} - \mu_{\zeta_{r+1}} \le \delta_r\}$. Notice that based on the analysis used in Equation (9) in the proof of Theorem 2, we can conclude that,

$$\mathcal{N}_r \ge r\left\lfloor \frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor.$$

Then, we can rewrite Equation (3) as follows,

$$\begin{aligned}
(3) &= \left(1-\mathbb{P}\left(\mu_{\zeta_1}-\mu_{\zeta_1'}>\delta_1 \text{ and } \bar{X}^1_{\zeta_1}<\bar{X}^1_{\zeta_1'}\right)\right)\times\\
&\quad \prod_{r=2}^{\log_2\hat{k}}\left(1-\mathbb{P}\left(\mu_{\zeta_r}-\mu_{\zeta_r'}>\delta_r \text{ and } \bar{X}^r_{\zeta_r}<\bar{X}^r_{\zeta_r'}\middle|\mathcal{Q}_1',\ldots,\mathcal{Q}_{r-1}'\right)\right)\\
&= \left(1-\mathbb{P}\left(\mu_{\zeta_1}-\mu_{\zeta_1'}>\delta_1 \text{ and } \frac{-\bar{X}^1_{\zeta_1}+\bar{X}^1_{\zeta_1'}+\mu_{\zeta_1}-\mu_{\zeta_1'}}{\sigma_{\zeta_1\zeta_1'}/\sqrt{\mathcal{N}_1}}>\sqrt{\mathcal{N}_1}\frac{\mu_{\zeta_1}-\mu_{\zeta_1'}}{\sigma_{\zeta_1\zeta_1'}}\right)\right)\times\\
&\quad \prod_{r=2}^{\log_2\hat{k}}\left(1-\mathbb{P}\left(\mu_{\zeta_r}-\mu_{\zeta_r'}>\delta_r \text{ and } \frac{-\bar{X}^r_{\zeta_r}+\bar{X}^r_{\zeta_r'}+\mu_{\zeta_r}-\mu_{\zeta_r'}}{\sigma_{\zeta_r\zeta_r'}/\sqrt{\mathcal{N}_r}}>\sqrt{\mathcal{N}_r}\frac{\mu_{\zeta_r}-\mu_{\zeta_r'}}{\sigma_{\zeta_r\zeta_r'}}\middle|\mathcal{Q}_1',\ldots,\mathcal{Q}_{r-1}'\right)\right)\\
&\ge \left(1-\mathbb{P}\left(\mu_{\zeta_1}-\mu_{\zeta_1'}>\delta_1 \text{ and } Z\ge\sqrt{\left\lfloor\frac{\eta_1'}{2\phi^2}\right\rfloor}\frac{\delta_1}{\sigma_{upper}}\right)\right)\times\\
&\quad \prod_{r=2}^{\log_2\hat{k}}\left(1-\mathbb{P}\left(\mu_{\zeta_r}-\mu_{\zeta_r'}>\delta_r \text{ and } Z>\sqrt{r\left\lfloor\frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\middle|\mathcal{Q}_1',\ldots,\mathcal{Q}_{r-1}'\right)\right)\\
&\ge \left(1-\mathbb{P}\left(Z\ge\sqrt{\left\lfloor\frac{\eta_1'}{2\phi^2}\right\rfloor}\frac{\delta_1}{\sigma_{upper}}\right)\right)\times\\
&\quad \prod_{r=2}^{\log_2\hat{k}}\left(1-\mathbb{P}\left(Z>\sqrt{r\left\lfloor\frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\middle|\mathcal{Q}_1',\ldots,\mathcal{Q}_{r-1}'\right)\right)\\
&= \prod_{r=1}^{\log_2\hat{k}}\left(1-\mathbb{P}\left(Z>\sqrt{r\left\lfloor\frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\right)\right),
\end{aligned} \tag{4}$$

where the last equality holds because event $\{Z>\sqrt{r\lfloor\eta_1'(2(\phi-1)/\phi)^r/(4\phi(\phi-1))\rfloor}\delta_r/\sigma_{upper}\}$ is independent of events $\{\mathcal{Q}_1',\ldots,\mathcal{Q}_{r-1}'\}$. Then, applying Lemma 3 to Equation (4), we have,

$$\begin{aligned}
(4) &\ge \prod_{r=1}^{\log_2\hat{k}}\left(1-\exp\left(-r\left\lfloor\frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor\frac{\delta_r^2}{2\sigma_{upper}^2}\right)\right)\\
&= \prod_{r=1}^{\log_2\hat{k}}\left(1-\exp\left(-r\left\lfloor\frac{\eta_1'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor\frac{\left(\frac{\sqrt{2(\phi-1)}-\sqrt{\phi}}{\sqrt{\phi}}\right)^2\left(\frac{\phi}{2(\phi-1)}\right)^r\delta^2}{2\sigma_{upper}^2}\right)\right)\\
&\ge \prod_{r=1}^{\log_2\hat{k}}\left(1-\exp\left(-r\frac{\eta_1'}{8\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\frac{\left(\frac{\sqrt{2(\phi-1)}-\sqrt{\phi}}{\sqrt{\phi}}\right)^2\left(\frac{\phi}{2(\phi-1)}\right)^r\delta^2}{2\sigma_{upper}^2}\right)\right)
\end{aligned}$$

$$= \prod_{r=1}^{\log_2 \hat{k}} \left(1 - \exp\left(-r \frac{\eta_1' \delta^2}{8\sigma_{upper}^2} \left(\frac{1}{\phi} - \frac{1}{\sqrt{2\phi(\phi-1)}}\right)^2\right)\right), \tag{5}$$

where the second inequality holds because if $a \geq 1$, then $\lfloor a \rfloor \geq a/2$. Applying Lemma 2 to Equation (5), it yields,

$$(5) \geq \exp\left(-\frac{8\pi^2 \sigma_{upper}^2 \phi^3 (\phi-1)}{3\eta_1' \delta^2 \left(\sqrt{2\phi(\phi-1)} - \phi\right)^2}\right).$$

Therefore, if $N/k \geq \eta_1'$, the PGS of the $\mathcal{FBKT}$ procedure is lower bounded by $\exp(-4\pi^2\sigma_{upper}^2/(3\eta_1'\delta^2(1/\phi - 1/\sqrt{2\phi(\phi-1)})^2))$. It concludes the proof. □

Appendix C: Proof of Theorems

C.1. Proof of Theorem 3

Let $R = \lceil \log_2 k/m \rceil$. We first show that if $\phi \geq 2$ and $N_r' = \left\lfloor \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} \right\rfloor$ for $r = 1, 2, \ldots, R$ and $N_r' = \left\lfloor \frac{r}{\phi-1} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} \right\rfloor$ for $r = R+1$, the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure can stop before $N_0 + N$ observations are used. It can be checked that

$$\begin{aligned}\sum_{r=1}^{R+1} N_r' &\leq \sum_{r=1}^{R} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} + \frac{R+1}{\phi-1} \left(\frac{\phi-1}{\phi}\right)^{R+1} \frac{N}{m} \\ &= \sum_{r=1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} + \frac{R+1}{\phi-1} \left(\frac{\phi-1}{\phi}\right)^{R+1} \frac{N}{m} - \sum_{r=R+1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m}.\end{aligned} \tag{6}$$

For the first term in Equation (6), we have,

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} &= \phi \left[\sum_{r=1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} - \sum_{r=1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^{r+1} \frac{N}{m}\right] \\ &= \phi \left[\frac{N}{\phi(\phi-1)m} \sum_{r=1}^{\infty} \left(\frac{\phi-1}{\phi}\right)^r\right] \\ &= \frac{N}{m}.\end{aligned} \tag{7}$$

For the third term in Equation (6), we have,

$$\begin{aligned}\sum_{r=R+1}^{\infty} \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} &= \sum_{r=1}^{\infty} \frac{r+R}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^{r+R} \frac{N}{m} \\ &= \frac{N}{\phi(\phi-1)m} \left(\frac{\phi-1}{\phi}\right)^R \left[\sum_{r=1}^{\infty} r \left(\frac{\phi-1}{\phi}\right)^r + R \sum_{r=1}^{\infty} \left(\frac{\phi-1}{\phi}\right)^r\right] \\ &= \frac{N}{\phi(\phi-1)m} \left(\frac{\phi-1}{\phi}\right)^R \left[\phi(\phi-1) + R(\phi-1)\right] \\ &= \frac{N}{m} \left(\frac{\phi-1}{\phi}\right)^R + \frac{NR}{\phi m} \left(\frac{\phi-1}{\phi}\right)^R.\end{aligned} \tag{8}$$

Plugging the results in Equations (7) and (8) into Equation (6) yields,

$$\begin{aligned}\sum_{r=1}^{R+1} N_r' &\le \sum_{r=1}^{\infty} \frac{r}{\phi(\phi-1)}\left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m} + \frac{R+1}{\phi-1}\left(\frac{\phi-1}{\phi}\right)^{R+1}\frac{N}{m} - \sum_{r=R+1}^{\infty}\frac{r}{\phi(\phi-1)}\left(\frac{\phi-1}{\phi}\right)^r\frac{N}{m}\\ &= \frac{N}{m} + \frac{(R+1)N}{\phi m}\left(\frac{\phi-1}{\phi}\right)^R - \frac{N}{m}\left(\frac{\phi-1}{\phi}\right)^R - \frac{NR}{\phi m}\left(\frac{\phi-1}{\phi}\right)^R\\ &= \frac{N}{m} - \frac{N}{m}\left(\frac{\phi-1}{\phi}\right)^{R+1}\\ &\le \frac{N}{m}. \end{aligned} \tag{9}$$

The results stated in Equation (9) suggests that if the sample allocation rule listed in Theorem 3 is used, processor s for $s=1,2,\ldots,m$ simulates fewer than $|\mathcal{I}_1^s|n_0 + N/m$ observations during the entire selection process. Notice that sets $\mathcal{I}_1^1,\mathcal{I}_1^2,\ldots,\mathcal{I}_1^m$ are mutually exclusive and $\cup_{s=1}^m \mathcal{I}_1^s = \mathcal{K}$. It implies that the total number of observations simulated by all processors is upper bounded by $\sum_{s=1}^m |\mathcal{I}_1^s|n_0 + N/m \le N_0 + N$. Therefore, we can conclude that the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure stops before all N_0+N observations are used. Then, we prove the first part of Theorem 3.

Proof of Part (1) Given that Assumptions 2 and 3 hold, we aim to show that for any positive constant $\eta_2 \ge 2\phi^2$, if $\phi \ge 2$ and $N/k \ge \eta_2$, the PCS of the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure can be lower bounded. To prove this, we first let $\hat{k}' = 2^R$. Notice that the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure equally allocates k alternatives to m processors. Each processor handles the selection of at most $\lceil k/m \rceil$ alternatives, and can identify the local best alternative in R rounds. For $1 \le r \le R$ and $s=1,2,\ldots,m$, at the beginning of round r in processor s, there are at most $\hat{k}'/2^{r-1}$ alternatives that are still in contention, i.e., $|\mathcal{I}_r^s| \le \hat{k}'/2^{r-1}$. Let s_k denote the processor where alternative k is assigned and k_r denote the alternative that competes with alternative k in round r in processor s_k. For $1 \le r \le R$, we define $\mathcal{E}_r$ as the event that alternative k eliminates alternative k_r in round r in processor s_k, i.e., $\mathcal{E}_r = \left\{\bar{X}_k^r \ge \bar{X}_{k_r}^r\right\}$. For $r=R+1$, we define $\mathcal{E}_r$ as the event that alternative k eliminates all other alternatives in $\mathcal{I}_{final}$, i.e., $\{\bar{X}_k^{R+1} \ge \max_{i\in\mathcal{I}_{final}\setminus\{k\}} \bar{X}_i^{R+1}\}$. For $1 \le r \le R+1$, we let $\mathcal{E}_r^c$ be the complement of event $\mathcal{E}_r$. Then, we can write the PCS of the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure as follows,

$$\text{PCS} = \mathbb{P}\left(\bigcap_{r=1}^{R+1}\mathcal{E}_r\right) = \mathbb{P}(\mathcal{E}_1)\prod_{r=2}^{R}\mathbb{P}(\mathcal{E}_r|\mathcal{E}_1,\ldots,\mathcal{E}_{r-1})\,\mathbb{P}(\mathcal{E}_{R+1}|\mathcal{E}_1,\ldots,\mathcal{E}_R). \tag{10}$$

For $1 \le r \le R$, we establish a lower bound for the sampling budget that should be allocated each alternative in round r in processor s_k as follows,

$$\begin{aligned}\mathcal{N}_r^{s_k} &= \left\lfloor \frac{N_r'}{|\mathcal{I}_r^{s_k}|}\right\rfloor\\ &= \left\lfloor \frac{r}{\phi(\phi-1)}\left(\frac{\phi-1}{\phi}\right)^r \frac{N}{m\,|\mathcal{I}_r^{s_k}|}\right\rfloor\end{aligned}$$

$$\geq \left\lfloor \frac{r}{\phi(\phi-1)} \left(\frac{\phi-1}{\phi}\right)^r \frac{\eta_2 k}{m \frac{\hat{k}'}{2^{r-1}}} \right\rfloor$$
$$= \left\lfloor \frac{r}{\phi(\phi-1)} \left(\frac{2(\phi-1)}{\phi}\right)^r \frac{\eta_2 k}{2m\hat{k}'} \right\rfloor$$
$$\geq r \left\lfloor \frac{\eta_2}{4\phi(\phi-1)} \left(\frac{2(\phi-1)}{\phi}\right)^r \right\rfloor, \tag{11}$$

where the first inequality holds because $N \geq \eta_2 k$ and $|\mathcal{I}_r^{s_k}| \leq \hat{k}'/2^{r-1}$ for $1 \leq r \leq R$. The second inequality holds because by the definition of $\hat{k}'$, $\hat{k}' < 2k/m$. Notice that $\eta_2 \geq 2\phi^2$ ensures that $\mathcal{N}_r^{s_k}$ is a positive integer no less than r. Then, based on the results listed in Equation (11), we can write the first two terms in Equation (10) as follows,

$$\mathbb{P}(\mathcal{E}_1) \prod_{r=2}^{R} \mathbb{P}(\mathcal{E}_r | \mathcal{E}_1, \ldots, \mathcal{E}_{r-1})$$
$$= (1 - \mathbb{P}(\mathcal{E}_1^c)) \prod_{r=2}^{R} (1 - \mathbb{P}(\mathcal{E}_r^c | \mathcal{E}_1, \ldots, \mathcal{E}_{r-1}))$$
$$= \left(1 - \mathbb{P}\left(\bar{X}_{k_1}^1 > \bar{X}_k^1\right)\right) \prod_{r=2}^{R} \left(1 - \mathbb{P}\left(\bar{X}_{k_r}^r > \bar{X}_k^r \middle| \mathcal{E}_1, \ldots, \mathcal{E}_{r-1}\right)\right)$$
$$= \left(1 - \mathbb{P}\left(\frac{\bar{X}_{k_1}^1 - \bar{X}_k^1 - \mu_{k_1} + \mu_k}{\frac{\sigma_{kk_1}}{\sqrt{\mathcal{N}_1^{s_k}}}} > \sqrt{\mathcal{N}_1^{s_k}} \frac{\mu_k - \mu_{k_1}}{\sigma_{kk_1}}\right)\right) \times$$
$$\prod_{r=2}^{R} \left(1 - \mathbb{P}\left(\frac{\bar{X}_{k_r}^r - \bar{X}_k^r - \mu_{k_r} + \mu_k}{\frac{\sigma_{kk_r}}{\sqrt{\mathcal{N}_r^{s_k}}}} > \sqrt{\mathcal{N}_r^{s_k}} \frac{\mu_k - \mu_{k_r}}{\sigma_{kk_r}} \middle| \mathcal{E}_1, \ldots, \mathcal{E}_{r-1}\right)\right)$$
$$\geq \left(1 - \mathbb{P}\left(Z > \sqrt{\left\lfloor \frac{\eta_2}{2\phi^2} \right\rfloor} \frac{\delta}{\sigma_{upper}}\right)\right) \prod_{r=2}^{R} \left(1 - \mathbb{P}\left(Z > \sqrt{r \left\lfloor \frac{\eta_2}{4\phi(\phi-1)} \left(\frac{2(\phi-1)}{\phi}\right)^r \right\rfloor} \frac{\delta}{\sigma_{upper}} \middle| \mathcal{E}_1, \ldots \mathcal{E}_{r-1}\right)\right)$$
$$\geq \left(1 - \mathbb{P}\left(Z > \sqrt{\left\lfloor \frac{\eta_2}{2\phi^2} \right\rfloor} \frac{\delta}{\sigma_{upper}}\right)\right) \prod_{r=2}^{R} \left(1 - \mathbb{P}\left(Z > \sqrt{r \left\lfloor \frac{\eta_2}{2\phi^2} \right\rfloor} \frac{\delta}{\sigma_{upper}} \middle| \mathcal{E}_1, \ldots \mathcal{E}_{r-1}\right)\right), \tag{12}$$

where the first inequality holds due to Equation (11) and Assumptions 2 and 3. The second inequality holds due to the fact that $[2(\phi-1)/\phi]^r \geq 2(\phi-1)/\phi$ when $r \geq 2$ and $\phi \geq 2$. Because event $\left\{Z > \sqrt{r \lfloor \eta_2/(2\phi^2) \rfloor} \delta/\sigma_{upper}\right\}$ is independent of events $\{\mathcal{E}_1, \ldots, \mathcal{E}_{r-1}\}$, Equation (12) yields

$$(12) = \prod_{r=1}^{R} \left(1 - \mathbb{P}\left(Z > \sqrt{r \left\lfloor \frac{\eta_2}{2\phi^2} \right\rfloor} \frac{\delta}{\sigma_{upper}}\right)\right)$$
$$\geq \prod_{r=1}^{\infty} \left(1 - \exp\left(-\frac{\lfloor \eta_2/(2\phi^2) \rfloor \delta^2}{2\sigma_{upper}^2} r\right)\right)$$
$$\geq \exp\left(-\frac{\pi^2 \sigma_{upper}^2}{3 \lfloor \eta_2/(2\phi^2) \rfloor \delta^2}\right)$$
$$\geq \exp\left(-\frac{4\phi^2 \pi^2 \sigma_{upper}^2}{3\eta_2 \delta^2}\right), \tag{13}$$

where the first inequality holds due to Lemma 1, and the second last inequality holds due to Lemma 2.

For processor $s=1,2,\dots,m$, $N'_{R+1}=\left\lfloor \frac{R+1}{\phi-1}\left(\frac{\phi-1}{\phi}\right)^{R+1}\frac{N}{m}\right\rfloor$ observations are generated for the local best alternative in round $R+1$. Let $\bar{X}_i^{R+1}$ be the sample mean of alternative $i\in\mathcal{I}_{final}$, calculated based on these N'_{R+1} observations. Then, we can write the third term in Equation (10) as follow,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{E}_{R+1}|\mathcal{E}_1,\dots,\mathcal{E}_R\right) &= \mathbb{P}\left(\bar{X}_k^{R+1} > \max_{i\in\mathcal{I}_{final}\setminus\{k\}} \bar{X}_i^{R+1} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&\geq 1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}_{final}\setminus\{k\}} \left\{\bar{X}_k^{R+1} < \bar{X}_i^{R+1}\right\} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&= 1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}_{final}\setminus\{k\}} \left\{\frac{\bar{X}_k^{R+1}-\bar{X}_i^{R+1}-\mu_k+\mu_i}{\sigma_{ki}/\sqrt{N'_{R+1}}} < \sqrt{N'_{R+1}}\frac{-\mu_k+\mu_i}{\sigma_{ki}}\right\} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&\geq 1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}_{final}\setminus\{k\}} \left\{\frac{\bar{X}_k^{R+1}-\bar{X}_i^{R+1}-\mu_k+\mu_i}{\sigma_{ki}/\sqrt{N'_{R+1}}} < \sqrt{N'_{R+1}}\frac{-\mu_k+\mu_i}{\sigma_{ki}}\right\} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&\geq 1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}_{final}\setminus\{k\}} \left\{\frac{\bar{X}_k^{R+1}-\bar{X}_i^{R+1}-\mu_k+\mu_i}{\sigma_{ki}/\sqrt{N'_{R+1}}} < \sqrt{N'_{R+1}}\frac{-\delta}{\sigma_{upper}}\right\} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&\geq 1-(m-1)\,\mathbb{P}\left(Z<\sqrt{N'_{R+1}}\frac{-\delta}{\sigma_{upper}} \middle| \mathcal{E}_1,\dots,\mathcal{E}_R\right)\\
&= 1-(m-1)\,\mathbb{P}\left(Z<\sqrt{N'_{R+1}}\frac{-\delta}{\sigma_{upper}}\right)\\
&\geq 1-(m-1)\exp\left(-\frac{N'_{R+1}\delta^2}{2\sigma_{upper}^2}\right),
\end{aligned}\tag{14}$$

where the last equality holds because event $\{Z<-\delta\sqrt{N'_{R+1}}/\sigma_{upper}\}$ is independent of events $\{\mathcal{E}_1,\dots,\mathcal{E}_R\}$ and the last inequality holds due to Lemma 3. For N'_{R+1}, we have,

$$\begin{aligned}
N'_{R+1} &= \left\lfloor \frac{R+1}{\phi-1}\left(\frac{\phi-1}{\phi}\right)^{R+1}\frac{N}{m}\right\rfloor\\
&\geq \left\lfloor \frac{\left(\log_2\frac{k}{m}+1\right)\eta_2 k}{m(\phi-1)}\left(\frac{\phi-1}{\phi}\right)^{\log_2 k/m+2}\right\rfloor\\
&= \left\lfloor \frac{\left(\log_2\frac{k}{m}+1\right)\eta_2 k}{m(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^{\log_2 k/m+2}\frac{m}{4k}\right\rfloor\\
&\geq \frac{\left(\log_2\frac{k}{m}+1\right)\eta_2}{8(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^{\log_2 k/m+2}.
\end{aligned}\tag{15}$$

Plugging the results in Equation (15) into Equation (14) yields,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{E}_{R+1}|\mathcal{E}_1,\dots,\mathcal{E}_R\right) &\geq 1-(m-1)\exp\left(-\frac{N'_{R+1}\delta^2}{2\sigma_{upper}^2}\right)\\
&\geq 1-(m-1)\exp\left(-\frac{\left(\log_2\frac{k}{m}+1\right)\eta_2\delta^2}{8\sigma_{upper}^2\phi}\left(\frac{2(\phi-1)}{\phi}\right)^{\log_2 k/m+1}\right).
\end{aligned}\tag{16}$$

Therefore, with the results listed in Equations (13) and (16), we can rewrite Equation (10) as,

$$\begin{aligned}\text{PCS} &= \mathbb{P}\left(\mathcal{E}_1\right)\prod_{r=2}^{R}\mathbb{P}\left(\mathcal{E}_r|\mathcal{E}_1,\ldots,\mathcal{E}_{r-1}\right)\mathbb{P}\left(\mathcal{E}_{R+1}|\mathcal{E}_1,\ldots,\mathcal{E}_R\right)\\ &\geq \exp\left(-\frac{4\phi^2\pi^2\sigma_{upper}^2}{3\eta_2\delta^2}\right)\left(1-(m-1)\exp\left(-\frac{\left(\log_2\frac{k}{m}+1\right)\eta_2\delta^2}{8\sigma_{upper}^2\phi}\left(\frac{2\left(\phi-1\right)}{\phi}\right)^{\log_2 k/m+1}\right)\right).\end{aligned}$$

It concludes the proof of part (1).

Proof of Part (2) Given that Assumptions 3 holds, we aim to show that for any positive constant $\eta_2'\geq 2\phi^2$ and IZ parameter δ, if $\phi\geq 3$ and $N/k\geq\eta_2'$, the PGS of the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure can be lower bounded. To prove this, we first let $\hat{k}'=2^R$, and

$$\delta_r=\frac{\sqrt{2\left(\phi-1\right)}-\sqrt{\phi}}{\sqrt{\phi}}\left(\frac{\phi}{2\left(\phi-1\right)}\right)^{\frac{r}{2}}\delta,$$

for $r\geq 1$. Because $\phi\geq 3$, $\delta_r>0$ for $r\geq 1$ and $\sum_{r=1}^{\infty}\delta_r=\delta$. Notice that, each processor can identify the local best alternative in R rounds and the procedure selects the final output in round $R+1$. For $1\leq r\leq R$, we let $\rho_r=\arg\max_{i\in\cup_{s=1}^m\mathcal{I}_s^r}\mu_i$, and define $\mathfrak{s}_r$ as the processor that contains alternative ρ_r. For $r=R+1$, we let $\rho_r=\arg\max_{i\in\mathcal{I}_{final}}\mu_i$. To ease the notation, we let $\rho_r=\varphi$ for $r=R+2$, where φ is the index of the alternative which is finally selected by the procedure. For $1\leq r\leq R$, we further define event $\mathcal{E}_r'=\{\mu_{\rho_r}-\mu_{\rho_r'}\leq\delta_r\}\cup\{\mu_{\rho_r}-\mu_{\rho_r'}>\delta_r \text{ and } \bar{X}_{\rho_r}^r\geq\bar{X}_{\rho_r'}^r\}$, where ρ_r' is the index of the alternative that competes with alternative ρ_r in processor $\mathfrak{s}_r$. For $r=R+1$, we define event $\mathcal{E}_r'=\cap_{i\in\mathcal{I}_{final}\setminus\{\rho_r\}}\left\{\{\mu_{\rho_r}-\mu_i\leq\delta_r\}\cup\{\mu_{\rho_r}-\mu_i>\delta_r \text{ and } \bar{X}_{\rho_r}^r\geq\bar{X}_i^r\}\right\}$. By letting $\mathcal{E}_r'^c$ be the complement of event $\mathcal{E}_r'$ for $1\leq r\leq R+1$, we can bound the PGS of the $\mathcal{FBKT}^{\mathcal{S}+}$ procedure as follows,

$$\text{PGS}\geq\mathbb{P}\left(\bigcap_{r=1}^{R+1}\left\{\mu_{\rho_r}-\mu_{\rho_{r+1}}\leq\delta_r\right\}\right)\geq\mathbb{P}\left(\bigcap_{r=1}^{R+1}\left\{\mathcal{E}_r'\right\}\right)=\mathbb{P}\left(\mathcal{E}_1'\right)\prod_{r=2}^{R}\mathbb{P}\left(\mathcal{E}_r'\middle|\mathcal{E}_1',\ldots,\mathcal{E}_{r-1}'\right)\mathbb{P}\left(\mathcal{E}_{R+1}'\middle|\mathcal{E}_1',\ldots,\mathcal{E}_R'\right),\tag{17}$$

where the first inequality holds because as long as the mean difference between alternatives ρ_r and ρ_{r+1} is less than δ_r for $1\leq r\leq R+1$ the mean difference between alternatives k and φ is less than $\sum_{r=1}^{\infty}\delta_r=\delta$, and the second inequality holds because event $\mathcal{E}_r'$ implies event $\{\mu_{\rho_r}-\mu_{\rho_{r+1}}\leq\delta_r\}$ for $1\leq r\leq R+1$. Notice that following the same arguments used in the analysis in Equation (11), we can conclude that

$$\mathcal{N}_r^{\mathfrak{s}_r}\geq r\left\lfloor\frac{\eta_2'}{4\phi\left(\phi-1\right)}\left(\frac{2\left(\phi-1\right)}{\phi}\right)^r\right\rfloor.\tag{18}$$

Because $\eta_2'\geq 2\phi^2$, $\mathcal{N}_r^{\mathfrak{s}_r}$ is a positive integer no less than r. Then, based on the results stated in Equation (18), we can rewrite the first two terms in Equation (17) as follows,

$$\mathbb{P}\left(\mathcal{E}_1'\right)\prod_{r=2}^{R}\mathbb{P}\left(\mathcal{E}_r'\middle|\mathcal{E}_1',\ldots,\mathcal{E}_{r-1}'\right)$$

$$
\begin{aligned}
&= \left(1-\mathbb{P}\left(\mathcal{E}_1'^c\right)\right)\prod_{r=2}^{R}\left(1-\mathbb{P}\left(\mathcal{E}_r'^c\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right)\\
&= \left(1-\mathbb{P}\left(\mu_{\rho_1}-\mu_{\rho_1'}>\delta_1 \text{ and } \bar{X}^1_{\rho_1}<\bar{X}^1_{\rho_1'}\right)\right)\times\\
&\prod_{r=2}^{R}\left(1-\mathbb{P}\left(\mu_{\rho_r}-\mu_{\rho_r'}>\delta_r \text{ and } \bar{X}^r_{\rho_r}<\bar{X}^r_{\rho_r'}\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right)\\
&= \left(1-\mathbb{P}\left(\mu_{\rho_1}-\mu_{\rho_1'}>\delta_1 \text{ and } \frac{-\bar{X}^1_{\rho_1}+\bar{X}^1_{\rho_1'}+\mu_{\rho_1}-\mu_{\rho_1'}}{\sigma_{\rho_1\rho_1'}/\sqrt{\mathcal{N}_1^{\mathfrak{s}_1}}}>\sqrt{\mathcal{N}_1^{\mathfrak{s}_1}}\frac{\mu_{\rho_1}-\mu_{\rho_1'}}{\sigma_{\rho_1\rho_1'}}\right)\right)\times\\
&\prod_{r=2}^{R}\left(1-\mathbb{P}\left(\mu_{\rho_r}-\mu_{\rho_r'}>\delta_r \text{ and } \frac{-\bar{X}^r_{\rho_r}+\bar{X}^r_{\rho_r'}+\mu_{\rho_r}-\mu_{\rho_r'}}{\sigma_{\rho_r\rho_r'}/\sqrt{\mathcal{N}_r^{\mathfrak{s}_r}}}>\sqrt{\mathcal{N}_r^{\mathfrak{s}_r}}\frac{\mu_{\rho_r}-\mu_{\rho_r'}}{\sigma_{\rho_r\rho_r'}}\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right)\\
&\ge \left(1-\mathbb{P}\left(\mu_{\rho_1}-\mu_{\rho_1'}>\delta_1 \text{ and } Z>\sqrt{\left\lfloor\frac{\eta_2'}{2\phi^2}\right\rfloor}\frac{\delta_1}{\sigma_{upper}}\right)\right)\times\\
&\prod_{r=2}^{R}\left(1-\mathbb{P}\left(\mu_{\rho_r}-\mu_{\rho_r'}>\delta_r \text{ and } Z>\sqrt{r\left\lfloor\frac{\eta_2'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right)\\
&\ge \left(1-\mathbb{P}\left(Z>\sqrt{\left\lfloor\frac{\eta_2'}{2\phi^2}\right\rfloor}\frac{\delta_1}{\sigma_{upper}}\right)\right)\prod_{r=2}^{R}\left(1-\mathbb{P}\left(Z>\sqrt{r\left\lfloor\frac{\eta_2'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right)\\
&= \prod_{r=1}^{R}\left(1-\mathbb{P}\left(Z>\sqrt{r\left\lfloor\frac{\eta_2'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor}\frac{\delta_r}{\sigma_{upper}}\middle|\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\right)\right), \qquad (19)
\end{aligned}
$$

where the last inequality holds because event $\{Z>\sqrt{r\lfloor\eta_2'/(4\phi(\phi-1))(2(\phi-1)/\phi)^r\rfloor}\delta_r/\sigma_{upper}\}$ is independent of events $\{\mathcal{E}_1',\dots,\mathcal{E}_{r-1}'\}$. Then, applying Lemmas 2 and 3 to Equation (19), we have,

$$
\begin{aligned}
(19) &\ge \prod_{r=1}^{R}\left(1-\exp\left(-r\left\lfloor\frac{\eta_2'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor\frac{\delta_r^2}{2\sigma_{upper}^2}\right)\right)\\
&= \prod_{r=1}^{\infty}\left(1-\exp\left(-r\left\lfloor\frac{\eta_2'}{4\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\right\rfloor\frac{\left(\frac{\sqrt{2(\phi-1)}-\sqrt{\phi}}{\sqrt{\phi}}\right)^2\left(\frac{\phi}{2(\phi-1)}\right)^r\delta^2}{2\sigma_{upper}^2}\right)\right)\\
&\ge \prod_{r=1}^{\log_2\hat{k}}\left(1-\exp\left(-r\frac{\eta_2'}{8\phi(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^r\frac{\left(\frac{\sqrt{2(\phi-1)}-\sqrt{\phi}}{\sqrt{\phi}}\right)^2\left(\frac{\phi}{2(\phi-1)}\right)^r\delta^2}{2\sigma_{upper}^2}\right)\right)\\
&= \prod_{r=1}^{\log_2\hat{k}}\left(1-\exp\left(-r\frac{\eta_2'\delta^2}{8\sigma_{upper}^2}\left(\frac{1}{\phi}-\frac{1}{\sqrt{2\phi(\phi-1)}}\right)^2\right)\right)\\
&\ge \exp\left(-\frac{8\pi^2\sigma_{upper}^2\phi^3(\phi-1)}{3\eta_2'\delta^2\left(\sqrt{2\phi(\phi-1)}-\phi\right)^2}\right), \qquad (20)
\end{aligned}
$$

where the first inequality holds due to Lemma 3, the second inequality holds because if $a\ge 1$, then $\lfloor a\rfloor \ge a/2$, and the third inequality holds due to Lemma 2.

For processor $s=1,2,\dots,m$, $N'_{R+1}=\left\lfloor \frac{R+1}{\phi-1}\left(\frac{\phi-1}{\phi}\right)^{R+1}\frac{N}{m}\right\rfloor$ observations are generated for the local best alternative in round $R+1$. Let $\bar{X}_i^{R+1}$ be the sample mean of alternative $i\in\mathcal{I}_{final}$ calculated based on these N'_{R+1} observations. Then, by letting $\mathcal{I}'_{final}=\mathcal{I}_{final}\backslash\{\rho_{R+1}\}$, we can write the third term in Equation (17) as,

$$
\begin{aligned}
&\mathbb{P}\left(\mathcal{E}'_{R+1}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&=\mathbb{P}\left(\bigcap_{i\in\mathcal{I}'_{final}}\left\{\{\mu_{\rho_{R+1}}-\mu_i\le\delta_{R+1}\}\cup\{\mu_{\rho_{R+1}}-\mu_i>\delta_{R+1}\text{ and }\bar{X}^{R+1}_{\rho_{R+1}}\ge\bar{X}_i^{R+1}\}\right\}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&=1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}'_{final}}\left\{\mu_{\rho_{R+1}}-\mu_i>\delta_{R+1}\text{ and }\bar{X}^{R+1}_{\rho_{R+1}}<\bar{X}_i^{R+1}\right\}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&=1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}'_{final}}\left\{\mu_{\rho_{R+1}}-\mu_i>\delta_{R+1}\text{ and }\frac{-\bar{X}^{R+1}_{\rho_{R+1}}+\bar{X}_i^{R+1}+\mu_{\rho_{R+1}}-\mu_i}{\sigma_{\rho_{R+1}i}/\sqrt{N'_{R+1}}}>\sqrt{N'_{R+1}}\frac{\mu_{\rho_{R+1}}-\mu_i}{\sigma_{\rho_{R+1}i}}\right\}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&\ge1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}'_{final}}\left\{\mu_{\rho_{R+1}}-\mu_i>\delta_{R+1}\text{ and }\frac{-\bar{X}^{R+1}_{\rho_{R+1}}+\bar{X}_i^{R+1}+\mu_{\rho_{R+1}}-\mu_i}{\sigma_{\rho_{R+1}i}/\sqrt{N'_{R+1}}}>\sqrt{N'_{R+1}}\frac{\delta_{R+1}}{\sigma_{upper}}\right\}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&\ge1-\mathbb{P}\left(\bigcup_{i\in\mathcal{I}'_{final}}\left\{\frac{-\bar{X}^{R+1}_{\rho_{R+1}}+\bar{X}_i^{R+1}+\mu_{\rho_{R+1}}-\mu_i}{\sigma_{\rho_{R+1}i}/\sqrt{N'_{R+1}}}>\sqrt{N'_{R+1}}\frac{\delta_{R+1}}{\sigma_{upper}}\right\}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&\ge1-(m-1)\,\mathbb{P}\left(Z>\sqrt{N'_{R+1}}\frac{\delta_{R+1}}{\sigma_{upper}}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&=1-(m-1)\,\mathbb{P}\left(Z>\sqrt{N'_{R+1}}\frac{\delta_{R+1}}{\sigma_{upper}}\right)\\
&\ge1-(m-1)\exp\left(-\frac{N'_{R+1}\delta^2_{R+1}}{2\sigma^2_{upper}}\right),
\end{aligned}
\tag{21}
$$

where the second equality holds due to De Morgan's Law, and the last inequality holds due to Lemma 3. According the results listed in Equation (15), for N'_{R+1}, we have,

$$
N'_{R+1}\ge\frac{\left(\log_2\frac{k}{m}+1\right)\eta'_2}{8(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^{\log_2 k/m+2}.
\tag{22}
$$

Plugging the results in Equation (22) into Equation (21), we have

$$
\begin{aligned}
&\mathbb{P}\left(\mathcal{E}'_{R+1}\middle|\mathcal{E}'_1,\dots,\mathcal{E}'_R\right)\\
&\ge1-(m-1)\exp\left(-\frac{N'_{R+1}\delta^2_{R+1}}{2\sigma^2_{upper}}\right)\\
&\ge1-(m-1)\exp\left(-\frac{\frac{\left(\log_2\frac{k}{m}+1\right)\eta'_2}{8(\phi-1)}\left(\frac{2(\phi-1)}{\phi}\right)^{\log_2 k/m+2}\left(\frac{\sqrt{2(\phi-1)}-\sqrt{\phi}}{\sqrt{\phi}}\right)^2\left(\frac{\phi}{2(\phi-1)}\right)^{\log_2 k/m+2}\delta^2}{2\sigma^2_{upper}}\right)
\end{aligned}
$$

$$= 1 - (m-1)\exp\left(-\frac{\left(\log_2 \frac{k}{m}+1\right)\eta_2'\left(3\phi - 2 + 2\sqrt{2\phi\left(\phi-1\right)}\right)}{16\sigma_{upper}^2\phi\left(\phi-1\right)}\delta^2\right). \tag{23}$$

Therefore, with the results listed in Equations (20) and (23), we can rewrite Equation (17) as

$$\begin{aligned}
\text{PGS} &\geq \mathbb{P}\left(\mathcal{E}_1'\right)\prod_{r=2}^{R}\mathbb{P}\left(\mathcal{E}_r'\middle|\mathcal{E}_1',\ldots,\mathcal{E}_{r-1}'\right)\mathbb{P}\left(\mathcal{E}_{R+1}'\middle|\mathcal{E}_1',\ldots,\mathcal{E}_R'\right)\\
&\geq \exp\left(-\frac{8\pi^2\sigma_{upper}^2\phi^3\left(\phi-1\right)}{3\eta_2'\delta^2\left(\sqrt{2\phi\left(\phi-1\right)}-\phi\right)^2}\right)\left(1-(m-1)\exp\left(-\frac{\left(\log_2\frac{k}{m}+1\right)\eta_2'\left(3\phi-2+2\sqrt{2\phi\left(\phi-1\right)}\right)}{16\sigma_{upper}^2\phi\left(\phi-1\right)}\delta^2\right)\right).
\end{aligned}$$

It concludes the proof of part (2). $\square$