Online Supplement of the paper “Cone Product Reformulation for Global Optimization”

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The online supplement of this paper contains five sections. Section A provides all multiplications of cone inequalities from Table 1 that are from the literature. In Section B, we describe the number of additional inequalities one would obtain when applying full RPT, i.e., the total additional conic inequalities resulting from the multiplication of all pairwise multiplications of the constraints in the two cones, including multiplications of all inequalities in the same cone. Section C presents the full formulations obtained when multiplying all constraints in the optimization problems used in the numerical experiments. Section D describes the problem instances used for the numerical experiment in Section 6.1. Finally, Section E provides, for each case studied, an example showing that incorporating the resulting constraints lead to a strict improvement in the optimal objective value.

A. Multiplication of cone inequalities from the literature

In this appendix, we provide all multiplications of cone inequalities from Table 1 that are from the literature.

A.1. Case 1 in Table 1: (L) × (L)

Consider two linear inequalities

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0.$$

Multiplying the two linear inequalities yields 1 additional linear inequality (Sherali and Alameddine 1992):

$$\begin{aligned}(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq 0 &\iff b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq 0 \\ &\implies b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq 0.\end{aligned}$$

A.2. Case 2 in Table 1: (L) × (Q)

Consider one linear inequality and one conic quadratic inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d}\right\|.$$

Multiplying the linear inequality with both sides of the conic quadratic inequality yields 1 additional conic quadratic inequality (Sturm and Zhang (2003)) :

$$\begin{aligned}&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d}\right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\ \iff &\left\|(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d})\right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\ \implies &\left\|b_1 \boldsymbol{D}\boldsymbol{x} + b_1 \boldsymbol{d} - \boldsymbol{D}\boldsymbol{U}\boldsymbol{a}_1 - \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{d}\right\| \leq b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U}\boldsymbol{a}_2.\end{aligned}$$

A.3. Case 5 in Table 1: (L) × (S)

Consider one linear inequality and one LMI respectively

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

We apply RPT to the multiplication of these inequalities, and obtain 1 additional LMI:

$$\begin{aligned}&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ \iff &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_0 + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_1 x_1 + \cdots + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_{n_x} x_{n_x} \succeq 0 \\ \implies &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_0 + (b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1)\boldsymbol{A}_1 + \cdots + (b_1 x_{n_x} - \boldsymbol{a}_1^\top \boldsymbol{u}_{n_x})\boldsymbol{A}_{n_x} \succeq 0.\end{aligned}$$

A.4. Case 6 in Table 1: (Q) $\times$ (Q)

Consider two conic quadratic inequalities

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1\right\| \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2\right\|. \tag{O-1}$$

We multiply the LHS of the first conic quadratic inequality with both sides of the second conic quadratic inequality and the LHS of the second conic quadratic inequality with both sides of the first conic quadratic inequality to obtain 2 additional conic quadratic inequalities, see Appendix A.2. Moreover, we multiply the LHSs and RHSs of the conic quadratic inequalities with each other and obtain 1 additional conic quadratic inequality:

$$(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq \left\|\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1\right\|\left\|\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2\right\| \tag{O-2}$$

$$\iff \quad b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq \left\|(\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)(\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top\right\|_F \tag{O-3}$$

$$\implies \quad b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq \left\|\boldsymbol{D}_1 \boldsymbol{U} \boldsymbol{D}_2^\top + \boldsymbol{p}_1 \boldsymbol{x}^\top \boldsymbol{D}^\top + \boldsymbol{D}\boldsymbol{x}\boldsymbol{p}_2^\top + \boldsymbol{p}_1 \boldsymbol{p}_2^\top\right\|_F. \tag{O-4}$$

This is Case 6(i) in Table 1.

In the literature also two LMIs are proposed. First observe that the two conic quadratic inequalities (O-1) can be written as

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1\right\| \iff \begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0$$

and

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2\right\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0.$$

We now assume that, without loss of generality, the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are of the same size. Indeed, suppose that $\boldsymbol{D}_1$ has less rows than $\boldsymbol{D}_2$, then we can extend matrix $\boldsymbol{D}_1$ by zero rows or by copying scaled versions of some of the original rows. Using the fact that the Kronecker product of two positive semidefinite matrices is also positive semidefinite (Horn and Johnson 1991, Theorem 4.2.12) and linearizing each element of the product, we obtain

$$\begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \otimes \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0$$

$$\implies \left[\begin{array}{cccccc} \alpha & \boldsymbol{\gamma}^\top & \delta_1 & \boldsymbol{\eta}_1^\top & \cdots & \delta_r \;\; \boldsymbol{\eta}_r^\top \\ \boldsymbol{\gamma} & \alpha I & \boldsymbol{\eta}_1 & \delta_1 I & & \boldsymbol{\eta}_r \;\; \delta_r I \\ \hline \delta_1 & \boldsymbol{\eta}_1^\top & \alpha & \boldsymbol{\gamma}^\top & & \\ \boldsymbol{\eta}_1 & \delta_1 I & \boldsymbol{\gamma} & \alpha I & & \\ \hline \vdots & & & & \ddots & \\ \hline \delta_r & \boldsymbol{\eta}_r^\top & & & & \alpha \;\; \boldsymbol{\gamma}^\top \\ \boldsymbol{\eta}_r & \delta_r I & & & & \boldsymbol{\gamma} \;\; \alpha I \end{array}\right] \succeq 0,$$

where

$$\alpha = b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top U a_2$$
$$\boldsymbol{\gamma} = b_1(\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2) - (a_1^\top \boldsymbol{x})\boldsymbol{p}_2 - \boldsymbol{D}_2\boldsymbol{U}\boldsymbol{a}_1$$
$$\delta_i = b_2(\boldsymbol{d}_{1i}^\top \boldsymbol{x}) + p_{1i}(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) - \boldsymbol{d}_{1i}^\top \boldsymbol{U}\boldsymbol{a}_1, \quad i \in [r]$$
$$\boldsymbol{\eta}_i = (\boldsymbol{d}_{1i}^\top \boldsymbol{x} + p_{1i})\boldsymbol{p}_2 + p_{1i}\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{D}_2\boldsymbol{U}\boldsymbol{d}_{1i}, \quad i \in [r]$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ denote the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. This is Case 6(ii) in Table 1. Another LMI is proposed by Jiang and Li (2019), using the Hadamard product instead of the Kronecker product. It follows for the Hadamard product that

$$\begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \circ \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0,$$

which implies

$$\begin{bmatrix} \alpha & \boldsymbol{\beta}^\top \\ \boldsymbol{\beta} & \alpha\boldsymbol{I} \end{bmatrix} \succeq 0, \tag{O-5}$$

where

$$\alpha = b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top \boldsymbol{U} a_2 \tag{O-6}$$
$$\beta_i = \boldsymbol{d}_{1i} U \boldsymbol{d}_{1i} + p_{1i}p_{2i} + p_{2i}\boldsymbol{d}_{1i}\boldsymbol{x} + p_{1i}\boldsymbol{d}_{2i}\boldsymbol{x}, \quad i \in [r] \tag{O-7}$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ is the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. Notice that the matrix in the LHS of (O-5) has an arrow structure, and hence LMI (O-5) is equivalent to the following conic quadratic inequality:

$$\|\beta\|_2 \le \alpha. \tag{O-8}$$

It can easily be verified that (O-8) is a weaker inequality than (O-4). This is Case 6(iii) in Table 1.

A.5. Case 9 in Table 1: (Q) $\times$ (S)

Consider one conic quadratic inequality and one LMI

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0. \tag{O-9}$$

First observe that the conic quadratic inequality can be formulated as an LMI (Anstreicher 2017):

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0.$$

We now multiply these inequalities. Using the fact that the Kronecker product of two positive semidefinite matrices is also positive semidefinite (Horn and Johnson 1991, Theorem 4.2.12), we obtain 1 additional LMI:

$$
\begin{aligned}
&\left(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} - \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\|\right) \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \implies \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \otimes \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\
&\iff \begin{bmatrix} (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & (\boldsymbol{d}_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & \cdots & (\boldsymbol{d}_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) \\ (d_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & & \\ \vdots & & \ddots & \\ (d_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) & & & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) \end{bmatrix} \succeq 0 \\
&\implies \begin{bmatrix} \boldsymbol{A}(b_2 \boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & \boldsymbol{A}(p_1 \boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \cdots & \boldsymbol{A}(p_r \boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) \\ \boldsymbol{A}(p_1 \boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \boldsymbol{A}(b_2 \boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & & \\ \vdots & & \ddots & \\ \boldsymbol{A}(p_r \boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) & & & \boldsymbol{A}(b_2 \boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \end{bmatrix} \succeq 0, \qquad \text{(O-10)}
\end{aligned}
$$

where $\boldsymbol{d}_i$ is the i-th row of $\boldsymbol{D}$. We could also directly multiply the LHS of the conic quadratic inequality with the LMI and obtain 1 additional LMI:

$$A(b_2 \boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \succeq 0,$$

which is also implied by (O-10).

A.6. Case 12 in Table 1: (P) × (S)

Consider one power cone inequality and one LMI

$$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

We multiply the nonnegativity constraints of the power cone with the LMI and obtain m additional LMIs:

$$x_i \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \quad \implies \quad \boldsymbol{A}(\boldsymbol{u}_i) \succeq 0, \qquad i \in [m].$$

A.7. Case 15 in Table 1: (S) × (S)

Consider two LMIs

$$\begin{cases} \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ \boldsymbol{B}(\boldsymbol{x}) \succeq 0. \end{cases}$$

If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of different sizes, it follows from (Horn and Johnson 1991, Theorem 4.2.12) that the Kronecker product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Kronecker product is the multiplication of two affine functions of

$\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(i) of Table 1. If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of the same size, it follows from the Schur Product Theorem (Schur 1911, Horn and Johnson 1991, Theorem 5.2.1) that the Hadamard product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Hadamard product is the multiplication of two affine functions of $\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(ii) of Table 1.

B. Full RPT

In this appendix, we describe the number of additional inequalities one would obtain when applying full RPT, i.e., the total additional conic inequalities resulting from the multiplication of all pairwise multiplications of the constraints in the two cones, including multiplications of all inequalities in the same cone.

Case 1 in Table O-1. Multiplying each linear inequality with itself yields 2 additional linear inequalities. Hence, with full RPT we would obtain in total *3 additional linear inequalities.*

Case 2 in Table O-1. Multiplying the linear inequality with itself yields 1 additional linear inequality, see Case 1 in Appendix A.1. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Hence, with full RPT we would obtain *1 additional linear inequality and 3 additional conic quadratic inequalities*.

Case 3(i) and 3(ii) in Table O-1. Multiplying the linear inequality with itself and the nonnegativity constraints of the power cone with themselves yields $1 + m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1. Multiplying the power cone inequality with the nonnegativity constraints yields m additional power cone inequalities for Case 3(i) and $\sum_{i=2}^{m+1} |\mathcal{V}_i|$ additional power cone inequalities for Case 3(ii), see Case 3 in Section 3.2. Multiplying the power cone inequality with itself yields 1 additional power cone inequality for Case 3(i) and $|\mathcal{V}_{m+2}|$ additional power cone inequalities for Case 3(ii), see Case 10 in Section 3.4. Hence, with full RPT we would obtain in total $m+1+m(m+1)/2$ *additional linear inequalities and* $m+2$ *additional power cone inequalities for Case 3(i), while we would obtain in total* $\sum_{i=1}^{m+2} |\mathcal{V}_i|$ *additional power cone inequalities for Case 3(ii).*

Case 4 in Table O-1. Multiplying the linear inequality with itself and the nonnegativity constraint of the exponential cone with itself yields 2 additional linear inequalities, see Case 1 in Appendix A.1. Multiplying the nonnegativity constraint of the exponential cone with the exponential cone inequality and the exponential cone inequality with itself yields 3 additional exponential cone inequalities, see Case 13 in Section 4.5. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 4 additional exponential cone inequalities.*

Case 5 in Table O-1. Multiplying the linear inequality with itself yields 1 additional linear inequality, see Case 1 in Appendix A.1. Multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we would obtain *1 additional linear inequality and 2 additional LMIs.*

Case 6(i) in Table O-1. We can further multiply both quadratic inequalities with themselves to obtain 4 additional quadratic inequalities, see Case 6(i) in Appendix A.4. Hence, with full RPT we would obtain in total *7 additional quadratic inequalities.*

Case 6(ii) and 6(iii) in Table O-1. We can multiply both quadratic inequalities with themselves as explained in Appendix A.4 for Case 6(ii) and 6(iii) to obtain 2 additional LMIs. Hence, with full RPT we would obtain in total *3 additional LMIs.*

Case 7(i) and 7(ii) in Table O-1. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Multiplying the nonnegativity constraints with each other yields $m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1. Further, multiplying the nonnegativity constraints with the power cone inequality yields m additional power cone inequalities for Case 7(i) and $\sum_{i=3}^{m+2} |\mathcal{V}_i|$ additional power cone inequalities for Case 7(ii), see Case 3 in Section 3.2. Multiplying the power cone inequality with itself yields 1 additional power cone inequality for Case 7(i) and $|\mathcal{V}_{m+3}|$ additional power cone inequalities for Case 7(ii), see Case 10 in Section 3.4. Hence, with full RPT, we would obtain $m(m+1)/2$ *additional linear inequalities,* $m+2$ *additional conic quadratic inequalities, and* $m+3$ *additional power cone inequalities for Case 7(i), while we would obtain* $\sum_{i=1}^{m+3} |\mathcal{V}_i|$ *additional power cone inequalities for Case 7(ii).*

Case 8(i) in Table O-1. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Multiplying the nonnegativity constraint with itself and the exponential cone inequality yields 1 additional linear inequality, see Case 1 in Appendix A.1, and 1 additional exponential cone inequality, see Case 4 in Section 4.2. Multiplying the exponential cone inequality with itself yields 2 additional exponential cone inequalities, see Case 13 in Section 4.5. Hence, with full RPT we would obtain *1 additional linear inequality, 4 additional conic quadratic inequalities, and 4 additional exponential cone inequalities.*

Case 8(ii) in Table O-1. We obtain 1 additional conic quadratic inequality and 1 additional exponential cone inequality from the multiplication of the derived linear inequality with the conic quadratic inequality and the exponential cone inequality, respectively. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, together with the inequalities in 8(i) in Table O-1, with full RPT we obtain in total *1 additional linear inequality, 5 additional conic quadratic inequalities and 5 additional exponential cone inequalities.*

Case 8(iii) in Table O-1. In addition to the inequalities from Case 8(ii), we also obtain 2 additional conic quadratic inequalities and 1 additional exponential cone inequality from the multiplication of the derived conic quadratic inequality with the conic quadratic inequality and the exponential cone inequality, respectively. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, in this case, with full RPT we would obtain in total *1 additional linear inequality, 7 additional conic quadratic inequalities, and 6 additional exponential cone inequalities.*

Case 9 in Table O-1. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Further, multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we would obtain in total *2 additional conic quadratic inequalities and 2 additional LMIs.*

Case 10(i) and 10(ii) in Table O-1. Multiplying the nonnegativity constraints of each cone with themselves and the power cone inequality of the same cone, results in $m_1(m_1+1)/2+m_2(m_2+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1, and m_1+m_2 additional power cone inequalities for Case 10(i) and $\sum_{i=m_1+m_2+2}^{2m_1+2m_2+1}|\mathcal{V}_i|$ additional power cone inequalities for Case 10(ii), see Case 3 in Section 3.2. Moreover, multiplying each power cone inequality with itself yields 2 additional power cone inequalities for Case 10(i) and $|\mathcal{V}_{2m_1+2m_2+2}|+|\mathcal{V}_{2m_1+2m_2+3}|$ additional power cone inequalities for Case 10(ii), see Case 10 in Section 3.4. Hence, with full RPT we would obtain in total $m_1(m_1+1)/2+m_2(m_2+1)/2+m_1m_2$ *additional linear inequalities and* $2m_1+2m_2+3$ *additional power cone inequalities for Case 10(i), while we would obtain* $\sum_{i=1}^{2m_1+2m_2+3}|\mathcal{V}_i|$ *additional power cone inequalities for Case 10(ii).*

Case 11(i) in Table O-1. Multiplying the nonnegativity constraints of the power cone with themselves and the power cone inequality yields $m(m+1)/2$ additional linear inequalities and m additional power cone inequalities, see Case 1 in Appendix A.1 and Case 3 in Section 3.2 respectively. Multiplying the power cone inequality with itself yields 1 additional power cone inequality, see Case 10 in Section 3.4. Also, multiplying the nonnegativity constraint of the exponential cone with itself and the exponential cone inequality and multiplying the exponential cone inequality with itself yields 1 linear and 3 additional new exponential cone inequalities, see Case 13 in Section 4.5.

Hence, with full RPT we would obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+3$ *additional exponential cone inequalities and* $m+3$ *additional power cone inequalities.*

Case 11(ii) in Table O-1. We obtain 1 additional exponential cone inequality and 1 additional power cone inequality from the multiplication of the derived linear inequality with the exponential cone inequality and power cone inequality, respectively. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, together with the inequalities in 11(i) in Table O-1, with full RPT we obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+4$ *additional exponential cone inequalities and* $m+4$ *additional power cone inequalities.*

Case 11(iii) in Table O-1. In addition to the inequalities from Case 11(ii), we obtain 1 additional exponential cone inequality and 2 additional power cone inequalities from the multiplication of the derived conic quadratic inequality with the exponential cone inequality and the power cone inequality, respectively. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, together with the inequalities in 11(ii) in Table O-1, with full RPT we obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+5$ *additional exponential cone inequalities, and* $m+6$ *additional power cone inequalities.*

Case 12 in Table O-1. Multiplying the nonnegativity constraints with themselves and the power cone inequality yields $m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1 and m additional power cone inequalities, see Case 3 in Section 3.2 Multiplying the power cone inequality with itself yields 1 additional power cone inequality, see Case 10 in Section 3.4. Moreover, multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we obtain in total $m(m+1)/2$ *additional linear inequalities,* $m+1$ *additional power cone inequalities, and* $m+1$ *additional LMIs.*

Case 13(i) in Table O-1. Multiplying the nonnegativity constraint of each exponential cone with itself and the exponential cone inequality of the same exponential cone yields 2 additional linear inequalities, see Case 1 in Appendix A.1, and 2 additional exponential cone inequalities, see Case 4 in Section 4.2. Multiplying the LHS of each exponential cone inequality with both sides of the same exponential cone inequality yields 2 additional exponential cone inequalities, see Case 4 in Section 4.2. Moreover, multiplying the LHSs and RHSs of both inequalities yields 2 additional exponential cone inequalities:

$$\begin{cases} x_1^2 \geq x_2^2 \exp\left(2x_3x_2/x_2^2\right) \\ x_4^2 \geq x_5^2 \exp\left(2x_6x_5/x_5^2\right) \end{cases} \implies \begin{cases} u_{11} \geq u_{22} \exp\left(2u_{23}/u_{22}\right) \\ u_{44} \geq u_{55} \exp\left(2u_{56}/u_{55}\right). \end{cases}$$

Hence, with full RPT we would obtain in total *3 additional linear inequalities and 11 additional exponential cone inequalities.*

Case 13(ii) in Table O-1. We multiply each of the derived linear inequalities $x_1 \geq x_2 + x_3$ and $x_4 \geq x_5 + x_6$ with the exponential cone inequalities and obtain 4 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 15 additional exponential cone inequalities.*

Case 13(iii) in Table O-1. We multiply each of the derived linear and conic quadratic inequalities with the exponential cone inequalities and obtain 8 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 19 additional exponential cone inequalities.*

Case 13(iv) in Table O-1. We multiply the derived linear and conic quadratic inequalities with the exponential cone inequalities and obtain 6 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 17 additional exponential cone inequalities.*

Case 14(i) in Table O-1. Multiplying the exponential cone with itself, yields 1 additional linear inequality and 3 additional exponential cone inequalities. Moreover, multiplying the LMI with itself yields 1 additional LMI. Hence, with full RPT we would obtain *1 additional linear inequality, 3 additional exponential cone inequalities, and 3 additional LMIs.*

Case 14(ii) in Table O-1. Multiplying the linear inequality resulting from the decomposition of the exponential cone with itself, the nonnegativity constraint of the exponential cone, and the exponential cone inequality gives 2 additional linear inequalities and 1 additional exponential cone inequality. Hence, together with the inequalities in 14(i) in Table O-1 and the inequalities resulting from multiplying this linear inequality with the LMI as explained in Section A.3, with full RPT we obtain in total *3 additional linear inequalities, 4 additional exponential cone inequalities, and 4 additional LMIs.*

Case 14(iii) in Table O-1. Multiplying the quadratic inequality resulting from the decomposition of the exponential cone with itself, the nonnegativity constraint of the exponential cone, the linear inequality resulting from the decomposition of the exponential cone, and the exponential cone inequality gives 5 additional quadratic inequalities and 1 additional exponential cone inequality. Hence, together with the inequalities in 14(ii) in Table O-1 and the inequalities resulting from multiplying this quadratic inequality with the LMI as explained in Section A.5, with full RPT we obtain in total *3 additional linear inequalities, 5 additional quadratic inequalities, 5 additional exponential cone inequalities and 5 LMIs*

Case	Cone-1	Cone-2	Full RPT
1	L	L	3L
2	L	Q	L + 3Q
3	L	P	(i) $(m+1+m(m+1)/2)$L + $(m+2)$P (ii) $(m+1+m(m+1)/2)$L + $\sum_{i=1}^{m+2}\|\mathcal{V}_i\|$P
4	L	E	3L + 4E
5	L	S	L + 2S
6	Q	Q	(i) 7Q (ii) 3S (iii) 3S
7	Q	P	(i) $m(m+1)/2$L + $(m+2)$Q + $(m+3)$P (ii) $m(m+1)/2$L + $(m+2)$Q + $\sum_{i=1}^{m+3}\|\mathcal{V}_i\|$P
8	Q	E	(i) L + 4Q + 4E (ii) L + 5Q + 5E (iii) L + 7Q + 6E
9	Q	S	2Q + 2S
10	P	P	(i) $(m_1(m_1+1)/2+m_2(m_2+1)/2+m_1m_2)$L + $(2m_1+2m_2+3)$P (ii) $(m_1(m_1+1)/2+m_2(m_2+1)/2+m_1m_2)$L + $\sum_{i=1}^{2m_1+2m_2+3}\|\mathcal{V}_i\|$P
11	P	E	(i) $((m+1)(m+2)/2)$L + $(m+3)$P + $(m+3)$E (ii) $((m+1)(m+2)/2+1)$L + $(m+4)$P + $(m+4)$E (iii) $((m+1)(m+2)/2+2)$L + 2Q + $(m+6)$P + $(m+5)$E
12	P	S	$(m(m+1)/2)$L + $(m+1)$P + $(m+1)$S
13	E	E	(i) 3L + 11E (ii) 3L + 15E (iii) 3L + 19E (iv) 3L + 17E
14	E	S	(i) L + 3E + 3S (ii) 3L + 4E + 4S (iii) 3L + 5Q + 5E + 5S
15	S	S	(i) 3S (ii) 3S

Table O-1 Results of multiplying the inequalities in two of the five basic cones as given in Section 2 when applying full RPT.

Case 15(i) and 15(ii) in Table O-1. Multiplying the LMIs with itself, yields 2 additional LMIs, hence with full RPT we would obtain in total *3 additional LMIs.*

C. RPT formulations of numerical experiments

In this section, we include the formulations obtained when multiplying all constraints in the problems encountered in the numerical experiments.

C.1. RPT formulation of Section 6.1

We linearize $\boldsymbol{x}\boldsymbol{x}^\top$ with $\boldsymbol{X}$, $\boldsymbol{z}\boldsymbol{z}^\top$ with $\boldsymbol{Z}$, $\boldsymbol{x}\boldsymbol{z}^\top$ with $\boldsymbol{V}$, $\boldsymbol{x}\boldsymbol{t}^\top$ with $\boldsymbol{W}$ and $\boldsymbol{z}\boldsymbol{t}^\top$ with $\boldsymbol{Q}$. When multiplying the constraints in Problem (38), without any additions, we obtain the following problem:

$$\min_{\substack{\boldsymbol{x},\boldsymbol{z},\boldsymbol{X},\\ \boldsymbol{V},\boldsymbol{Z}}} \quad \mathrm{Tr}\left(\boldsymbol{A}_0\boldsymbol{X}\right)+\boldsymbol{b}_0^\top\boldsymbol{x}+c_0 \tag{O-11a}$$

$$\text{s.t.} \quad \sum_{i=1}^{n_x} z_i \le 1, \tag{O-11b}$$

$$\exp(-x_i-a)\le z_i, \qquad i\in[n_x], \tag{O-11c}$$

$$\sum_{i=1}^{n_x} t_i \le \beta, \tag{O-11d}$$

$$\exp(x_i)\le t_i, \qquad i\in[n_x], \tag{O-11e}$$

$$\left(1-\sum_{j=1}^{n_x} z_j\right)\exp\left(\frac{-x_i-\alpha+\sum_{j=1}^{n_x}V_{ij}+\alpha\sum_{j=1}^{n_x}z_j}{1-\sum_{j=1}^{n_x}z_j}\right)\le z_i-\sum_{j=1}^{n_x}Z_{ij} \qquad i\in[n_x], \tag{O-11f}$$

$$\beta-\sum_{j=1}^{n_x}t_j-\beta\sum_{j=1}^{n_x}z_j+\sum_{i,j=1}^{n_x}Q_{ij}\ge 0, \tag{O-11g}$$

$$\left(1-\sum_{j=1}^{n_x} z_j\right)\exp\left(\frac{x_i-\sum_{j=1}^{n_x}V_{ij}}{1-\sum_{j=1}^{n_x}z_j}\right)\le t_i-\sum_{j=1}^{n_x}Q_{ji}, \qquad i\in[n_x], \tag{O-11h}$$

$$\sum_{i,j=1}^{n_x}Z_{ij}-2\sum_{i=1}^{n_x}z_i+1\ge 0, \tag{O-11i}$$

$$\left(\beta-\sum_{j=1}^{n_x} t_j\right)\exp\left(\frac{-\beta x_i-\alpha\beta+\sum_{j=1}^{n_x}W_{ij}+\alpha\sum_{j=1}^{n_x}t_j}{\beta-\sum_{j=1}^{n_x}t_j}\right)\le z_i\beta-\sum_{j=1}^{n_x}Q_{ij}, \qquad i\in[n_x], \tag{O-11j}$$

$$\left(\beta-\sum_{j=1}^{n_x} t_j\right)\exp\left(\frac{\beta x_i-\sum_{j=1}^{n_x}W_{ij}}{\beta-\sum_{j=1}^{n_x}t_j}\right)\le \beta t_i-\sum_{j=1}^{n_x}T_{ij}, \qquad i\in[n_x], \tag{O-11k}$$

$$\sum_{i,j=1}^{n_x}T_{ij}-2\sum_{j=1}^{n_x}t_j+\beta^2\ge 0, \tag{O-11l}$$

$$z_j\exp\left(\frac{V_{ij}}{z_j}\right)\le Q_{ji}, \qquad i,j\in[n_x], \tag{O-11m}$$

$$\exp(-x_i-x_j-2\alpha)\le Z_{ij}, \qquad i\le j\in[n_x], \tag{O-11n}$$

$$\exp\left(-x_i-a+x_j\right)\le Q_{ij}, \qquad i,j\in[n_x], \tag{O-11o}$$

$$z_j\exp\left(\frac{-V_{ij}-az_j}{z_j}\right)\le Z_{ij}, \qquad i,j\in[n_x], \tag{O-11p}$$

$$t_j\exp\left(\frac{-W_{ij}-at_j}{t_j}\right)\le Q_{ij}, \qquad i,j\in[n_x], \tag{O-11q}$$

$$t_j\exp\left(\frac{W_{ij}}{t_j}\right)\le T_{ij}, \qquad i,j\in[n_x], \tag{O-11r}$$

$$\exp\left(x_i+x_j\right)\le T_{ij}, \qquad i\le j\in[n_x]. \tag{O-11s}$$

Further, from the decomposition of the exponential cone we obtain the following additional constraints:

$$-x_i-\alpha+1\le z_i, \qquad i\in[n_x], \tag{O-12a}$$

$$x_i+1\le t_i, \qquad i\in[n_x], \tag{O-12b}$$

$$z_i-\sum_{j=1}^{n_x}Z_{ij}+x_i+\alpha-1-\sum_{j=1}^{n_x}V_{ij}-\alpha\sum_{j=1}^{n_x}z_j+\sum_{j=1}^{n_x}z_j\ge 0, \qquad i\in[n_x], \tag{O-12c}$$

$$(z_j+x_j+\alpha-1)\exp\left(\frac{-V_{ij}-X_{ij}-\alpha x_i+x_i-\alpha(z_j+x_j+\alpha-1)}{z_j+x_j+\alpha-1}\right)\le Z_{ij}+V_{ji}+\alpha z_i-z_i, \qquad i,j\in[n_x], \quad \text{(O-12d)}$$

$$(z_j+x_j+\alpha-1)\exp\left(\frac{V_{ij}+X_{ij}+\alpha x_i-x_i}{z_j+x_j+\alpha-1}\right)\le Q_{ji}+W_{ji}+\alpha t_i-t_i, \qquad i,j\in[n_x], \quad \text{(O-12e)}$$

$$(\alpha-1)(z_j+x_j+\alpha-1)+Z_{ij}+V_{ji}+(\alpha-1)z_i+V_{ij}+X_{ij}+(\alpha-1)x_i\ge 0, \qquad i,j\in[n_x], \quad \text{(O-12f)}$$

$$t_i-\sum_{j=1}^{n_x}Q_{ji}-x_i+\sum_{j=1}^{n_x}V_{ij}-1+\sum_{j=1}^{n_x}z_j\ge 0, \qquad i\in[n_x], \quad \text{(O-12g)}$$

$$(t_j-x_j-1)\exp\left(\frac{-W_{ij}-\alpha t_j+X_{ij}+\alpha(x_j+1)+x_i}{t_j-x_j-1}\right)\le Q_{ij}-V_{ji}-z_i, \qquad i,j\in[n_x], \quad \text{(O-12h)}$$

$$(t_j-x_j-1)\exp\left(\frac{W_{ij}-X_{ij}-x_i}{t_j-x_j-1}\right)\le T_{ij}-Q_{ji}-t_i, \qquad i,j\in[n_x], \quad \text{(O-12i)}$$

$$T_{ij}-W_{ij}-t_j-W_{ji}+X_{ij}+x_j-t_i+x_i+1\ge 0, \qquad i,j\in[n_x], \quad \text{(O-12j)}$$

$$Q_{ji}+W_{ji}+\alpha t_i-t_i-V_{ij}-X_{ij}-\alpha x_i+x_i-z_j-x_j-\alpha+1\ge 0, \qquad i,j\in[n_x], \quad \text{(O-12k)}$$

$$\beta z_j-\sum_i Q_{ji}+\beta x_j-\sum_i W_{ji}-\alpha\sum_i t_i+\alpha\beta-\beta+\sum_i t_i\ge 0, \qquad j\in[n_x], \quad \text{(O-12l)}$$

$$\beta t_j-\sum_i T_{ij}-\beta x_j+\sum_i W_{ji}-\beta+\sum_i t_i\ge 0, \qquad j\in[n_x]. \quad \text{(O-12m)}$$

C.2. RPT formulation of Section 6.2

We linearize $\boldsymbol{r}\boldsymbol{r}^\top$, $\boldsymbol{t}\boldsymbol{t}^\top$, $\boldsymbol{r}\boldsymbol{t}^\top$ with $\boldsymbol{R},\boldsymbol{T}$ and $\boldsymbol{V}$ respectively. We multiply all constraints in problem (40) to obtain the following problem:

$$\max_{\substack{\boldsymbol{r},\boldsymbol{t},\boldsymbol{R},\\ \boldsymbol{T},\boldsymbol{V}}} \quad \boldsymbol{r}^\top\boldsymbol{A}\boldsymbol{r}+\boldsymbol{b}^\top\boldsymbol{r}+d \quad \text{(O-13a)}$$

$$\text{s.t.} \quad \boldsymbol{c}^\top\boldsymbol{r}\le W, \quad \text{(O-13b)}$$

$$\boldsymbol{\beta}_l^\top\boldsymbol{r}\ge\eta_l, \qquad l\in\mathcal{L}\setminus\{m\}, \quad \text{(O-13c)}$$

$$\sum_k t_k=\frac{1}{\rho}(\boldsymbol{\beta}_m^\top\boldsymbol{r}-\eta_m), \quad \text{(O-13d)}$$

$$t_k^{1/q}\left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top\boldsymbol{r}-\eta_m)\right)^{1-1/q}\ge r_k, \qquad k\in\mathcal{K}, \quad \text{(O-13e)}$$

$$\boldsymbol{\beta}_l^\top\boldsymbol{R}_k\ge\eta_l r_k, \qquad l\in\mathcal{L}\setminus\{m\},\ k\in\mathcal{K}, \quad \text{(O-13f)}$$

$$\left(\boldsymbol{\beta}_l^\top\boldsymbol{V}_k-\eta_l t_k\right)^{\frac{1}{q}}\left(\frac{1}{\rho}\left(\boldsymbol{\beta}_1^\top\boldsymbol{R}\boldsymbol{\beta}_l-\eta_l\boldsymbol{\beta}_1^\top\boldsymbol{r}-\eta_1\boldsymbol{\beta}_l^\top\boldsymbol{r}+\eta_1\eta_l\right)\right)^{1-1/q} \ge\boldsymbol{\beta}_l^\top\boldsymbol{R}_k-\eta_l r_k, \qquad l\in\mathcal{L}\setminus\{m\},\ k\in\mathcal{K}, \quad \text{(O-13g)}$$

$$\boldsymbol{c}^\top\boldsymbol{R}_k\le W r_k, \qquad k\in\mathcal{K}, \quad \text{(O-13h)}$$

$$W^2-2W\boldsymbol{c}^\top\boldsymbol{r}+\boldsymbol{c}^\top\boldsymbol{R}\boldsymbol{c}\ge 0, \quad \text{(O-13i)}$$

$$W\boldsymbol{\beta}_l^\top\boldsymbol{r}+\eta_l\boldsymbol{c}^\top\boldsymbol{r}-\boldsymbol{c}^\top\boldsymbol{R}\boldsymbol{\beta}_l-W\eta_l\ge 0, \qquad l\in\mathcal{L}\setminus\{m\}, \quad \text{(O-13j)}$$

$$\left(Wt_k-\boldsymbol{c}^\top\boldsymbol{V}_k\right)^{\frac{1}{q}}\left(\frac{1}{\rho}\left(W\boldsymbol{\beta}_m^\top\boldsymbol{r}-W\eta_m+\eta_m\boldsymbol{c}^\top\boldsymbol{r}-\boldsymbol{c}^\top\boldsymbol{R}\boldsymbol{\beta}_m\right)\right)^{1-1/q}$$

$$\geq Wr_k - \boldsymbol{c}^\top \boldsymbol{R}_k \qquad k \in \mathcal{K}, \tag{O-13k}$$

$$\sum_k \boldsymbol{V}_k = \frac{1}{\rho}(\boldsymbol{R}\boldsymbol{\beta}_m - \eta_m \boldsymbol{r}), \tag{O-13l}$$

$$\sum_k \boldsymbol{T}_k = \frac{1}{\rho}(\boldsymbol{V}^\top \boldsymbol{\beta}_m - \eta_m \boldsymbol{t}), \tag{O-13m}$$

$$V_{k'k}^{1/q}\left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{R}_{k'} - \eta_m r_{k'})\right)^{1-1/q} \geq R_{kk'}, \qquad k, k' \in \mathcal{K}, \tag{O-13n}$$

$$T_{kk'}^{\theta_{11}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_k - \eta_m t_k\right)\right)^{\theta_{12}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_{k'} - \eta_m t_{k'}\right)\right)^{\theta_{21}} \cdot \left(\frac{1}{\rho^2}\left(\boldsymbol{\beta}_m^\top \boldsymbol{R}\boldsymbol{\beta}_m - 2\eta_m \boldsymbol{\beta}_m^\top \boldsymbol{r} + \eta_m^2\right)\right)^{\theta_{22}} \geq R_{kk'}, \qquad k, k' \in \mathcal{K}, \tag{O-13o}$$

$$\boldsymbol{r}, \boldsymbol{R} \geq \boldsymbol{0}, \tag{O-13p}$$

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{V} & \boldsymbol{r} \\ \boldsymbol{V}^\top & \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{r}^\top & \boldsymbol{t}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}. \tag{O-13q}$$

For the multiplication of the power cone constraints we generate a feasible $\boldsymbol{\theta}$ that satisfies the following:

$$\theta_{11} + \theta_{21} = 1/q, \ \theta_{12} + \theta_{22} = 1 - 1/q,$$
$$\theta_{11} + \theta_{12} = 1/q, \ \theta_{21} + \theta_{22} = 1 - 1/q.$$

D. Data generation of Section 6.1

In problem instance 1 the objective is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i + 5)^2$ and in problem instance 2 it is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i + 7)^2$. In problem instances 3, 4, 5, and 6, the matrix $\boldsymbol{A}_0$ is generated as $\boldsymbol{L}^\top \boldsymbol{L}$, where $\boldsymbol{L} \in \mathbb{R}^{n \times n}$, with $L_{ij} \sim [0, 1]$, and further $\boldsymbol{b}_0 = \boldsymbol{0}$, $c_0 = 0$. We summarize all parameters describing each instance in Table O-2.

Instance	n_x	α	β
1	5	2	20
2	5	2	20
3	10	2	3
4	20	3	4
5	50	3	4
6	100	13	20

Table O-2 Problem (37) parameters for each instance. n_x refers to the number of variables and α, β to the constraint parameters.

E. Value of constraint multiplications: examples

In this appendix, we demonstrate that the additional constraints obtained from multiplying part of one cone inequality with part of the other cone inequality can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities. In particular, for each case we have found an example demonstrating that incorporating the resulting constraints leads to a strict improvement in the optimal objective value.

Remark O-1. In all convex relaxations discussed in the examples in this appendix we consider the additional inequalities $u_{ii} \geq 0$, where u_{ii} linearizes the product term x_i^2 for $i \in [n_x]$.

Product with a power cone inequality

E.1. Case 3: (L) $\times$ (P)

The following example demonstrates that the additional constraints resulting from multiplying the linear with the power cone inequality as discussed in Section 3.2 can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Example O-1. We consider the following example

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(O-14a)}\\
\text{s.t.} \quad & x_3 \geq 5, && \text{(O-14b)}\\
& x_1^{0.5} x_2^{0.5} \geq x_1 + x_2, && \text{(O-14c)}\\
& x_1, x_2 \geq 0. && \text{(O-14d)}
\end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && && \text{(O-15a)}\\
& \text{(O-14b)} - \text{(O-14d)}, && && \\
& u_{33} - 10x_3 + 25 \geq 0, && && \text{(O-15b)}\\
& (u_{13} - 5x_1)^{0.5}(u_{23} - 5x_2)^{0.5} \geq u_{13} + u_{23} - 5x_1 - 5x_2, && && \text{(O-15c)}\\
& u_{i3} - 5x_i \geq 0, && i \in [2], && \text{(O-15d)}\\
& u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i1} + u_{i2}, && i \in [2], && \text{(O-15e)}\\
& u_{11}^{0.5} u_{22}^{0.5} \geq u_{11} + 2u_{12} + u_{22}, && && \text{(O-15f)}\\
& u_{11}, u_{12}, u_{22} \geq 0, && && \text{(O-15g)}
\end{aligned}$$

with optimal objective value 0.00. The most valuable additional constraints in this case are (O-15c), (O-15d), which result from the multiplication of the linear inequality (O-14b) with the power cone inequalities (O-14c) and (O-14d), as without these multiplications we get an optimal objective value of $-\infty$. □

We refer to Example 1 demonstrating that the additional constraints derived from the best reformulation (Case 3(ii)) can outperform all other possible additional constraints.

E.2. Case 7: (Q) × (P)

In Table O-3 we give an overview of the constraint multiplications as discussed Section 3.3. Moreover, in this section we provide examples demonstrating that these additional constraints can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Case	Constraints 1	Constraints 2	Example
7(i)	1. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	O-2
	2. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0$	$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	O-3
	3. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$x_i \geq 0,\ i \in [m]$	O-4
7(ii)	Best reformulation		1

Table O-3 Overview of the constraint multiplications for Case 7 as discussed in Section 3.3.

Example O-2. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_3 x_4 \tag{O-16a}$$

$$\text{s.t.} \quad 3x_1 \geq \lVert \operatorname{diag}(\boldsymbol{a})\boldsymbol{x} \rVert, \tag{O-16b}$$

$$x_1^{0.5} x_2^{0.5} \geq |x_3|, \tag{O-16c}$$

$$x_1, x_2 \geq 0, \tag{O-16d}$$

$$x_1 \exp(x_1) \leq 5, \tag{O-16e}$$

$$x_1 \exp(x_2) \leq 5, \tag{O-16f}$$

where $\boldsymbol{a} = (1,0,0,1)^\top$. We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && \text{(O-17a)}\\
\text{s.t.} \quad & \text{(O-16b)} - \text{(O-16d)}, \\
& x_1 \exp\left(\frac{u_{11}}{x_1}\right) \le 5, && \text{(O-17b)}\\
& x_1 \exp\left(\frac{u_{12}}{x_1}\right) \le 5, && \text{(O-17c)}\\
& 9u_{11} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top\right\|_F, && \text{(O-17d)}\\
& u_{11}^{0.5}u_{12}^{0.5} \ge |u_{13}|, && \text{(O-17e)}\\
& u_{12}^{0.5}u_{22}^{0.5} \ge |u_{23}|, && \text{(O-17f)}\\
& u_{11}^{0.5}u_{22}^{0.5} \ge |u_{33}|, && \text{(O-17g)}\\
& 3u_{1i} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, \quad i \in [2], && \text{(O-17h)}\\
& (3u_{11})^{0.5}(3u_{12})^{0.5} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_3\right\|, && \text{(O-17i)}\\
& u_{11}, u_{12}, u_{22} \ge 0, && \text{(O-17j)}
\end{aligned}
$$

with optimal objective value -5.52. The most valuable additional constraint multiplication is (O-17i), which results from the multiplication of the quadratic inequality (O-16b) with the power cone inequality (O-16c). Without this constraint we get an optimal objective value of $-\infty$.

□

EXAMPLE O-3. Consider the following example

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_3x_4 && \text{(O-18a)}\\
\text{s.t.} \quad & 3x_3 \ge \|\operatorname{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(O-18b)}\\
& x_1^{0.5}x_2^{0.5} \ge |x_4|, && \text{(O-18c)}\\
& x_1 \exp(x_3) \le 5, && \text{(O-18d)}\\
& x_2 \exp(x_3) \le 5, && \text{(O-18e)}\\
& x_1, x_2 \ge 0, && \text{(O-18f)}
\end{aligned}
$$

where $\boldsymbol{a} = (1,1,0,0)^\top$. We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && \text{(O-19a)}\\
\text{s.t.} \quad & \text{(O-18b)} - \text{(O-18c)}, \text{(O-18f)} \\
& x_1 \exp\left(\frac{u_{13}}{x_1}\right) \le 5, && \text{(O-19b)}
\end{aligned}
$$

$$x_2 \exp\left(\frac{u_{23}}{x_2}\right) \leq 5, \tag{O-19c}$$

$$3u_{33} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{u}_3\right\|, \tag{O-19d}$$

$$9u_{33} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{U}\text{diag}(\boldsymbol{a})^\top\right\|_F, \tag{O-19e}$$

$$u_{1i}^{0.5}u_{2i}^{0.5} \geq |u_{i4}|, \qquad i \in [2], \tag{O-19f}$$

$$u_{11}^{0.5}u_{22}^{0.5} \geq |u_{44}|, \tag{O-19g}$$

$$\left\|\text{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\| \leq 3u_{i3}, \qquad i \in [2], \tag{O-19h}$$

$$(3u_{13})^{0.5}(3u_{23})^{0.5} \geq |3u_{34}|, \tag{O-19i}$$

$$(3u_{13})^{0.5}(3u_{23})^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{u}_4\right\|, \tag{O-19j}$$

$$u_{11}, u_{12}, u_{22} \geq 0, \tag{O-19k}$$

with optimal objective value -1.84. The most valuable constraint multiplication in this case is (O-19i), which results from the multiplication of the LHS of the conic quadratic inequality (O-18b) with the power cone inequality (O-18c). Without this constraint we would obtain an optimal objective value of $-\infty$. □

EXAMPLE O-4. Consider the following example

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1x_3 && \text{(O-20a)}\\
\text{s.t.} \quad & x_1\exp(x_1) \leq 5, && \text{(O-20b)}\\
& x_2\exp(x_2) \leq 5, && \text{(O-20c)}\\
& 3x_2 \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{x}\right\|, && \text{(O-20d)}\\
& x_1^{0.5}x_2^{0.5} \geq 2x_2, && \text{(O-20e)}\\
& x_1, x_2 \geq 0, && \text{(O-20f)}
\end{aligned}$$

where $\boldsymbol{a} = (0,1,1)^\top$. We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && && \text{(O-21a)}\\
\text{s.t.} \quad & \text{(O-20d)} - \text{(O-20f)} && && \\
& x_1\exp\left(\frac{u_{11}}{x_1}\right) \leq 5, && && \text{(O-21b)}\\
& x_2\exp\left(\frac{u_{22}}{x_2}\right) \leq 5, && && \text{(O-21c)}\\
& 3u_{i2} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, && i \in [2], && \text{(O-21d)}\\
& \left\|\text{diag}(\boldsymbol{a})\boldsymbol{U}\text{diag}(\boldsymbol{a})^\top\right\|_F \leq 9u_{22}, && && \text{(O-21e)}\\
& u_{i1}^{0.5}u_{i2}^{0.5} \geq 2u_{i2}, && i \in [2], && \text{(O-21f)}
\end{aligned}$$

$$u_{11}, u_{12}, u_{22} \geq 0, \tag{O-21g}$$

$$u_{11}^{0.5} u_{22}^{0.5} \geq 4u_{22}, \tag{O-21h}$$

$$(3u_{12})^{0.5}(3u_{22})^{0.5} \geq \left\|2\text{diag}(\boldsymbol{a})\boldsymbol{u}_2\right\|, \tag{O-21i}$$

with optimal objective value -1.30. The most valuable constraint multiplications in this case are (O-21d), as without this constraint we would obtain an optimal objective value of $-\infty$, and (O-21f) & (O-21g), as without these constraints we would also obtain an optimal objective value of $-\infty$. (O-21d) results from the multiplication of the conic quadratic inequality (O-20d) with the nonnegativity constraints of the power cone (O-20f). (O-21f) & (O-21g) result from the multiplications of the nonnegativity constraints (O-20f) with the power cone inequalities (O-20d) and (O-20f). □

E.3. Case 10: (P) × (P)

In Table O-4 we give an overview of the constraints multiplications discussed in Section 3.4. Moreover, in this section we provide examples demonstrating that these additional constraints can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Case	Constraints 1	Constraints 2	Example
10(i)	1. $\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	$\prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2}$	O-5
	2. $x_i \geq 0,\ i \in [m_1]$	$\begin{cases} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\ x_{\sigma(j)} \geq 0,\ j \in [m_2], \end{cases}$	O-6 / O-7 / O-8
	3. $\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \geq 0,\ i \in [m_1] \end{cases}$	$x_{\sigma(j)} \geq 0,\ j \in [m_2]$	O-6 / O-7 / O-8
10(ii)	Best reformulation		1

Table O-4 Overview of the constraint multiplications for Case 10 as discussed in Section 3.4.

Example O-5. Consider the following example

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 \exp(x_1) && \text{(O-22a)}\\
\text{s.t.} \quad & x_1 x_1 \geq 10, && \text{(O-22b)}\\
& x_2 x_2 \geq 5, && \text{(O-22c)}\\
& x_3 x_3 \geq 10, && \text{(O-22d)}\\
& x_2 \exp(x_1) \leq 25, && \text{(O-22e)}\\
& x_1^{0.5} x_2^{0.5} \geq x_3, && \text{(O-22f)}\\
& x_1, x_2 \geq 0, && \text{(O-22g)}\\
& x_3 \geq 0. && \text{(O-22h)}
\end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & x_1 \exp\left(\frac{u_{11}}{x_1}\right) && && \text{(O-23a)}\\
\text{s.t.} \quad & u_{11} \geq 10, && && \text{(O-23b)}\\
& u_{22} \geq 5, && && \text{(O-23c)}\\
& u_{33} \geq 10, && && \text{(O-23d)}\\
& x_2 \exp\left(\frac{u_{22}}{x_2}\right) \leq 25, && && \text{(O-23e)}\\
& \text{(O-22f)} - \text{(O-22h)} && && \\
& u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i3}, && i \in [2], && \text{(O-23f)}\\
& u_{11}^{0.5} u_{22}^{0.5} \geq u_{33}, && && \text{(O-23g)}\\
& u_{13}^{0.5} u_{23}^{0.5} \geq u_{33}, && && \text{(O-23h)}\\
& u_{11}, u_{12}, u_{22} \geq 0, && && \text{(O-23i)}\\
& u_{13}, u_{23} \geq 0, && && \text{(O-23j)}\\
& u_{33} \geq 0, && && \text{(O-23k)}
\end{aligned}$$

with optimal objective value 29.56.

The most valuable constraint multiplication is (O-23g), resulting from the multiplication of the power cone inequality (O-22f) with itself. Without this constraint we would obtain an optimal objective value of 27.18. □

Example O-6. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \qquad \text{(O-24a)}$$

$$\text{s.t.} \quad x_1 x_2 \geq 1, \tag{O-24b}$$

$$x_2 x_3 \geq 1, \tag{O-24c}$$

$$x_3 x_3 \geq 5, \tag{O-24d}$$

$$x_1^{0.5} x_2^{0.5} \geq x_3, \tag{O-24e}$$

$$x_1, x_2 \geq 0, \tag{O-24f}$$

$$x_3 \geq \exp(x_2). \tag{O-24g}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{O-25a}$$

$$\text{s.t.} \quad u_{12} \geq 1, \tag{O-25b}$$

$$u_{23} \geq 1, \tag{O-25c}$$

$$u_{33} \geq 5, \tag{O-25d}$$

$$\text{(O-24e)} - \text{(O-24g)},$$

$$u_{11}, u_{22}, u_{12} \geq 0, \tag{O-25e}$$

$$u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i3}, \qquad i \in [2], \tag{O-25f}$$

$$u_{11}^{0.5} u_{22}^{0.5} \geq u_{33}, \tag{O-25g}$$

$$u_{33} \geq x_3 \exp\left(\frac{u_{23}}{x_3}\right), \tag{O-25h}$$

$$u_{33} \geq \exp(2x_2), \tag{O-25i}$$

$$x_i \exp\left(\frac{u_{i2}}{x_i}\right) \leq u_{i3}, \qquad i \in [2], \tag{O-25j}$$

$$u_{13}^{0.5} u_{23}^{0.5} \geq u_{33}, \tag{O-25k}$$

with optimal objective value 13.65.

The most valuable additional constraints in this case are (O-25e) - (O-25f), (O-25j), and (O-25k). Without these constraints we would obtain an optimal objective value of 13.59, 13.59, and 9.07 respectively. (O-25e) - (O-25f) result from the multiplication of the nonnegativity constraints (O-24f) of the power cone inequality with themselves and the power cone inequality (O-24e). (O-25j) results from the multiplication of the nonnegativity constraints (O-24f) of the power cone inequality with the exponential cone inequality (O-24g). (O-25k) results from the multiplication of the power cone inequality (O-24e) with the LHS of the exponential cone inequality (O-24g). □

Example O-7. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_3 x_4 \tag{O-26a}$$

$$\begin{aligned}\text{s.t.}\quad & x_1^{0.5}x_2^{0.5} \geq \|\text{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(O-26b)}\\ & x_1, x_2 \geq 0, && \text{(O-26c)}\\ & x_2 + x_3 \geq \exp(x_1+x_2), && \text{(O-26d)}\\ & (x_1+x_2)\exp(x_i) \leq 5, \quad i \in [4]. && \text{(O-26e)}\end{aligned}$$

where $\boldsymbol{a} = (0,1,0,1)^\top$. We obtain the following RPT relaxation

$$\begin{aligned}\min_{\boldsymbol{x},\boldsymbol{U}}\quad & u_{34} && \text{(O-27a)}\\ \text{s.t.}\quad & \text{(O-26b)} - \text{(O-26d)}, && \\ & (x_1+x_2)\exp\left(\frac{u_{1i}+u_{2i}}{x_1+x_2}\right) \leq 5, \quad i \in [4], && \text{(O-27b)}\\ & u_{1i}^{0.5}u_{2i}^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, \quad i \in [2], && \text{(O-27c)}\\ & u_{11}, u_{12}, u_{22} \geq 0, && \text{(O-27d)}\\ & u_{11}^{0.5}u_{22}^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})\boldsymbol{U}\text{diag}(\boldsymbol{a})^\top\right\|_F, && \text{(O-27e)}\\ & u_{22}+2u_{23}+u_{33} \geq (x_2+x_3)\exp\left(\frac{u_{12}+u_{22}+u_{13}+u_{23}}{x_2+x_3}\right), && \text{(O-27f)}\\ & u_{22}+2u_{23}+u_{33} \geq \exp(2x_1+2x_2), && \text{(O-27g)}\\ & u_{i2}+u_{i3} \geq x_i\exp\left(\frac{u_{1i}+u_{2i}}{x_i}\right), \quad i \in [2], && \text{(O-27h)}\\ & (u_{12}+u_{13})^{0.5}(u_{22}+u_{23})^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})(\boldsymbol{u}_2+\boldsymbol{u}_3)\right\|, && \text{(O-27i)}\\ & x_3 \geq x_1+1+y, && \text{(O-27j)}\\ & 1+y \geq \left\|\left(\sqrt{2}(x_1+x_2),\ 1-y\right)\right\|, && \text{(O-27k)}\\ & (u_{13}-u_{11}-x_1-z_1)^{0.5}(u_{23}-u_{12}-x_2-z_2)^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})(\boldsymbol{u}_3-\boldsymbol{u}_1-\boldsymbol{x}-\boldsymbol{z})\right\|, && \text{(O-27l)}\\ & (x_1+z_1)^{0.5}(x_2+z_2)^{0.5} \geq \left\|\text{diag}(\boldsymbol{a})\left(\boldsymbol{x}+\boldsymbol{z}\right)\right\|, && \text{(O-27m)}\\ & (x_1+z_1)^{0.5}(x_2+z_2)^{0.5} \geq \left\|\begin{pmatrix}\sqrt{2}(u_{12}+u_{22}) & \sqrt{2}(u_{14}+u_{24})\\ x_2-z_2 & x_4-z_4\end{pmatrix}\right\|, && \text{(O-27n)}\\ & u_{23}+u_{33}-u_{12}-u_{13}-x_2-x_3-z_2-z_3 \geq && \\ & \qquad (x_3-x_1-1-y)\exp\left(\frac{u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2}{x_3-x_1-1-y}\right), && \text{(O-27o)}\\ & x_2+x_3+z_2+z_3 \geq (1+y)\exp\left(\frac{x_1+x_2+z_1+z_2}{1+y}\right), && \text{(O-27p)}\end{aligned}$$

with optimal objective value -0.65.

The most valuable additional constraints in this case are (O-27c)-(O-27d) and (O-27l)-(O-27n). Without these constraints we would obtain an optimal objective value of -0.79 and -0.76 respectively. (O-27c)-(O-27d) result from the multiplication of the nonnegativity constraints (O-26c) of the power cone with the power cone inequalities (O-26b) and (O-26c). (O-27l)-(O-27n) result from the multiplication of the additional inequalities (O-27j) and (O-27k) with the power cone inequality (O-26b). □

Example O-8. Consider the following example

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_3x_4 && \text{(O-28a)}\\
\text{s.t.} \quad & x_1^{0.5}x_2^{0.5} \geq |x_3|, && \text{(O-28b)}\\
& x_1, x_2 \geq 0. && \text{(O-28c)}\\
& x_2 \geq \exp(-x_4), && \text{(O-28d)}\\
& x_2 \exp(x_1) \leq 5, && \text{(O-28e)}\\
& x_2 \exp(x_2) \leq 10, && \text{(O-28f)}\\
& x_1 \exp(x_4) \leq 10, && \text{(O-28g)}\\
& x_2 \exp(x_4) \leq 10, && \text{(O-28h)}
\end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && && \text{(O-29a)}\\
\text{s.t.} \quad & \text{(O-28b)} - \text{(O-28d)} && &&\\
& x_2 \exp\left(\frac{u_{12}}{x_2}\right) \leq 5, && && \text{(O-29b)}\\
& x_2 \exp\left(\frac{u_{22}}{x_2}\right) \leq 10, && && \text{(O-29c)}\\
& x_1 \exp\left(\frac{u_{14}}{x_1}\right) \leq 10, && && \text{(O-29d)}\\
& x_2 \exp\left(\frac{u_{24}}{x_2}\right) \leq 10, && && \text{(O-29e)}\\
& u_{1i}^{0.5}u_{2i}^{0.5} \geq |u_{i3}|, && i \in [2], && \text{(O-29f)}\\
& u_{11}, u_{12}, u_{22} \geq 0, && && \text{(O-29g)}\\
& u_{11}^{0.5}u_{22}^{0.5} \geq |u_{33}|, && && \text{(O-29h)}\\
& u_{22} \geq \exp(-2x_4), && && \text{(O-29i)}\\
& u_{i2} \geq x_i \exp\left(\frac{-u_{i4}}{x_i}\right), && i \in [2], && \text{(O-29j)}\\
& x_2 \geq -x_4 + 1, && && \text{(O-29k)}\\
& (u_{12} + u_{14} - x_1)^{0.5}(u_{22} + u_{24} - x_2)^{0.5} \geq |u_{23} + u_{34} - x_3|, && && \text{(O-29l)}\\
& u_{22} + u_{24} - x_2 \geq (x_2 + x_4 - 1)\exp\left(\frac{-u_{24} - u_{44} + x_4}{x_2 + x_4 - 1}\right), && && \text{(O-29m)}
\end{aligned}$$

with optimal objective value -8.43.

The most valuable additional constraints in this case are (O-29f) - (O-29g), and (O-29l). Without these constraints we would obtain an optimal objective value of $-\infty$ in both cases. (O-29f) - (O-29g) result from the multiplication of the nonnegativity constraints (O-28c) with the power cone inequalities (O-28b) and (O-28c). (O-29l) results from the multiplication of inequality (O-29k), resulting from the decomposition of the exponential cone, with the power cone inequality (O-28b).

□

Product with an exponential cone inequality

E.4. Case 4: (L) × (E)

Example O-9 demonstrates that the additional constraints obtained from multiplying the linear inequality with the exponential cone inequality as discussed in Section 4.2 can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Example O-9. Consider the following example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(O-30a)} \\ \text{s.t.} \quad & x_2 \exp(x_1) \leq 5, && \text{(O-30b)} \\ & x_1 \geq 4, && \text{(O-30c)} \\ & x_2 \geq \exp(-x_3). && \text{(O-30d)} \end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(O-31a)} \\ \text{s.t.} \quad & x_2 \exp\left(\frac{u_{12}}{x_2}\right) \leq 5, && \text{(O-31b)} \\ & \text{(O-30c)} - \text{(O-30d)}, && \\ & u_{11} - 8x_1 + 16 \geq 0, && \text{(O-31c)} \\ & u_{22} \geq \exp\left(-2x_3\right), && \text{(O-31d)} \\ & u_{22} \geq x_2 \exp\left(\frac{-u_{23}}{x_2}\right), && \text{(O-31e)} \\ & u_{12} - 4x_2 \geq (x_1 - 4) \exp\left(\frac{-u_{13} + 4x_3}{x_1 - 4}\right), && \text{(O-31f)} \end{aligned}$$

with optimal objective value 9.56. The most valuable additional constraint in this case is (O-30d), resulting from the multiplication of the linear inequality (O-30c) with the exponential cone inequality (O-30d). Without this constraint we would obtain an optimal objective value of $-\infty$.

□

Case	Constraints 1	Constraints 2	Example
8(i)	1. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\Vert$	$x_2 \geq 0$	O-10
	2. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\Vert$	$x_1 \geq 0$	O-11
	3. $b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0$	$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	O-12
8(ii)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\Vert$	$x_1 \geq x_2 + x_3$	O-13
8(iii)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\Vert$	$\begin{cases} x_1 \geq x_2 + x_3 + y \\ \left\Vert (\sqrt{2}x_3, x_2 - y) \right\Vert_2 \leq x_2 + y \end{cases}$	O-12

Table O-5 Overview of the constraint multiplications for Case 8 as discussed in Section 4.3.

E.5. Case 8: (Q) $\times$ (E)

In Table O-5 we give an overview of the constraint multiplications discussed in Section 4.3. Moreover, in this section we provide examples demonstrating the value of each proposed multiplication.

EXAMPLE O-10. Consider the following example

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(O-32a)}\\
\text{s.t.} \quad & 3x_2 \geq \left\Vert \operatorname{diag}(\boldsymbol{a})\boldsymbol{x} \right\Vert, && \text{(O-32b)}\\
& x_3 \exp\left(\frac{-x_2}{x_3}\right) \leq 5, && \text{(O-32c)}\\
& x_3 \geq 0, && \text{(O-32d)}\\
& x_3 \exp(x_2) \leq 10, && \text{(O-32e)}
\end{aligned}$$

where $\boldsymbol{a} = (1,1,0)^\top$. We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(O-33a)}\\
\text{s.t.} \quad & \text{(O-32b)} - \text{(O-32d)}, && \text{(O-33b)}\\
& x_3 \exp\left(\frac{u_{23}}{x_3}\right) \leq 10, && \text{(O-33c)}\\
& 9u_{22} \geq \left\Vert 3\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_2 \right\Vert, && \text{(O-33d)}\\
& 9u_{22} \geq \left\Vert \operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top \right\Vert_F, && \text{(O-33e)}\\
& u_{33} \exp\left(\frac{-2u_{23}}{u_{33}}\right) \leq 25, && \text{(O-33f)}\\
& u_{33} \exp\left(\frac{-u_{23}}{u_{33}}\right) \leq 5x_3, && \text{(O-33g)}
\end{aligned}$$

$$3u_{23} \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_3\|, \tag{O-33h}$$

$$u_{23}\exp\left(\frac{-u_{22}}{u_{23}}\right) \leq 5x_2, \tag{O-33i}$$

$$u_{33} \geq 0, \tag{O-33j}$$

with optimal objective value -10.41.

The most valuable additional constraint in this case is (O-33h), resulting from the multiplication of the conic quadratic inequality (O-32b) with the nonnegativity constraint (O-32d) of the exponential cone inequality. Without this constraint we would obtain an optimal objective value of $-\infty$. □

Example O-11. Consider the following example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(O-34a)}\\ \text{s.t.} \quad & 3x_1 \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(O-34b)}\\ & x_3 \geq \exp(x_1), && \text{(O-34c)}\\ & x_2 x_3 \geq 3, && \text{(O-34d)} \end{aligned}$$

where $\boldsymbol{a} = (1,1,0)^\top$. We obtain the following RPT relaxation

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(O-35a)}\\ \text{s.t.} \quad & \text{(O-34b)} - \text{(O-34c)} && \text{(O-35b)}\\ & u_{23} \geq 3, && \text{(O-35c)}\\ & 9u_{11} \geq \|3\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_1\|, && \text{(O-35d)}\\ & 9u_{11} \geq \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{U}\mathrm{diag}(\boldsymbol{a})^\top\right\|_F, && \text{(O-35e)}\\ & u_{33} \geq \exp(2x_1), && \text{(O-35f)}\\ & u_{33} \geq x_3\exp\left(\frac{u_{13}}{x_3}\right), && \text{(O-35g)}\\ & 3u_{13} \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_3\|, && \text{(O-35h)}\\ & u_{13} \geq x_1\exp\left(\frac{u_{11}}{x_1}\right), && \text{(O-35i)} \end{aligned}$$

with objective value 1.06. observe that The most valuable additional constraint in this case is (O-35h), which results from the multiplication of the conic quadratic inequality (O-34b) with the LHS of the exponential cone inequality (O-34c). Without this constraint we would obtain an optimal objective value of 0.00. □

Example O-12. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_3x_4 \tag{O-36a}$$
$$\text{s.t.} \quad x_1+x_2 \geq \|\text{diag}(\boldsymbol{a})\boldsymbol{x}\|, \tag{O-36b}$$
$$x_2+x_3 \geq \exp(x_1+x_2), \tag{O-36c}$$
$$(x_1+x_2)\exp(x_i) \leq 5, \quad i\in[4]. \tag{O-36d}$$

where $\boldsymbol{a}=(0,1,0,1)^\top$. We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{34} \tag{O-37a}$$
$$\text{s.t.} \quad \text{(O-36b)} - \text{(O-36c)},$$
$$(x_1+x_2)\exp\left(\frac{u_{1i}+u_{2i}}{x_1+x_2}\right) \leq 5, \quad i\in[4], \tag{O-37b}$$
$$\left\|\text{diag}(\boldsymbol{a})(\boldsymbol{u}_1+\boldsymbol{u}_2)\right\| \leq u_{11}+2u_{12}+u_{22}, \tag{O-37c}$$
$$\left\|\text{diag}(\boldsymbol{a})\boldsymbol{U}\text{diag}(\boldsymbol{a})^\top\right\|_F \leq u_{11}+2u_{12}+u_{22}, \tag{O-37d}$$
$$\left\|\text{diag}(\boldsymbol{a})(\boldsymbol{u}_2+\boldsymbol{u}_3)\right\| \leq u_{12}+u_{22}+u_{13}+u_{23}, \tag{O-37e}$$
$$(x_2+x_3)\exp\left(\frac{u_{12}+u_{22}+u_{13}+u_{23}}{x_2+x_3}\right) \leq u_{22}+2u_{23}+u_{33}, \tag{O-37f}$$
$$\exp(2x_1+2x_2) \leq u_{22}+2u_{23}+u_{33}, \tag{O-37g}$$
$$(x_1+x_2)\exp\left(\frac{u_{11}+2u_{12}+u_{22}}{x_1+x_2}\right) \leq u_{12}+u_{22}+u_{13}+u_{23}, \tag{O-37h}$$
$$x_3 \geq x_1+1+y, \tag{O-37i}$$
$$\left\|\left(\sqrt{2}(x_1+x_2),\ 1-y\right)\right\| \leq 1+y, \tag{O-37j}$$
$$\left\|\text{diag}(\boldsymbol{a})\left(\boldsymbol{u}_3-\boldsymbol{u}_1-\boldsymbol{x}-\boldsymbol{z}\right)\right\| \leq u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2, \tag{O-37k}$$
$$\left\|\left(\sqrt{2}(u_{11}+2u_{12}+u_{22}),\ x_1+x_2-z_1-z_2\right)\right\| \leq x_1+x_2+z_1+z_2, \tag{O-37l}$$
$$\left\|\text{diag}(\boldsymbol{a})\left(\boldsymbol{x}+\boldsymbol{z}\right)\right\| \leq x_1+x_2+z_1+z_2, \tag{O-37m}$$
$$\left\|\begin{pmatrix}\sqrt{2}(u_{12}+u_{22}) & \sqrt{2}(u_{14}+u_{24})\\ x_2-z_2 & x_4-z_4\end{pmatrix}\right\| \leq x_1+x_2+z_1+z_2, \tag{O-37n}$$
$$(x_3-x_1-1-y)\exp\left(\frac{u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2}{x_3-x_1-1-y}\right)$$
$$\leq u_{23}+u_{33}-u_{12}-u_{13}-x_2-x_3-z_2-z_3, \tag{O-37o}$$
$$(1+y)\exp\left(\frac{x_1+x_2+z_1+z_2}{1+y}\right) \leq x_2+x_3+z_2+z_3, \tag{O-37p}$$

with optimal objective value -2.52.

Some of the most valuable additional constraints in this case are (O-37h) and (O-37k)-(O-37n). Without these constraints the optimal objective value would be -2.58 and -2.90 respectively. (O-37h) results from the multiplication of the exponential cone inequality (O-36c) with the LHS of the quadratic inequality (O-36b). (O-37k)-(O-37n) result from the multiplication of the inequalities (O-29i) and (O-29j), resulting from the multiplication of the decomposition of the exponential cone, with the conic quadratic inequality (O-36b). □

Example O-13. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{O-38a}$$

$$\text{s.t.} \quad x_2 \exp(x_i) \leq 5,\ i \in [3], \tag{O-38b}$$

$$2x_2 \geq \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{x}\right\|, \tag{O-38c}$$

$$x_2 \geq \exp(-x_3). \tag{O-38d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{O-39a}$$

$$\text{s.t.} \quad x_2 \exp\left(\frac{u_{i2}}{x_2}\right) \leq 5,\ i \in [3], \tag{O-39b}$$

$$(\text{O-38c}) - (\text{O-38d}),$$

$$2u_{22} \geq \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_2\right\|, \tag{O-39c}$$

$$4u_{22} \geq \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top\right\|_F, \tag{O-39d}$$

$$x_2 \exp\left(\frac{-u_{23}}{x_2}\right) \leq u_{22}, \tag{O-39e}$$

$$\exp(-2x_3) \leq u_{22}, \tag{O-39f}$$

$$x_2 \geq 1 - x_3, \tag{O-39g}$$

$$2u_{22} - 2x_2 + 2u_{23} \geq \left\|\operatorname{diag}(\boldsymbol{a})\left(\boldsymbol{u}_2 - \boldsymbol{x} + \boldsymbol{u}_3\right)\right\|, \tag{O-39h}$$

$$(x_2 - 1 + x_3)\exp\left(\frac{-u_{23} + x_3 - u_{33}}{x_2 - 1 + x_3}\right) \leq u_{22} - x_2 + u_{23}, \tag{O-39i}$$

with optimal objective value -8.21.

The most valuable additional constraint in this case is (O-39h), resulting from the multiplication of the conic quadratic inequality (O-38c) with the additional inequality (O-39g) resulting from the decomposition of the exponential cone inequality. Without this constraint we would obtain an optimal objective value of $-\infty$. □

E.6. Case 11: (P) $\times$ (E)

In Table O-6 we give an overview of the constraint multiplications discussed in Section 4.4. Moreover, in this section we provide examples demonstrating that these additional constraints can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Case	Constraints 1	Constraints 2	Example
11(i)	1. $x_1, \cdots, x_m \geq 0$	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	O-6
	2. $\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_2 \geq 0$	O-14
	3. $\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_1 \geq 0$	O-6
11(ii)	1. $\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_1 \geq x_2 + x_3$	O-8
	2. $x_1, \cdots, x_m \geq 0$	$x_1 \geq x_2 + x_3$	Redundant by Lemma 1
11(iii)	1. $\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$\begin{cases} x_1 \geq x_2 + x_3 + y \\ \left\|(\sqrt{2}x_3, x_2 - y)\right\|_2 \leq x_2 + y \end{cases}$	O-7
	2. $x_1, \cdots, x_m \geq 0$	$\begin{cases} x_1 \geq x_2 + x_3 + y \\ \left\|(\sqrt{2}x_3, x_2 - y)\right\|_2 \leq x_2 + y \end{cases}$	Redundant by Lemma 1

Table O-6 Overview of the constraint multiplications for Case 11 as discussed in Section 4.4.

EXAMPLE O-14. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{O-40a}$$

$$\text{s.t.} \quad x_1 x_2 \geq 1, \tag{O-40b}$$

$$x_2 x_3 \geq 1, \tag{O-40c}$$

$$x_1^{0.5} x_2^{0.5} \geq x_3, \tag{O-40d}$$

$$5 \geq x_3 \exp\left(\frac{x_2}{x_3}\right), \tag{O-40e}$$

$$x_1, x_2 \geq 0, \tag{O-40f}$$

$$x_3 \geq 0. \tag{O-40g}$$

We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && && \text{(O-41a)}\\
\text{s.t.} \quad & u_{12} \geq 1, && && \text{(O-41b)}\\
& u_{23} \geq 1, && && \text{(O-41c)}\\
& \text{(O-40d)} - \text{(O-40g)}, && && \\
& u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i3}, && i \in [2], && \text{(O-41d)}\\
& u_{11}, u_{12}, u_{22} \geq 0, && && \text{(O-41e)}\\
& u_{11}^{0.5} u_{22}^{0.5} \geq u_{33}, && && \text{(O-41f)}\\
& 5x_3 \geq u_{33} \exp\left(\frac{u_{23}}{u_{33}}\right), && && \text{(O-41g)}\\
& 25 \geq u_{33} \exp\left(\frac{2u_{23}}{u_{33}}\right), && && \text{(O-41h)}\\
& 5x_i \geq u_{i3} \exp\left(\frac{u_{i2}}{u_{i3}}\right), && i \in [2], && \text{(O-41i)}\\
& u_{13}, u_{23} \geq 0, && && \text{(O-41j)}\\
& u_{13}^{0.5} u_{23}^{0.5} \geq u_{33}, && && \text{(O-41k)}\\
& u_{33} \geq 0, && && \text{(O-41l)}
\end{aligned}
$$

with optimal objective value 0.27.

The most valuable additional constraints in this case are (O-41h), (O-41i) and (O-41j). Without these constraints we would obtain an optimal objective of 0.05 and 0.05 respectively. (O-41h) results from the multiplication of the exponential inequality (O-41g) with itself. (O-41i) and (O-41j) result from the multiplication of the power cone inequalities (O-40d) and (O-40f) with the nonnegativity constraint (O-40g) of the exponential cone inequality. □

E.7. Case 13: (E) × (E)

In Table O-7 we give an overview of the constraint multiplications as discussed in Section 4.5. Moreover, in this section we provide examples demonstrating that these additional constraints can outperform all other potential constraints derived from different pairwise multiplications of parts of two out of the five basic cone inequalities.

Case	Constraints 1	Constraints 2	Example
13(i)	1. $x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	O-14
	2. $x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$x_4 \geq 0$	O-15
	3. $\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$x_5 \geq 0$	O-16
	4. $x_1 \geq 0$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	O-15
	5. $x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	O-16
13(ii)	1. $x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	O-17
	2. $x_1 \geq x_2 + x_3$	$x_4 \geq x_5 + x_6$	Redundant by Lemma 1
	3. $\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$x_4 \geq x_5 + x_6$	O-17
13(iii)	1. $\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\|(\sqrt{2}x_3, x_2 - y_1)\right\|_2 \leq x_2 + y_1 \end{cases}$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	O-18
	2. $\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\|(\sqrt{2}x_3, x_2 - y_1)\right\|_2 \leq x_2 + y_1 \end{cases}$	$x_5 \geq 0$	Redundant by Lemma 1
	3. $\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\|(\sqrt{2}x_3, x_2 - y_1)\right\|_2 \leq x_2 + y_1 \end{cases}$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\|(\sqrt{2}x_6, x_5 - y_2)\right\|_2 \leq x_5 + y_2 \end{cases}$	Redundant by Lemma 1
	4. $x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\|(\sqrt{2}x_6, x_5 - y_2)\right\|_2 \leq x_5 + y_2 \end{cases}$	O-18
	5. $x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\|(\sqrt{2}x_6, x_5 - y_2)\right\|_2 \leq x_5 + y_2 \end{cases}$	Redundant by Lemma 1
13(iv)	1. $x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	O-17
	2. $x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\|(\sqrt{2}x_6, x_5 - y)\right\|_2 \leq x_5 + y \end{cases}$	Redundant by Lemma 1
	3. $x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\|(\sqrt{2}x_6, x_5 - y)\right\|_2 \leq x_5 + y \end{cases}$	O-18
	4. $x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\|(\sqrt{2}x_6, x_5 - y)\right\|_2 \leq x_5 + y \end{cases}$	Redundant by Lemma 1

Table O-7 Overview of the constraint multiplications for Case 13 as discussed in Section 4.5.

EXAMPLE O-15. Consider the following example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_2 && \text{(O-42a)} \\ \text{s.t.} \quad & x_1 x_3 \geq 5, && \text{(O-42b)} \\ & x_1 \geq \exp(x_3), && \text{(O-42c)} \\ & x_2 \geq \exp(x_3). && \text{(O-42d)} \end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{12} && \text{(O-43a)} \\ \text{s.t.} \quad & u_{13} \geq 5, && \text{(O-43b)} \\ & \text{(O-42c)} - \text{(O-42d)}, \\ & u_{11} \geq x_1 \exp\left(\frac{u_{13}}{x_1}\right), && \text{(O-43c)} \\ & u_{12} \geq x_2 \exp\left(\frac{u_{23}}{x_2}\right), && \text{(O-43d)} \\ & u_{12} \geq x_1 \exp\left(\frac{u_{13}}{x_1}\right), && \text{(O-43e)} \\ & u_{22} \geq x_2 \exp\left(\frac{u_{23}}{x_2}\right), && \text{(O-43f)} \\ & u_{11} \geq \exp\left(2x_3\right), && \text{(O-43g)} \\ & u_{22} \geq \exp\left(2x_3\right), && \text{(O-43h)} \\ & u_{12} \geq \exp\left(2x_3\right), && \text{(O-43i)} \end{aligned}$$

with optimal objective value 2.72. The most valuable additional constraints in this case are (O-43c) - (O-43f), resulting from the multiplication of the LHS of inequality (O-42c) with itself and inequality (O-42d) and the multiplication of the LHS of inequality (O-42d) with itself and inequality (O-42c). Without these constraints we would obtain an optimal objective value of 0.00. □

EXAMPLE O-16. Consider the following example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(O-44a)} \\ \text{s.t.} \quad & x_1 x_2 \geq 1, && \text{(O-44b)} \\ & x_3 \geq x_1 \exp\left(\frac{x_2}{x_1}\right), && \text{(O-44c)} \\ & x_1 \geq 0. && \text{(O-44d)} \end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(O-45a)}\\
\text{s.t.} \quad & u_{12} \geq 1, && \text{(O-45b)}\\
& \text{(O-44c)} - \text{(O-44d)}, &&\\
& u_{33} \geq u_{13} \exp\left(\frac{u_{23}}{u_{13}}\right), && \text{(O-45c)}\\
& u_{33} \geq u_{11} \exp\left(\frac{2u_{12}}{u_{11}}\right), && \text{(O-45d)}\\
& u_{13} \geq u_{11} \exp\left(\frac{u_{12}}{u_{11}}\right), && \text{(O-45e)}\\
& u_{11} \geq 0, && \text{(O-45f)}
\end{aligned}$$

with optimal objective value 2.72.

The most valuable additional constraints in this case are (O-45e) & (O-45f), which result from the multiplication of the nonnegativity constraint (O-44d) of the exponential cone inequality with the exponential cone inequalities (O-44c) and (O-44d). Without these constraints we would obtain an optimal objective value of 0.00. □

Example O-17. We consider the group of constraints resulting from the following multiplications of constraints from numerical experiment 6.1:

$$\left\{\begin{array}{l}
(z_j + x_j + a - 1)\ \exp(-x_i - a) \ \leq\ z_i\,(z_j + x_j + a - 1)\,,\\
(z_j + x_j + a - 1)\ \exp(x_i) \ \leq\ t_i\,(z_j + x_j + a - 1)\,,\\
(t_j - x_j - 1)\ \exp(-x_i - a) \ \leq\ z_i\,(t_j - x_j - 1)\,,\\
(t_j - x_j - 1)\ \exp(x_i) \ \leq\ t_i\,(t_j - x_j - 1)
\end{array}\right\}, \qquad \text{(O-46)}$$

which consists of all multiplications of a linear inequality derived from the decomposition of one of the exponential cones with all other exponential cones. We list the values obtained when including/excluding this group of constraints in Table O-8 for Instances 1 and 2.

Constraint	Multiplication	Without	With
Instance 1	E × E	-119.98	-102
Instance 2	E × E	-191.27	-175.82

Table O-8 Comparison of the optimal values for Problem (38), with and without each of the proposed groups of constraint multiplications, for Instance 1.

□

Example O-18. Consider the following example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{O-47a}$$

$$x_1 \geq x_3, \tag{O-47b}$$

$$x_2 + x_3 \geq \exp(x_2), \tag{O-47c}$$

$$x_2 \geq 0. \tag{O-47d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{O-48a}$$

$$\text{s.t.} \quad \text{(O-47b)} - \text{(O-47d)}$$

$$u_{12} \geq u_{23}, \tag{O-48b}$$

$$u_{11} - 2u_{13} + u_{33} \geq 0, \tag{O-48c}$$

$$u_{22} + u_{23} \geq x_2 \exp\left(\frac{u_{22}}{x_2}\right), \tag{O-48d}$$

$$u_{22} + 2u_{23} + u_{33} \geq \exp(2x_2), \tag{O-48e}$$

$$u_{22} + 2u_{23} + u_{33} \geq (x_2 + x_3)\exp\left(\frac{u_{22} + u_{23}}{x_2 + x_3}\right), \tag{O-48f}$$

$$u_{12} - u_{23} + u_{13} - u_{33} \geq (x_1 - x_3)\exp\left(\frac{u_{12} - u_{23}}{x_1 - x_3}\right), \tag{O-48g}$$

$$x_3 \geq 1 + y, \tag{O-48h}$$

$$\left\| \left(\sqrt{2}x_2, 1 - y\right) \right\| \leq 1 + y, \tag{O-48i}$$

$$u_{23} + u_{33} - x_2 - x_3 - z_2 - z_3 \geq (x_3 - 1 - y)\exp\left(\frac{u_{23} - x_2 - z_2}{x_3 - 1 - y}\right), \tag{O-48j}$$

$$x_2 + x_3 + z_2 + z_3 \geq (1 + y)\exp\left(\frac{x_2 + z_2}{1 + y}\right), \tag{O-48k}$$

with optimal objective value 1.00.

The most valuable additional constraints in this case are (O-48j) & (O-48k), which result from the multiplication of the inequalities (O-48h) and (O-48i) resulting from the decomposition of the exponential cone with the exponential cone inequality (O-47c). Without these constraints we would obtain an optimal objective value of 0.61. □

References

K. Anstreicher. Kronecker product constraints with an application to the two-trust-region subproblem. *SIAM Journal on Optimization*, 27(1):368–378, 2017.

R. Horn and C. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.

R. Jiang and D. Li. Second order cone constrained convex relaxations for nonconvex quadratically constrained quadratic programming. *Journal of Global Optimization*, 75:461–494, 2019.

I. Schur. Bemerkungen zur theorie der beschränkten bilinearformen mit unendlich vielen veränderlichen. *Journal für die Reine und Angewandte Mathematik*, 140:1–28, 1911.

H. Sherali and A. Alameddine. A new reformulation-linearization technique for bilinear programming problems. *Journal of Global Optimization*, 2(4):379–410, 1992.

J. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28(2):246–267, 2003.