Online Supplement to Finding Feasible Systems for Subjective Constraints Using Recycled Observations

Yuwei Zhou

Sigrún Andradóttir

Seong-Hee Kim
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332, USA, yzhou332@gatech.edu, sigrun.andradottir@isye.gatech.edu, seonghee.kim@isye.gatech.edu

Chuljin Park
Department of Industrial Engineering
Hanyang University
Seoul, 04763, South Korea, parkcj@hanyang.ac.kr

Online Supplements EC.1, EC.2, and EC.3 provide the algorithm statements and statistical validity proofs for the three restart procedures, namely Restart$^{\text{prod}}$, Restart$^{\text{sum}}$, and Restart$^{\text{max}}$, that we compared with $\mathcal{RF}$. We give the algorithmic statement and proof of Recycle$^{\mathcal{B}}$ in Online Supplement EC.4. We discuss the existence and uniqueness of the algorithm parameter η in Online Supplement EC.5. Online Supplement EC.6 proves Lemma 1 for $c=\infty$. We also provide a discussion about the implementation parameters for systems when multiple constraints are present in Online Supplement EC.7, and a discussion about the comparison between our proposed procedures and alternative procedures in Online Supplement EC.8.

EC.1. Algorithm Statement and Proof of Statistical Validity for the Restart$^{\text{prod}}$ Procedure

In this section, we provide the full description of the Restart$^{\text{prod}}$ procedure and discuss its statistical validity. The full description of Restart$^{\text{prod}}$ is presented in Algorithm S.1.

Notice that Restart$^{\text{prod}}$ performs feasibility check for each combination of thresholds and each combination has one threshold on each constraint. Thus, unlike $\mathcal{RF}$, Recycle$^{\mathcal{B}}$, and Restart$^{\text{sum}}$, Restart$^{\text{prod}}$ has only one way of setting the implementation parameter β_ℓ for all $\ell = 1, \ldots, s$.

We now show the statistical validity of the Restart$^{\text{prod}}$ by proving the following theorem.

Theorem 1. *Procedure Restart*$^{\text{prod}}$ *guarantees* $\text{PCD} \geq 1-\alpha$.

Algorithm S.1 Procedure Restart$^{\text{prod}}$

[**Setup**:] Choose confidence level $1-\alpha$, tolerance level ϵ_ℓ, and thresholds $\{q_{\ell 1}, q_{\ell 2}, \dots, q_{\ell d_\ell}\}$ for constraint $\ell = 1, \dots, s$. Choose the value of c and set $\Theta = \{1, 2, \dots, k\}$. Set $D = \prod_{\ell=1}^{s} d_\ell$ and formulate threshold vectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \dots, \boldsymbol{q}^{(D)}$, where $d = 1, \dots, D$, as discussed in Section 3. Set η such that $g(\eta) = \beta$, where

$$\beta = \begin{cases} [1-(1-\alpha)^{1/(kD)}]/s & \text{when systems are independent,} \\ [1-(1-\alpha)^{1/D}]/(ks) & \text{when systems are dependent.} \end{cases}$$

.

for each system $i \in \Theta$ **do**
  for $d = 1, \dots, D$ **do**
    [**Initialization**:]
    - Set $r_i = n_0$ and $\text{ON} = \{1, \dots, s\}$.
    - Obtain n_0 observations $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \dots, Y_{i\ell n_0}^{(d)}$ for $\ell \in \text{ON}$.
    - Compute the sample variance of $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \dots, Y_{i\ell n_0}^{(d)}$ as $S_{i\ell d}^2(n_0)$ for $\ell \in \text{ON}$.

    [**Feasibility Check**:]
    for $\ell \in \text{ON}$ **do**
     If $\sum_{n=1}^{r_i}(Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \geq R(r_i; \epsilon_\ell, \eta, S_{i\ell d}^2(n_0))$, set $Z_{i\ell}^{(d)} = 0$ and $\text{ON} = \text{ON} \setminus \{\ell\}$;
     Else if $\sum_{n=1}^{r_i}(Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \leq -R(r_i; \epsilon_\ell, \eta, S_{i\ell d}^2(n_0))$, set $Z_{i\ell}^{(d)} = 1$ and $\text{ON} = \text{ON} \setminus \{\ell\}$.
    end for
    [**Stopping Condition:**]
     If $\text{ON} = \emptyset$, return $Z_{i\ell}^{(d)}$ for $\ell = 1, \dots, s$. Otherwise, set $r_i = r_i + 1$, take one additional observation $Y_{i\ell r_i}^{(d)}$ for $\ell \in \text{ON}$ and go to [**Feasibility Check**].
  end for
end for

Proof. We let $\boldsymbol{q}^{(d)} = (q_1^{(d)}, \dots, q_s^{(d)})$, where $d = 1, \dots, D$, denote the dth combination of the thresholds. Then for one particular threshold vector $\boldsymbol{q}^{(d)}$, we have the probability of correct decision for system i as

$$\Pr\left(\cap_{\ell=1}^{s} \text{CD}_{i\ell}(q_\ell^{(d)})\right) \geq 1 - \sum_{\ell=1}^{s} \beta_\ell, \tag{1}$$

where the inequality holds due to the Bonferroni inequality and Lemma 1. Restart$^{\text{prod}}$ performs feasibility check for each combination of thresholds independently. We then consider the cases when systems are simulated independently or with correlation.

If the systems are simulated independently, then equation (1) yields

$$\begin{aligned} \text{PCD} = \Pr\left(\cap_{d=1}^{D} \cap_{i=1}^{k} \cap_{\ell=1}^{s} \text{CD}_{i\ell}(q_\ell^{(d)})\right) &\geq \prod_{d=1}^{D}\prod_{i=1}^{k}\left(1 - \sum_{\ell=1}^{s} \beta_\ell\right) \\ = \prod_{d=1}^{D}\prod_{i=1}^{k}\left(1 - \sum_{\ell=1}^{s}\left[1-(1-\alpha)^{1/(kD)}\right]/s\right) &= \prod_{d=1}^{D}\prod_{i=1}^{k}(1-\alpha)^{1/(kD)} \end{aligned}$$

$$= 1-\alpha.$$

If the systems are simulated with CRN, then the Bonferroni inequality and Lemma 1 yields

$$\begin{aligned}
\mathrm{PCD} &= \Pr\left(\cap_{d=1}^{D}\cap_{i=1}^{k}\cap_{\ell=1}^{s}\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) = \prod_{d=1}^{D}\Pr\left(\cap_{i=1}^{k}\cap_{\ell=1}^{s}\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right)\\
&\geq \prod_{d=1}^{D}\left[1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\Pr(\mathrm{ICD}_{i\ell}(q_\ell^{(d)}))\right] = \prod_{d=1}^{D}\left[1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\beta_\ell\right]\\
&= \prod_{d=1}^{D}\left(1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\left[1-(1-\alpha)^{1/D}\right]/(ks)\right) = 1-\alpha. \quad \square
\end{aligned}$$

EC.2. Algorithm Statement and Proof of Statistical Validity for the Restart$^{\text{sum}}$ Procedure

In this section, we provide the full description of the Restart$^{\text{sum}}$ procedure and discuss its statistical validity. Restart$^{\text{sum}}$ is motivated by the following considerations (relative to Restart$^{\text{prod}}$). Restart$^{\text{sum}}$ makes feasibility decisions iteratively for each threshold value of each constraint, while Restart$^{\text{prod}}$ may make multiple decisions for each such threshold value (because Restart$^{\text{prod}}$ determines feasibility for each combination of threshold values of all constraints). Therefore, Restart$^{\text{sum}}$ usually performs fewer restarts than Restart$^{\text{prod}}$, and thus usually needs fewer observations compared with Restart$^{\text{prod}}$. The full description of Restart$^{\text{sum}}$ is provided in Algorithm S.2.

We show the statistical validity of the Restart$^{\text{sum}}$ procedure based on the following theorem.

THEOREM 2. *Procedure* Restart$^{\text{sum}}$ *guarantees* $\mathrm{PCD}\geq 1-\alpha$.

Proof. For the Restart$^{\text{sum}}$ procedure, as we restart a feasibility check for each system, each constraint, and each threshold value, the $\mathrm{CD}_{i\ell}(q_\ell^{(d)})$ events are independent from $\mathrm{CD}_{i\ell}(q_\ell^{(d')})$ for all $\ell = 1,\ldots,s$ and $d\neq d'$. We prove the theorem based on whether systems are simulated independently or with correlation.

If the systems are simulated independently, then with β_ℓ from the first choice (i) of Algorithm S.2, Lemma 1 yields

$$\mathrm{PCD} = \Pr\left(\cap_{i=1}^{k}\cap_{\ell=1}^{s}\cap_{d=1,q_\ell^{(d)}\neq\mathrm{NC}}^{D}\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) = \prod_{i=1}^{k}\prod_{\ell=1}^{s}\prod_{d=1,q_\ell^{(d)}\neq\mathrm{NC}}^{D}\Pr\left(\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right)$$

Algorithm S.2 Procedure Restart$^{\text{sum}}$

[**Setup**:] Choose confidence level $1-\alpha$, tolerance level ϵ_ℓ, and thresholds $\{q_{\ell 1}, q_{\ell 2}, \ldots, q_{\ell d_\ell}\}$ for constraint $\ell = 1, \ldots, s$. Choose the value of c and set $\Theta = \{1, 2, \ldots, k\}$. Set $D = \sum_{\ell=1}^{s} d_\ell$ and formulate threshold vectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \ldots, \boldsymbol{q}^{(D)}$, where $d = 1, \ldots, D$, as discussed in Section 3. For $\ell = 1, \ldots, s$, set η_ℓ such that $g(\eta_\ell) = \beta_\ell$, where either

(i)

$$\beta_\ell = \begin{cases} 1 - \sqrt[ksd_\ell]{1-\alpha}, & \text{when systems are independent,} \\ (1 - \sqrt[sd_\ell]{1-\alpha})/k, & \text{when systems are dependent,} \end{cases}$$

or

(ii)

$$\beta_\ell = \begin{cases} 1 - \sqrt[kD]{1-\alpha}, & \text{when systems are independent,} \\ (1 - \sqrt[D]{1-\alpha})/k, & \text{when systems are dependent.} \end{cases}$$

for each system $i \in \Theta$ **do**
  for each threshold $d = 1, \ldots, D$ **do**
    [**Initialization**:]
    - Set $r_i = n_0$ and $\text{ON} = \{\ell = 1, \ldots, s | q_\ell^{(d)} \neq \text{NC}\}$.
    - Obtain n_0 observations $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \ldots, Y_{i\ell n_0}^{(d)}$ for $\ell \in \text{ON}$.
    - Compute the sample variance of $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \ldots, Y_{i\ell n_0}^{(d)}$ as $S_{i\ell d}^2(n_0)$ for $\ell \in \text{ON}$.

    [**Feasibility Check**:]
    for $\ell \in \text{ON}$ **do**
      If $\sum_{n=1}^{r_i} (Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \geq R(r_i; \epsilon_\ell, \eta_\ell, S_{i\ell d}^2(n_0))$, return $Z_{i\ell}^{(d)} = 0$;
      Else if $\sum_{n=1}^{r_i} (Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \leq -R(r_i; \epsilon_\ell, \eta_\ell, S_{i\ell d}^2(n_0))$, return $Z_{i\ell}^{(d)} = 1$;
      Else, set $r_i = r_i + 1$, take one additional observation $Y_{i\ell r_i}^{(d)}$ and go to [**Feasibility Check**].
    end for
  end for
end for

$$\geq \prod_{i=1}^{k} \prod_{\ell=1}^{s} \prod_{d=1, q_\ell^{(d)} \neq \text{NC}}^{D} (1-\beta_\ell) = \prod_{i=1}^{k} \prod_{\ell=1}^{s} \prod_{m=1}^{d_\ell} (1-\beta_\ell) = \prod_{i=1}^{k} \prod_{\ell=1}^{s} (1-\beta_\ell)^{d_\ell}$$

$$= \prod_{i=1}^{k} \prod_{\ell=1}^{s} \left(1 - (1 - \sqrt[ksd_\ell]{1-\alpha})\right)^{d_\ell} = \prod_{i=1}^{k} \prod_{\ell=1}^{s} \sqrt[ks]{1-\alpha} = 1-\alpha.$$

With β_ℓ from the second choice (ii) of Algorithm S.2, Lemma 1 yields

$$\text{PCD} = \Pr\left(\cap_{i=1}^{k} \cap_{d=1}^{D} \cap_{\ell=1, q_\ell^{(d)} \neq \text{NC}}^{s} \text{CD}_{i\ell}\left(q_\ell^{(d)}\right)\right) = \prod_{i=1}^{k} \prod_{d=1}^{D} \prod_{\ell=1, q_\ell^{(d)} \neq \text{NC}}^{s} \Pr\left(\text{CD}_{i\ell}\left(q_\ell^{(d)}\right)\right)$$

$$\geq \prod_{i=1}^{k} \prod_{d=1}^{D} \prod_{\ell=1, q_\ell^{(d)} \neq \text{NC}}^{s} (1-\beta_\ell) = \prod_{i=1}^{k} \prod_{d=1}^{D} (1-\beta_\ell) = \prod_{i=1}^{k} \prod_{d=1}^{D} \left(1 - (1 - \sqrt[kD]{1-\alpha})\right)$$

$$= \left(\sqrt[kD]{1-\alpha}\right)^{kD} = 1-\alpha.$$

If the systems are simulated with CRN, then with β_ℓ from the first choice (i) of Algorithm S.2, the Bonferroni inequality yield and Lemma 1 yield

$$\begin{aligned}\mathrm{PCD} &= \Pr\left(\cap_{i=1}^{k}\cap_{\ell=1}^{s}\cap_{d=1,q_\ell^{(d)}\neq \mathrm{NC}}^{D}\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) \geq \prod_{\ell=1}^{s}\prod_{d=1,q_\ell^{(d)}\neq \mathrm{NC}}^{D}\left(1-\sum_{i=1}^{k}\Pr(\mathrm{ICD}_{i\ell}(q_\ell^{(d)}))\right)\\ &\geq \prod_{\ell=1}^{s}\prod_{d=1,q_\ell^{(d)}\neq \mathrm{NC}}^{D}\left(1-\sum_{i=1}^{k}\beta_\ell\right) = \prod_{\ell=1}^{s}\prod_{m=1}^{d_\ell}\left(1-(1-\sqrt[sd_\ell]{1-\alpha})\right) = \prod_{\ell=1}^{s}\sqrt[s]{1-\alpha} = 1-\alpha.\end{aligned}$$

With β_ℓ from the second choice (ii) of Algorithm S.2, the Bonferroni inequality and Lemma 1 yield

$$\begin{aligned}\mathrm{PCD} &= \Pr\left(\cap_{i=1}^{k}\cap_{d=1}^{D}\cap_{\ell=1,q_\ell^{(d)}\neq \mathrm{NC}}^{s}\mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) \geq \prod_{d=1}^{D}\prod_{\ell=1,q_\ell^{(d)}\neq \mathrm{NC}}^{s}\left(1-\sum_{i=1}^{k}\Pr(\mathrm{ICD}_{i\ell}(q_\ell^{(d)}))\right)\\ &\geq \prod_{d=1}^{D}\prod_{\ell=1,q_\ell^{(d)}\neq \mathrm{NC}}^{s}\left(1-\sum_{i=1}^{k}\beta_\ell\right) = \prod_{d=1}^{D}\left(1-\sum_{i=1}^{k}\beta_\ell\right) = \prod_{d=1}^{D}\left(1-(1-\sqrt[D]{1-\alpha})\right)\\ &= \left(\sqrt[D]{1-\alpha}\right)^{D} = 1-\alpha. \quad \square\end{aligned}$$

EC.3. Algorithm Statement and Proof of Statistical Validity for the Restart$^{\max}$ Procedure

In this section, we provide the full description of the Restart$^{\max}$ procedure and discuss its statistical validity. Restart$^{\max}$ performs feasibility checks for each threshold vector which is formed by choosing one threshold from each constraint in a pre-defined order. Once the feasibility of all thresholds on some constraints is determined, the threshold vector is formed based on the thresholds from the remaining constraints by omitting the constraints whose thresholds have all received feasibility decisions. The procedure terminates when all the thresholds on all the constraints have received their feasibility decisions. Therefore, Restart$^{\max}$ requires $\max_{\ell=1,\dots,s} d_\ell$ restarts and performs better compared with Restart$^{\mathrm{prod}}$ and Restart$^{\mathrm{sum}}$. Similar to Restart$^{\mathrm{prod}}$, each restart of Restart$^{\max}$ involves one threshold on each remaining constraint. Therefore, Restart$^{\max}$ only has one way of setting the implementation parameter β_d where $d = 1\dots,d$. The full description of Restart$^{\max}$ is provided in Algorithm S.3.

We show the statistical validity of the Restart$^{\max}$ procedure in the following theorem.

THEOREM 3. *Procedure* Restart$^{\max}$ *guarantees* $\mathrm{PCD} \geq 1-\alpha$.

Algorithm S.3 Procedure Restart$^{\max}$

[**Setup**:] Choose confidence level $1-\alpha$, tolerance level ϵ_ℓ, and thresholds $\{q_{\ell 1}, q_{\ell 2}, \ldots, q_{\ell d_\ell}\}$ for constraint $\ell = 1, \ldots, s$. Choose the value of c and set $\Theta = \{1, 2, \ldots, k\}$. Set $D = \max_{\ell=1,\ldots,s} d_\ell$ and formulate threshold vectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \ldots, \boldsymbol{q}^{(D)}$, where $d = 1, \ldots, D$, as discussed in Section 3. For $d = 1, \ldots, D$, set η_d such that $g(\eta_d) = \beta_d$, where

$$\beta_d = \begin{cases} (1 - \sqrt[kD]{1-\alpha}) / \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d), & \text{when systems are independent,} \\ (1 - \sqrt[D]{1-\alpha}) / (k \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d)), & \text{when systems are dependent.} \end{cases} \tag{2}$$

for each system $i \in \Theta$ **do**
  for $d = 1, \ldots, D$ **do**
    [**Initialization**:]
    - Set $r_i = n_0$ and $\mathrm{ON} = \{\ell = 1, \ldots, s | q_\ell^{(d)} \neq \mathrm{NC}\}$.
    - Obtain n_0 observations $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \ldots, Y_{i\ell n_0}^{(d)}$ for $\ell \in \mathrm{ON}$.
    - Compute the sample variance of $Y_{i\ell 1}^{(d)}, Y_{i\ell 2}^{(d)}, \ldots, Y_{i\ell n_0}^{(d)}$ as $S_{i\ell d}^2(n_0)$ for $\ell \in \mathrm{ON}$.

    [**Feasibility Check**:]
    for $\ell \in \mathrm{ON}$ **do**
      If $\sum_{n=1}^{r_i} (Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \geq R(r_i; \epsilon_\ell, \eta_d, S_{i\ell d}^2(n_0))$, set $Z_{i\ell}^{(d)} = 0$ and $\mathrm{ON} = \mathrm{ON} \setminus \{\ell\}$;
      Else if $\sum_{n=1}^{r_i} (Y_{i\ell n}^{(d)} - q_\ell^{(d)}) \leq -R(r_i; \epsilon_\ell, \eta_d, S_{i\ell d}^2(n_0))$, set $Z_{i\ell}^{(d)} = 1$ and $\mathrm{ON} = \mathrm{ON} \setminus \{\ell\}$.
    end for
    [**Stopping Condition**:]
    If $\mathrm{ON} = \emptyset$, return $Z_{i\ell}^{(d)}$ for $\ell = 1, \ldots, s$. Otherwise, set $r_i = r_i + 1$, take one additional observation $Y_{i\ell r_i}^{(d)}$ and go to [**Feasibility Check**].
  end for
end for

Proof. For the Restart$^{\max}$ procedure, as we restart a feasibility check for D threshold vectors, the $\mathrm{CD}_{i\ell}(q_\ell^{(d)})$ events are independent for all $\mathrm{CD}_{i\ell}(q_\ell^{(d')})$ where $d' \neq d$ when i, ℓ are fixed. We prove the theorem based on whether systems are simulated independently or with correlation.

If the systems are simulated independently, then the Bonferroni inequality yields

$$\begin{aligned} \mathrm{PCD} &= \Pr\left(\cap_{d=1}^{D} \cap_{i=1}^{k} \cap_{\ell=1, d_\ell \geq d}^{s} \mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) = \prod_{d=1}^{D} \prod_{i=1}^{k} \left(1 - \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \cdot \Pr(\mathrm{ICD}_{i\ell}(q_\ell^{(d)}))\right) \\ &\geq \prod_{d=1}^{D} \prod_{i=1}^{k} \left(1 - \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \beta_d\right) = \prod_{d=1}^{D} \prod_{i=1}^{k} \left(1 - \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \cdot \frac{1 - \sqrt[kD]{1-\alpha}}{\sum_{\ell'=1}^{s} \mathcal{I}(d_{\ell'} \geq d)}\right) \\ &= \prod_{d=1}^{D} \prod_{i=1}^{k} \sqrt[kD]{1-\alpha} = 1 - \alpha. \end{aligned}$$

If the systems are simulated with CRN, then the Bonferroni inequality yields

$$\begin{aligned} \mathrm{PCD} &= \Pr\left(\cap_{d=1}^{D} \cap_{i=1}^{k} \cap_{\ell=1, d_\ell \geq d}^{s} \mathrm{CD}_{i\ell}(q_\ell^{(d)})\right) \geq \prod_{m=1}^{D} \left(1 - \sum_{i=1}^{k} \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \Pr(\mathrm{ICD}_{i\ell}(q_\ell^{(d)}))\right) \\ &\geq \prod_{d=1}^{D} \left(1 - \sum_{i=1}^{k} \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \beta_d\right) = \prod_{d=1}^{D} \left(1 - \sum_{i=1}^{k} \sum_{\ell=1}^{s} \mathcal{I}(d_\ell \geq d) \cdot \frac{1 - \sqrt[D]{1-\alpha}}{k \sum_{\ell'=1}^{s} \mathcal{I}(d_{\ell'} \geq d)}\right) \end{aligned}$$

$$= \prod_{d=1}^{D} \left(1 - k \frac{1 - \sqrt[D]{1-\alpha}}{k} \right) = 1 - \alpha. \quad \square$$

EC.4. Algorithm Statement and Proof of Statistical Validity for the Recycle$^{\mathcal{B}}$ Procedure

We discuss the statistical validity of the Recycle$^{\mathcal{B}}$ procedure in this section. The full description of Recycle$^{\mathcal{B}}$ is provided in Algorithm S.4. The Recycle$^{\mathcal{B}}$ procedure is essentially the same as the $\mathcal{RF}$ procedure except that β_ℓ is defined differently.

Algorithm S.4 Procedure Recycle$^{\mathcal{B}}$

[**Setup**:] Choose confidence level $1-\alpha$, tolerance level ϵ_ℓ, and thresholds $\{q_{\ell 1}, q_{\ell 2}, \ldots, q_{\ell d_\ell}\}$ for constraint $\ell = 1, 2, \ldots, s$. Also, choose the value of c and set $\Theta = \{1, 2, \ldots, k\}$. For $\ell = 1, \ldots, s$, set η_ℓ such that $g(\eta_\ell) = \beta_\ell$, where β satisfies (1), and either
(i) $\beta_\ell = \beta/(s \cdot d_\ell)$ for $\ell = 1, \ldots, s$, or
(ii) $\beta_\ell = \beta/D$ and $D = \sum_{\ell=1}^{s} d_\ell$ for $\ell = 1, \ldots, s$.
for each system $i \in \Theta$ **do**
  [**Initialization**:]
  - Obtain n_0 observations $Y_{i\ell 1}, Y_{i\ell 2}, \ldots, Y_{i\ell n_0}$ for $\ell = 1, 2, \ldots, s$.
  - Compute $S_{i\ell}^2(n_0)$ for $\ell = 1, 2, \ldots, s$.
  - Set $r_i = n_0, \mathrm{ON} = \{1, 2, \ldots, s\}$, and $\mathrm{ON}_\ell = \{1, 2, \ldots, d_\ell\}$ for $\ell = 1, 2, \ldots, s$.

  [**Feasibility Check**:]
  for $\ell \in \mathrm{ON}$ **do**
    for $m \in \mathrm{ON}_\ell$ **do**
      If $\sum_{n=1}^{r_i}(Y_{i\ell n} - q_{\ell m}) \geq R(r_i; \epsilon_\ell, \eta_\ell, S_{i\ell}^2(n_0))$, set $Z_{i\ell m} = 0$ and $\mathrm{ON}_\ell = \mathrm{ON} \backslash \{m\}$.
      If $\sum_{n=1}^{r_i}(Y_{i\ell n} - q_{\ell m}) \leq -R(r_i; \epsilon_\ell, \eta_\ell, S_{i\ell}^2(n_0))$, set $Z_{i\ell m} = 1$ and $\mathrm{ON}_\ell = \mathrm{ON} \backslash \{m\}$.
    end for
    If $\mathrm{ON}_\ell = \emptyset$, set $\mathrm{ON} = \mathrm{ON} \setminus \{\ell\}$.
  end for
  [**Stopping Condition**:]
  If $\mathrm{ON} = \emptyset$, return $Z_{i\ell m}$ for $\ell = 1, 2, \ldots, s$ and $m = 1, 2, \ldots, d_\ell$. Otherwise, set $r_i = r_i + 1$, take one additional observation $Y_{i\ell r_i}$ for $\ell \in \mathrm{ON}$, and go to [**Feasibility Check**].
end for

We now show the statistical validity of the Recycle$^{\mathcal{B}}$ procedure in the following theorem.

THEOREM 4. *Procedure Recycle$^{\mathcal{B}}$ guarantees* $\mathrm{PCD} \geq 1 - \alpha$.

Proof. We prove the theorem based on whether systems are simulated independently or with correlation.

If the systems are simulated independently (i.e., no CRN), then the Bonferroni inequality yields

$$\mathrm{PCD} = \Pr\left(\cap_{i=1}^{k} \cap_{\ell=1}^{s} \cap_{m=1}^{d_\ell} \mathrm{CD}_{i\ell}(q_{\ell m}) \right) \geq \prod_{i=1}^{k} \left[1 - \sum_{\ell=1}^{s} \sum_{m=1}^{d_\ell} \Pr(\mathrm{ICD}_{i\ell}(q_{\ell m})) \right].$$

With β_ℓ from the first choice (i) of Algorithm S.4, Lemma 1 and equation (1) yield

$$\begin{aligned}
\text{PCD} &\geq \prod_{i=1}^{k}\left(1-\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\beta_\ell\right)=\prod_{i=1}^{k}\left(1-\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\frac{\beta}{sd_\ell}\right)\\
&=\prod_{i=1}^{k}\left(1-\sum_{\ell=1}^{s}\frac{\beta}{s}\right)=\prod_{i=1}^{k}(1-\beta)\\
&=\prod_{i=1}^{k}\left(1-\left(1-(1-\alpha)^{1/k}\right)\right)=1-\alpha.
\end{aligned}$$

With β_ℓ from the second choice (ii) of Algorithm S.4, Lemma 1 and equation (1) yield

$$\begin{aligned}
\text{PCD} &\geq \prod_{i=1}^{k}\left(1-\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\frac{\beta}{D}\right)\\
&=\prod_{i=1}^{k}\left(1-\sum_{\ell=1}^{s}\frac{\beta}{D}d_\ell\right)=\prod_{i=1}^{k}(1-\frac{\beta}{D}\sum_{\ell=1}^{s}d_\ell)\\
&=\prod_{i=1}^{k}(1-\beta)=\prod_{i=1}^{k}\left(1-\left(1-(1-\alpha)^{1/k}\right)\right)=1-\alpha.
\end{aligned}$$

If the systems are simulated with correlation, then the Bonferroni inequality yields

$$\text{PCD}=\Pr\left(\cap_{i=1}^{k}\cap_{\ell=1}^{s}\cap_{m=1}^{d_\ell}\text{CD}_{i\ell}(q_{\ell m})\right)\geq 1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\Pr(\text{ICD}_{i\ell}(q_{\ell m})).$$

If β_ℓ is set based on the first choice (i), then Lemma 1 and equation (1) yield

$$\begin{aligned}
\text{PCD} &\geq 1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\beta_\ell=1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\frac{\beta}{sd_\ell}\\
&=1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\frac{\beta}{s}=1-\sum_{i=1}^{k}\beta\\
&=1-k\beta=1-k(\alpha/k)=1-\alpha.
\end{aligned}$$

If β_ℓ is set based on the second choice (ii), then Lemma 1 and equation (1) yield

$$\begin{aligned}
\text{PCD} &\geq 1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\beta_\ell=1-\sum_{i=1}^{k}\sum_{\ell=1}^{s}\sum_{m=1}^{d_\ell}\frac{\beta}{D}\\
&=1-\sum_{i=1}^{k}\frac{\beta}{D}\left(\sum_{\ell=1}^{s}d_\ell\right)=1-\sum_{i=1}^{k}\beta\\
&=1-k\beta=1-k(\alpha/k)=1-\alpha. \quad \square
\end{aligned}$$

EC.5. The Existence of η_ℓ

In this section, we show the existence of a unique solution of η_ℓ to $g(\eta_\ell)=\beta_\ell$. As our experimental results are based on the cases when $c=1$ and $c=\infty$, we only discuss these two cases in this section.

When $c=1$, we have

$$g(\eta_\ell)=\frac{1}{2}\left(1+2\eta_\ell\right)^{-(n_0-1)/2}.$$

Then we have

$$\frac{\partial g(\eta_\ell)}{\partial \eta_\ell}=-\frac{(n_0-1)}{2}(1+2\eta_\ell)^{-(n_0+1)/2}<0,$$

for all $\eta_\ell>0$. We also know that $\lim_{\eta_\ell\to\infty} g(\eta_\ell)=0$ and $g(0)=\frac{1}{2}$. Therefore, $g(\eta_\ell)=\beta_\ell$ has a unique solution $\eta_\ell=\frac{1}{2}\left((2\beta_\ell)^{-2/(n_0-1)}-1\right)$ for all $\beta_\ell\in(0,\frac{1}{2}]$.

When $c=\infty$, we use $f(x)$ to denote the probability density function of a chi-squared distribution with n_0-1 degrees of freedom. Then, by taking derivative with respect to η_ℓ, we have

$$\begin{aligned}
\frac{\partial}{\partial \eta_\ell} g(\eta_\ell) &= \frac{\partial}{\partial \eta_\ell}\int_0^\infty \frac{f(x)}{1+\exp(2\eta_\ell x)}dx \\
&= \int_0^\infty \frac{\partial}{\partial \eta_\ell}\frac{f(x)}{1+\exp(2\eta_\ell x)}dx \\
&= -\int_0^\infty \frac{2x\exp(2\eta_\ell x)f(x)}{\left(1+\exp(2\eta_\ell x)\right)^2}dx \\
&\le -\mathrm{E}\left[\frac{2\chi^2_{n_0-1}\exp(2\eta_\ell \chi^2_{n_0-1})}{\left(1+\exp(2\eta_\ell\chi^2_{n_0-1})\right)^2}\right] \\
&< 0.
\end{aligned}$$

The second equality holds due to the fact that

$$\left|\frac{\partial}{\partial \eta_\ell}\frac{f(x)}{1+\exp(2\eta_\ell x)}\right| = \left|\frac{2x\exp(2\eta_\ell x)f(x)}{\left[1+\exp(2\eta_\ell x)\right]^2}\right| = 2xf(x)\frac{\exp(2\eta_\ell x)}{\left[1+\exp(2\eta_\ell x)\right]^2}\le 2xf(x),\ \text{ for } x,\eta_\ell\ge 0,$$

and $\int_0^\infty 2xf(x)dx=2\mathrm{E}[\chi^2_{n_0-1}]=2(n_0-1)<\infty$ (Billingsley 1986). This means that $g(\eta_\ell)$ is decreasing when $c=\infty$. However, $g(0)=1$ and

$$\lim_{\eta_\ell\to\infty} g(\eta_\ell)=\lim_{\eta_\ell\to\infty}\int_0^\infty \frac{f(x)}{1+\exp(2\eta_\ell x)}dx=\int_0^\infty \lim_{\eta_\ell\to\infty}\frac{f(x)}{1+\exp(2\eta_\ell x)}dx=0,$$

where the second equality holds due to bounded convergence theorem. Therefore, $g(\eta_\ell)=\beta_\ell$ has a unique solution for all $\beta_\ell\in(0,1]$. Thus, a simple search method such as the bi-section search will find the unique η_ℓ value when $c=\infty$.

EC.6. Proof of Lemma 1 for $c = \infty$

We prove Lemma 1 when $c = \infty$ in this section. We first present the following lemma that is useful for our proof.

LEMMA 1. *(Karlin and Taylor 1975) (Theorem 7.5.2) Let $\{\mathcal{B}(t, \Delta, \sigma^2, x) : t \geq 0\}$ be a Brownian motion process with drift $\Delta \neq 0$, variance σ^2, and the starting point x when $t = 0$. The probability that the process reaches the level $a > x$ before hitting $-a < x$ is given by*

$$\Pr\{\mathcal{B}(T, \Delta, \sigma^2, x) = a\} = \frac{\exp(-2\Delta x/\sigma^2) - \exp(2\Delta a/\sigma^2)}{\exp(-2\Delta a/\sigma^2) - \exp(2\Delta a/\sigma^2)},$$

where $T = \min\{t : \mathcal{B}(t, \Delta, \sigma^2, x) \notin (-a, a)\}$, i.e. the first time when the drifted Brownian motion hits $-a$ or a.

We now prove Lemma 1 when $c = \infty$.

Proof. Consider system i and constraint ℓ with mean $y_{i\ell}$ and threshold value $q_{\ell m}$ where $m = 1, \ldots, d_\ell$. Let ϵ_ℓ be the fixed tolerance level and $(-R, R)$ be the straight-line continuation region, where $R = (n_0 - 1)\eta_\ell S^2_{i\ell}(n_0)/\epsilon_\ell$.

Assume system i is unacceptable with respect to constraint ℓ for threshold $q_{\ell m}$, i.e., $y_{i\ell} \geq q_{\ell m} + \epsilon_\ell$. Define T_d and T_c as follows:

$T_d = \min\left\{t \in \mathbb{Z}^+, t \geq n_0 : \mathcal{B}\left(t, y_{i\ell} - q_{\ell m}, \sigma^2_{i\ell}, 0\right) \notin (-R, R)\right\}$ where $\mathbb{Z}^+$ denotes the set of positive integers,

$T_c = \min\left\{t \in \mathbb{R}^+, t \geq n_0 : \mathcal{B}\left(t, y_{i\ell} - q_{\ell m}, \sigma^2_{i\ell}, 0\right) \notin (-R, R)\right\}$ where $\mathbb{R}^+$ denotes the set of positive real numbers.

That is we define T_d/T_c as the first integer/continuous passage time of the drifted Brownian motion $\mathcal{B}\left(t, y_{i\ell} - q_{\ell m}, \sigma^2_{i\ell}, 0\right)$, respectively. Then we have

$$\begin{aligned}
\Pr\left(\mathrm{CD}_{i\ell}(q_{\ell m})\right) &= \Pr\left(\sum_{n=1}^{T_d}(Y_{i\ell n} - q_{\ell m}) \geq R\right) = \mathrm{E}\left[\Pr\left(\sum_{n=1}^{T_d}\left(\frac{Y_{i\ell n} - q_{\ell m}}{\sigma_{i\ell}}\right) \geq \frac{R}{\sigma_{i\ell}} \middle| S^2_{i\ell}(n_0)\right)\right] \\
&\geq \mathrm{E}\left[\Pr\left(\mathcal{B}\left(T_c, \frac{y_{i\ell} - q_{\ell m}}{\sigma_{i\ell}}, 1, 0\right) \geq \frac{R}{\sigma_{i\ell}} \middle| S^2_{i\ell}(n_0)\right)\right] \geq \mathrm{E}\left[\Pr\left(\mathcal{B}\left(T_c, \frac{\epsilon_\ell}{\sigma_{i\ell}}, 1, 0\right) \geq \frac{R}{\sigma_{i\ell}} \middle| S^2_{i\ell}(n_0)\right)\right].
\end{aligned}$$

The first inequality holds because of the fact that the sample mean $\bar{Y}_{i\ell}(n_0)$ and sample variance $S^2_{i\ell}(n_0)$ of normal random variables are independent, and because observing at discrete time reduces the chance of error (Jennison et al. 1982). The second inequality holds due to the assumption that $y_{i\ell} - q_{\ell m} \geq \epsilon_\ell$.

We then have the following derivation,

$$\mathrm{E}\left[\Pr\left(\mathcal{B}\left(T_c, \frac{\epsilon_\ell}{\sigma_{i\ell}}, 1, 0\right) \geq \frac{(n_0-1)\eta_\ell S^2_{i\ell}(n_0)}{\epsilon_\ell \sigma_{i\ell}} \middle| S^2_{i\ell}(n_0)\right)\right] = \mathrm{E}\left[\frac{1-\exp\left(2\eta_\ell \frac{(n_0-1)S^2_{i\ell}(n_0)}{\sigma^2_{i\ell}}\right)}{\exp\left(-2\eta_\ell \frac{(n_0-1)S^2_{i\ell}(n_0)}{\sigma^2_{i\ell}}\right) - \exp\left(2\eta_\ell \frac{(n_0-1)S^2_{i\ell}(n_0)}{\sigma^2_{i\ell}}\right)}\right] \quad (3)$$

due to Lemma 1. By the fact that $\frac{(n_0-1)S^2_{i\ell}(n_0)}{\sigma^2_{i\ell}}$ follows a χ^2 distribution with $n_0 - 1$ degrees of freedom, we have

$$\begin{aligned}
(3) &= \mathrm{E}\left[\frac{1-\exp(2\eta_\ell \chi^2_{n_0-1})}{\exp(-2\eta_\ell \chi^2_{n_0-1}) - \exp(2\eta_\ell \chi^2_{n_0-1})}\right] \\
&= \mathrm{E}\left[\frac{\left(1-\exp(2\eta_\ell \chi^2_{n_0-1})\right)\exp(2\eta_\ell \chi^2_{n_0-1})}{\left(\exp(-2\eta_\ell \chi^2_{n_0-1}) - \exp(2\eta_\ell \chi^2_{n_0-1})\right)\exp(2\eta_\ell \chi^2_{n_0-1})}\right] \\
&= \mathrm{E}\left[\frac{\left(1-\exp(2\eta_\ell \chi^2_{n_0-1})\right)\exp(2\eta_\ell \chi^2_{n_0-1})}{1-\left[\exp(2\eta_\ell \chi^2_{n_0-1})\right]^2}\right] \\
&= \mathrm{E}\left[\frac{\exp(2\eta_\ell \chi^2_{n_0-1})}{1+\exp(2\eta_\ell \chi^2_{n_0-1})}\right] = 1 - \mathrm{E}\left[\frac{1}{1+\exp(2\eta_\ell \chi^2_{n_0-1})}\right] \\
&= 1 - \int_0^\infty \frac{1}{1+\exp(2\eta_\ell x)} \times \frac{1}{2^{(n_0-1)/2}\Gamma((n_0-1)/2)} x^{(n_0-1)/2-1} e^{-x/2} dx \\
&= 1 - \beta_\ell,
\end{aligned}$$

where the last equality holds because of the definition of $g(\cdot)$ in (2) and the fact that η_ℓ is the solution to $g(\eta_\ell) = \beta_\ell$.

The above results also hold for $y_{i\ell} \leq q_{\ell m} - \epsilon_\ell$. Finally, $\Pr\left(\mathrm{CD}_{i\ell}(q_{\ell m})\right) = 1 \geq 1 - \beta_\ell$ when $q_{\ell m} - \epsilon_\ell < y_{i\ell} < q_{\ell m} + \epsilon_\ell$. Hence, when $c = \infty$, Lemma 1 follows. $\square$

EC.7. Implementation Parameters for Systems with Multiple Constraints

In this section, we provide a numerical example and a discussion about the performance of $\mathcal{RF}$, Recycle$^{\mathcal{B}}$, and Restart$^{\text{sum}}$ as a function of the two different ways of setting the implementation parameters β_ℓ (see Section 4.2). As Restart$^{\text{prod}}$ and Restart$^{\text{max}}$ only have one way of setting the implementation parameter, we omit them in this section. We use $\mathcal{RF}_1, \text{Recycle}_1^{\mathcal{B}}$, and $\text{Restart}_1^{\text{sum}}$ to denote the versions of the procedures that set the parameters $\beta_1, \ldots, \beta_s$ based on choice (i), and use $\mathcal{RF}_2, \text{Recycle}_2^{\mathcal{B}}$, and $\text{Restart}_2^{\text{sum}}$ to denote the corresponding procedures with choice (ii) in each algorithm.

We first test the performance of the four procedures applied to four configurations. We consider a single system with two constraints, where the first constraint has one fixed threshold and the second constraint has two thresholds. In all the configurations shown below, we choose $\boldsymbol{y} = (0, 0)$ and $\epsilon = 0.1$. The observations of the two constraints are independent standard normal random variables.

Configuration 1: Set $q_{1,1} = -\epsilon, q_{2,1} = -\epsilon$, and $q_{2,2} = \epsilon$.

Configuration 2: Set $q_{1,1} = -\epsilon, q_{2,1} = -2\epsilon$, and $q_{2,2} = 2\epsilon$.

Configuration 3: Set $q_{1,1} = -\epsilon, q_{2,1} = -4\epsilon$, and $q_{2,2} = 4\epsilon$.

Configuration 4: Set $q_{1,1} = -4\epsilon, q_{2,1} = -\epsilon$, and $q_{2,2} = \epsilon$.

One may notice that the mean performance of the first constraint is at the boundary of the unacceptable region in the first three configurations, which is a "most difficult" case for one constraint with a single threshold. Configuration 1 sets the mean performance of the second constraint at either the boundary of the unacceptable region ($q_{2,1}$) or the boundary of the desirable region ($q_{2,2}$), while Configurations 2 and 3 set the mean performance further from the boundaries of the unacceptable (desirable) region. Configuration 4 has the same (difficult) thresholds on the second constraint as in Configuration 1 but sets the mean performance of the first constraint far from the boundary of the unacceptable region. Notice that the performance of $\mathcal{RF}$ and $\text{Recycle}^{\mathcal{B}}$ are expected to be identical as $d_\ell \leq 2$, where $\ell = 1, 2$. Table 1 shows the estimated PCD and OBS of the $\mathcal{RF}, \text{Recycle}^{\mathcal{B}}$, and $\text{Restart}^{\text{sum}}$ procedures under triangular-shaped continuation regions for all configurations.

We see that under both choices (i) and (ii), all the procedures guarantee statistical validity of all configurations. Choice (i) requires more observations than choice (ii) under Configurations 1

Table 1 **Average number of observations and observed PCD (reported in parentheses) for implementation parameters (i) and (ii)**

	$\mathcal{RF}_1(\text{Recycle}_1^{\mathcal{B}})$	$\mathcal{RF}_2(\text{Recycle}_2^{\mathcal{B}})$	$\text{Restart}_1^{\text{sum}}$	$\text{Restart}_2^{\text{sum}}$
Configuration 1	408.5 (0.954)	396.1 (0.954)	833.3 (0.953)	815.4 (0.953)
Configuration 2	285.6 (0.977)	303.3 (0.983)	596.9 (0.975)	602.2 (0.983)
Configuration 3	242.4 (0.976)	276.6 (0.984)	435.1 (0.977)	456.1 (0.984)
Configuration 4	385.5 (0.977)	352.4 (0.968)	679.9 (0.977)	635.8 (0.968)

and 4 for all three procedures, while choice (i) requires fewer observations than choice (ii) under Configurations 2 and 3.

In Configuration 1, since all the thresholds for both constraints are considered as “most difficult”, assigning error evenly to each feasibility check, which follows choice (ii), is plausible. However, as Configuration 2 has a “difficult” threshold on the first constraint but “easy” thresholds on the second constraint, allocating more error to the threshold for the first constraint and less error to the threshold for the second constraint, which follows choice (i), is beneficial. Configuration 3 has even “easier” thresholds on the second constraint, suggesting that choice (i) is more beneficial for all three procedures. In these cases, choosing between (i) and (ii) depends on the difficulty of the feasibility checks. Although Configuration 4 has the same number of thresholds as the other three configurations, it has an “easy” threshold on the first constraint but two “difficult” thresholds on the second constraint. Choice (ii) allows more error allocation to the second constraint and less error allocation to the first constraint, which performs better than choice (i).

It is clear that the total number of required observations depends on the difficulty of the feasibility checks and the number of thresholds on each constraint. Of course, the decision maker may not have the information about the mean configuration before she performs feasibility check. Thus it is difficult to predict in advance whether (i) or (ii) will result in better performance.

EC.8. Comparison between $\text{Restart}^{\text{prod}}$, $\text{Restart}^{\text{sum}}$, and the other Procedures

In this section, we compare the performance of $\text{Restart}^{\text{prod}}$ and $\text{Restart}^{\text{sum}}$ with the other procedures. Based on the decriptions shown in Appendix EC.1 for $\text{Restart}^{\text{prod}}$, EC.2 for $\text{Restart}^{\text{sum}}$ and EC.3 for $\text{Restart}^{\text{max}}$, the number of "restarts" depends highly on the number of thresholds on each constraint. As $\text{Restart}^{\text{prod}}$ performs feasibility checks for each combination of thresholds of all constraints, it requires $\prod_{\ell=1}^{s} d_\ell$ "restarts" to determine feasibility for one system, whereas $\text{Restart}^{\text{sum}}$ performs feasibility checks independently for each system, each constraint, and each threshold, and hence requires $\sum_{\ell=1}^{s} d_\ell$ "restarts" for one system. On the other hand, $\text{Restart}^{\text{max}}$ performs feasibility check by restarting independently for threshold vectors that contain thresholds from all constraints with thresholds that have not yet received feasibility decisions until the feasibility of each threshold on each constraint is determined. This requires $\max_{\ell=1,\dots,s} d_\ell$ "restarts" for one system.

We consider three configurations of a single system with two constraints and independent standard normal observations. The first configuration has one threshold on the first constraint and two thresholds on the second constraints. The second configuration has two thresholds on both constraints, and the third configuration has two thresholds on the first constraint and four thresholds on the second constraint. In all configurations, we set $\boldsymbol{y} = (0, 0)$ and $\epsilon = 0.1$. More specifically, we have

Configuration 1 (C1): $\boldsymbol{q}^1 = (-\epsilon), \boldsymbol{q}^2 = (-\epsilon, \epsilon)$.

Configuration 2 (C2): $\boldsymbol{q}^1 = (-\epsilon, \epsilon), \boldsymbol{q}^2 = (-\epsilon, \epsilon)$.

Configuration 3 (C3): $\boldsymbol{q}^1 = (-\epsilon, \epsilon), \boldsymbol{q}^2 = (-1.25\epsilon, -\epsilon, \epsilon, 1.25\epsilon)$.

The experimental results for the average number of observations and observed PCD for $\mathcal{RF}$, $\text{Recycle}^{\mathcal{B}}$, $\text{Restart}^{\text{sum}}$, $\text{Restart}^{\text{max}}$ and $\text{Restart}^{\text{prod}}$ under triangular-shaped continuation regions are shown in Table 2.

We can easily see that Configuration 1 has two combinations of thresholds, while Configurations 2 and 3 have four and eight combinations, respectively. This means that $\text{Restart}^{\text{prod}}$ needs to

Table 2 Average number of observations and observed PCD (reported in parentheses) for $\text{Restart}^{\text{prod}}$, $\text{Restart}^{\text{sum}}$ and the other procedures

	$\mathcal{RF}_1$	$\mathcal{RF}_2$	$\text{Recycle}_1^{\mathcal{B}}$	$\text{Recycle}_2^{\mathcal{B}}$	$\text{Restart}^{\max}$	$\text{Restart}_1^{\text{sum}}$	$\text{Restart}_2^{\text{sum}}$	$\text{Restart}^{\text{prod}}$
C1	408.5 (0.954)	396.1 (0.954)	408.5 (0.954)	396.1 (0.954)	622.7 (0.953)	833.3 (0.953)	815.4 (0.953)	765.4 (0.953)
C2	466.6 (0.954)	466.6 (0.954)	466.6 (0.954)	466.6 (0.954)	766.3 (0.954)	1201.1 (0.954)	1201.1 (0.954)	1871.0 (0.954)
C3	467.2 (0.953)	467.2 (0.953)	523.4 (0.965)	524.7 (0.969)	1476.8 (0.964)	1981.9 (0.960)	1953.2 (0.964)	4315.2 (0.962)

perform two, four, and eight restarts to conclude feasibility checks for Configuration 1, 2, and 3, respectively. However, $\text{Restart}^{\text{sum}}$ performs three, four, and six restarts as it performs feasibility check independently for each system, constraint, and threshold. One the other hand, $\text{Restart}^{\max}$ performs two restarts for Configurations 1 and 2, and performs four restarts for Configuration 3. One may notice that when $\sum_{\ell=1}^{s} d_\ell$ is smaller than $\prod_{\ell=1}^{s} d_\ell$, $\text{Restart}^{\text{sum}}$ is likely to perform better than $\text{Restart}^{\text{prod}}$. As $\max_{\ell=1,\dots,s} d_\ell$ is always smaller than $\sum_{\ell=1}^{s} d_\ell$ and $\prod_{\ell=1}^{s} d_\ell$, $\text{Restart}^{\max}$ is superior compared with $\text{Restart}^{\text{prod}}$ and $\text{Restart}^{\text{sum}}$.

References

Billingsley P (1986) *Probability and Measure* (John Wiley and Sons), second edition.

Jennison C, Johnstone IM, Turnbull BW (1982) Asymptotically optimal procedures for sequential adaptive selection of the best of several normal means. Gupta SS, Berger JO, eds., *Statistical Decision Theory and Related Topics III*, volume 2, 55–86 (Academic Press).

Karlin S, Taylor HM (1975) *A First Course in Stochastic Processes* (Academic Press).