Dynamic Basis Function Generation for Network Revenue Management

Online Supplement

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A Proofs

A.1 Proof of Proposition 1

Let (D_K) denote the dual problem with K basis functions. Let (D_{K+1}) be the dual problem where a violated flow-balance constraint has been added; i.e. $|\ell_t(\vec{\beta}_{K+1})| > 0$ and the following violated constraint for t has been added. Then, (D.FB) reads:

$$\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t} \phi(\vec{x};\vec{\beta}_{K+1})\lambda_{t,(\vec{x},\vec{u})} = \begin{cases} -\phi(\vec{c};\vec{\beta}_{K+1}) & \text{if } t=1 \\ \sum_{(\vec{x},\vec{u})\in\mathcal{F}_{t-1}} \Big(\sum_{j=1}^J p_{j,t-1}\phi(\vec{x}-\vec{a}_j u_j;\vec{\beta}_{K+1}) & \\ \qquad +(1-\sum_{j=1}^J p_{j,t-1})\phi(\vec{x};\vec{\beta}_{K+1})\Big)\lambda_{t-1,(\vec{x},\vec{u})} & \text{otherwise.} \end{cases} \tag{A.1}$$

Let $\tilde{\Lambda}^K$ and $\tilde{\Lambda}^{K+1}$ be the optimal solutions to (D_K) and (D_{K+1}), respectively. The solution $\tilde{\Lambda}^K$ leads to $|\ell_t(\vec{\beta}_{K+1})| > 0$ and hence violates (A.1). Thus, $\tilde{\Lambda}^K$ is not a feasible solution for (D_{K+1}). In contrast, $\tilde{\Lambda}^{K+1}$ is feasible to (D_K). Because $\tilde{\Lambda}^K$ is the unique and optimal solution to (D_K),

$$\sum_{t=1}^{\tau-1}\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t} \tilde{\lambda}^K_{t,(\vec{x},\vec{u})} \left\{ \sum_{j=1}^J p_{j,t}\left[f_j u_j + \psi_{t+1}(\vec{x}-\vec{a}_j u_j)\right] + \Big(1-\sum_{j=1}^J p_{j,t}\Big)\psi_{t+1}(\vec{x}) - \psi_t(\vec{x}) \right\}$$

$$
\begin{aligned}
&+\sum_{(\vec{x},\vec{u})\in\mathcal{F}_\tau}\tilde{\lambda}^K_{\tau,(\vec{x},\vec{u})}\Big(\sum_{j=1}^J p_{j,\tau}f_ju_j-\psi_\tau(\vec{x})\Big)\\
\neq &\sum_{t=1}^{\tau-1}\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t}\tilde{\lambda}^{K+1}_{t,(\vec{x},\vec{u})}\left\{\sum_{j=1}^J p_{j,t}\left[f_ju_j+\psi_{t+1}(\vec{x}-\vec{a}_ju_j)\right]+\Big(1-\sum_{j=1}^J p_{j,t}\Big)\psi_{t+1}(\vec{x})-\psi_t(\vec{x})\right\}\\
&+\sum_{(\vec{x},\vec{u})\in\mathcal{F}_\tau}\tilde{\lambda}^{K+1}_{\tau,(\vec{x},\vec{u})}\Big(\sum_{j=1}^J p_{j,\tau}f_ju_j-\psi_\tau(\vec{x})\Big)
\end{aligned}
$$

Because the feasible region of (D_{K+1}) does not contain $\tilde{\lambda}^K$, the optimal objective value of (D_{K+1}) is (strictly) smaller than that of (D_K). Strong duality then implies that linearized (AP) with the new variables $V_{t,K+1}$ and new basis function $\phi(\vec{x},\vec{\beta}_{K+1})$ reduces the previous upper bound.

A.2 Proof of Theorem 1

The following lemma is essential for proving Theorem 1.

Lemma 1. *For any value function approximation $v(\vec{x})=\xi_t+\psi_t(\vec{x})+\sum_{k+1}^K V_{t,k}\phi(\vec{x};\vec{\beta})$ and any dual variables $\lambda_{t,(\vec{x},\vec{u})}\geq 0$ the following decomposition applies*

$$
\begin{aligned}
\Xi(\xi;\lambda)+\Psi(\lambda)+\Phi(V,\tilde{\vec{\beta}};\lambda)=\sum_{t=1}^{\tau}\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t}\lambda_{t,(\vec{x},\vec{u})}&\left\{\tilde{v}_t(\vec{x})-\left\{\sum_{j=1}^J p_{t,j}\left[f_ju_j+\mathbf{1}_{\{t<\tau\}}\tilde{v}_{t+1}(\vec{x}-\vec{a}_ju_j)\right]\right.\right.\\
&\left.\left.+\left(1-\sum_{j=1}^J p_{t,j}\right)\mathbf{1}_{\{t<\tau\}}\tilde{v}_{t+1}(\vec{x})\right\}\right\},
\end{aligned}
\tag{A.2}
$$

where

$$
\Xi(\xi;\lambda):=-\sum_{t=1}^{\tau}\sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)}\lambda_{t,(\vec{x},\vec{u})}\sum_{j=1}^J p_{t,j}f_ju_j+\xi_1,
$$

$$
\Psi(\lambda):=\sum_{t=1}^{\tau}\sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)}\lambda_{t,(\vec{x},\vec{u})}\left\{\psi_t(\vec{x})-\left[\sum_{j=1}^J p_{t,j}\mathbf{1}_{\{t<\tau\}}\psi_{t+1}(\vec{x}-\vec{a}_ju_j)+\left(1-\sum_{j=1}^J p_{t,j}\right)\mathbf{1}_{\{t<\tau\}}\psi_{t+1}(\vec{x})\right]\right\},
$$

$$
\Phi(V,\vec{\beta};\lambda):=-\sum_{k=1}^K V_{1,k}\phi(\vec{c};\vec{\beta}_k)-\sum_{t=2}^{\tau}\sum_{k=1}^K V_{t,k}\ell_t(\vec{\beta}_k;\lambda).
$$

and $\mathcal{Q}_t(\lambda):=\{(\vec{x},\vec{u})\in\mathcal{F}_t:\ \lambda_{t,\vec{x},\vec{u})}>0\}$.

Proof. Introducing the set $\mathcal{Q}_t(\tilde{\lambda}):=\{(\vec{x},\vec{u})\in\mathcal{F}_t:\ \tilde{\lambda}_{t,\vec{x},\vec{u})}>0\}$, we can decompose the terms in the right hand side of (A.2) as

$$
\begin{aligned}
\Xi(\xi;\lambda) &= \sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \left\{ \xi_t - \left\{ \sum_{j=1}^{J} p_{t,j} \left[f_j u_j + \mathbf{1}_{\{t<\tau\}} \xi_{t+1} \right] + \left(1 - \sum_{j=1}^{J} p_{t,j} \right) \mathbf{1}_{\{t<\tau\}} \xi_{t+1} \right\} \right\} \\
&= -\sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \sum_{j=1}^{J} p_{t,j} f_j u_j + \sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \left(\xi_t - \mathbf{1}_{\{t<\tau\}} \xi_{t+1} \right) \\
&= -\sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \sum_{j=1}^{J} p_{t,j} f_j u_j + \sum_{t=1}^{\tau} \left(\xi_t - \mathbf{1}_{\{t<\tau\}} \xi_{t+1} \right) \\
&= -\sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \sum_{j=1}^{J} p_{t,j} f_j u_j + \xi_1,
\end{aligned}
$$

$$
\Psi(\lambda) = \sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \left\{ \psi_t(\vec{x}) - \left[\sum_{j=1}^{J} p_{t,j} \mathbf{1}_{\{t<\tau\}} \psi_{t+1}(\vec{x} - \vec{a}_j u_j) + \left(1 - \sum_{j=1}^{J} p_{t,j} \right) \mathbf{1}_{\{t<\tau\}} \psi_{t+1}(\vec{x}) \right] \right\},
$$

$$
\begin{aligned}
\Phi(V,\vec{\beta};\lambda) &= \sum_{t=1}^{\tau} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \Bigg\{ -\sum_{k=1}^{K} V_{t,k} \phi(\vec{x};\vec{\beta}_k) - \Bigg[-\sum_{j=1}^{J} p_{t,j} \mathbf{1}_{\{t<\tau\}} \sum_{k=1}^{K} V_{t+1,k} \phi(\vec{x} - \vec{a}_j u_j;\vec{\beta}_k) \\
&\quad - \left(1 - \sum_{j=1}^{J} p_{t,j} \right) \mathbf{1}_{\{t<\tau\}} \sum_{k=1}^{K} V_{t+1,k} \phi(\vec{x};\vec{\beta}_k) \Bigg] \Bigg\} \\
&= -\sum_{t=1}^{\tau} \sum_{k=1}^{K} V_{t,k} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) \\
&\quad + \sum_{t=1}^{\tau-1} \sum_{k=1}^{K} V_{t+1,k} \sum_{j=1}^{J} p_{t,j} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \phi(\vec{x} - \vec{a}_j u_j;\vec{\beta}_k) \\
&\quad + \sum_{t=1}^{\tau-1} \sum_{k=1}^{K} V_{t+1,k} (1 - \sum_{j=1}^{J} p_{t,j}) \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) \\
&= -\sum_{k=1}^{K} V_{1,k} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_1} \lambda_{1,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) - \sum_{t=2}^{\tau} \sum_{k=1}^{K^*} V_{t,k} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_t(\lambda)} \lambda_{t,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) \\
&\quad + \sum_{t=2}^{\tau} \sum_{k=1}^{K} V_{t,k} \sum_{j=1}^{J} p_{t-1,j} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_{t-1}} \lambda_{t-1,(\vec{x},\vec{u})} \phi(\vec{x} - \vec{a}_j u_j;\vec{\beta}_k) \\
&\quad + \sum_{t=2}^{\tau} \sum_{k=1}^{K} V_{t,k} (1 - \sum_{j=1}^{J} p_{t-1,j}) \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_{t-1}} \lambda_{t-1,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) \\
&= -\sum_{k=1}^{K} V_{1,k} \sum_{(\vec{x},\vec{u})\in\mathcal{Q}_1} \lambda_{1,(\vec{x},\vec{u})} \phi(\vec{x};\vec{\beta}_k) - \sum_{t=2}^{\tau} \sum_{k=1}^{K} V_{t,k} \ell_t(\vec{\beta}_k;\lambda)
\end{aligned}
$$

$$= -\sum_{k=1}^{K} V_{1,k}\phi(\vec{c};\vec{\beta}_k) - \sum_{t=2}^{\tau}\sum_{k=1}^{K} V_{t,k}\ell_t(\vec{\beta}_k;\lambda).$$

□

We now proceed to prove Theorem 1. Using Lemma 1, complementary slackness yields

$$0 = \sum_{t=1}^{\tau}\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t} \tilde{\lambda}_{t,(\vec{x},\vec{u})}\left\{\tilde{v}_t(\vec{x}) - \left\{\sum_{j=1}^{J} p_{t,j}\left[f_j u_j + \mathbf{1}_{\{t<\tau\}}\tilde{v}_{t+1}(\vec{x}-\vec{a}_j u_j)\right] + \left(1-\sum_{j=1}^{J} p_{t,j}\right)\mathbf{1}_{\{t<\tau\}}\tilde{v}_{t+1}(\vec{x})\right\}\right\}$$
$$= \Xi(\tilde{\xi};\tilde{\lambda}) + \Psi(\tilde{\lambda}) + \Phi(\tilde{V},\tilde{\vec{\beta}};\tilde{\lambda}), \tag{A.3}$$

On the other hand, because $\tilde{\lambda}_t,(\vec{x},\vec{u}) \geq 0$ and $v_t^*(\vec{x}) \geq \sum_{j=1}^{J} p_{t,j}\left[f_j u_j + \mathbf{1}_{\{t<\tau\}} v_{t+1}^*(\vec{x}-\vec{a}_j u_j)\right] + \left(1-\sum_{j=1}^{J} p_{t,j}\right)\mathbf{1}_{\{t<\tau\}} v_{t+1}^*(\vec{x})$ for all $(\vec{x},\vec{u}) \in \mathcal{F}_t$, we have that

$$0 \leq \sum_{t=1}^{\tau}\sum_{(\vec{x},\vec{u})\in\mathcal{F}_t} \tilde{\lambda}_{t,(\vec{x},\vec{u})}\left\{v_t^*(\vec{x}) - \left\{\sum_{j=1}^{J} p_{t,j}\left[f_j u_j + \mathbf{1}_{\{t<\tau\}} v_{t+1}^*(\vec{x}-\vec{a}_j u_j)\right] + \left(1-\sum_{j=1}^{J} p_{t,j}\right)\mathbf{1}_{\{t<\tau\}} v_{t+1}^*(\vec{x})\right\}\right\}.$$
$$= \Xi(\xi^*;\tilde{\lambda}) + \Psi(\tilde{\lambda}) + \Phi(V^*,\vec{\beta}^*;\tilde{\lambda}) \tag{A.4}$$

where the equality comes from Lemma 1. Subtracting (A.3) from (A.4), yields

$$0 \leq \left[\Xi(\xi^*;\tilde{\lambda}) + \Psi(\tilde{\lambda}) + \Phi(V^*,\vec{\beta}^*;\tilde{\lambda})\right] - \left[\Xi(\tilde{\xi};\tilde{\lambda}) + \Psi(\tilde{\lambda}) + \Phi(\tilde{V},\vec{\beta};\tilde{\lambda})\right]$$
$$= \Xi(\xi^*;\tilde{\lambda}) - \Xi(\tilde{\xi};\tilde{\lambda}) + \Phi(V^*,\vec{\beta}^*;\tilde{\lambda}) - \Phi(\tilde{V},\tilde{\vec{\beta}};\tilde{\lambda})$$
$$= \xi_1^* - \sum_{k=1}^{K^*} V_{1,k}^*\phi(\vec{c};\vec{\beta}_k^*) - \left(\tilde{\xi}_1 - \sum_{k=1}^{\tilde{K}} \tilde{V}_{1,k}\phi(\vec{c};\tilde{\vec{\beta}}_k)\right) - \sum_{t=2}^{\tau}\sum_{k=1}^{K^*} V_{t,k}^*\ell_t(\vec{\beta}_k^*;\tilde{\lambda}) + \sum_{t=2}^{\tau}\sum_{k=1}^{\tilde{K}} \tilde{V}_{t,k}\ell_t(\tilde{\vec{\beta}}_k;\tilde{\lambda})$$
$$= \left[v_1^*(\vec{c}) - \tilde{v}_1(\vec{c})\right] - \sum_{t=2}^{\tau}\sum_{k=1}^{K^*} V_{t,k}^*\ell_t(\vec{\beta}_k^*;\tilde{\lambda}).$$

where the last equality holds due to $\ell_t(\tilde{\vec{\beta}}_k;\tilde{\lambda}) = 0$. Since $v_1^*(\vec{c}) - \tilde{v}_1(\vec{c}) < 0$ by assumption, we cannot have $\ell_t(\vec{\beta}_k^*;\tilde{\lambda}) = 0$ for all $t = 1, ..., \tau$, $k = 1, ..., K^*$. In fact, we need

$$-\sum_{t=2}^{\tau}\sum_{k=1}^{K^*} V_{t,k}^*\ell_t(\vec{\beta}_k^*;\tilde{\lambda}) \geq v_1^*(\vec{c}) - \tilde{v}_1(\vec{c}).$$

A.3 Proof of Proposition 2

Proof. Using the convexity of g, we obtain

$$\phi(\lambda\vec{x} + (1-\lambda)\vec{x}') = g(\vec{\beta}^\top(\lambda\vec{x} + (1-\lambda)\vec{x}')) = g(\lambda\vec{\beta}^\top\vec{x} + (1-\lambda)\vec{\beta}^\top\vec{x}')$$

$$\leq \lambda g(\vec{\beta}^\top \vec{x}) + (1-\lambda) g(\vec{\beta}^\top \vec{x}') = \lambda \phi(\vec{x}) + (1-\lambda)\phi(\vec{x}').$$

□

A.4 Proof of Proposition 3

Proof. For given k, a lower bound of the basis function (7) can be found by solving

$$\min_{\vec{\beta}_k, \vec{x} \in \{0,1,\ldots,c_i\}^I} \quad e^{-\sum_i \beta_{i,k} x_i} \quad \text{s.t.} \quad \sum_i c_i |\beta_{i,k}| = 1$$

Since the objective function is monotone decreasing in $\sum_i \beta_{i,k} x_i$, we can write this problem as

$$\max_{\vec{\beta}_k, \vec{x} \in \{0,1,\ldots,c_i\}^I} \quad \sum_i \beta_{i,k} x_i \quad \text{s.t.} \quad \sum_i c_i |\beta_{i,k}| = 1$$

This objective function is monotone increasing in every x_i with $\beta_{i,k} \geq 0$ and monotone decreasing in x_i given $\beta_{i,k} < 0$. Hence, the optimal value of the previous problem is upper bounded by the optimal objective of the following program

$$\max_{\vec{\beta}_k, \vec{x} \in \{0,1,\ldots,c_i\}^I} \quad \sum_i \beta_{i,k} x_i \quad \text{s.t.} \quad \sum_i c_i \beta_{i,k} = 1, \quad \beta_{i,k} \geq 0 \qquad \forall i = 1, \ldots, I$$

Given the non-negativity constraint on the $\vec{\beta}_k$ and the norm constraint, the objective is upper bounded as follows $\sum_i \beta_{i,k} x_i \leq \sum_i \beta_{i,k} c_i = 1$. Hence, a lower bound of the basis function (7) is given by e^{-1} as long as $\vec{\beta}_k$ is constrained by (8). Similarly, an upper bound of e^1 can be obtained. □

B Additional Computational Results

L	τ	c	SPLA	NSEP	AA	H-2PIAlg	H-2PIAlg+AA	K H-2PIAlg
			Hub-and-Spoke instances (H&S)					
2	20	3	15	18 498	66	6366	6432	32
	50	8	30	1860	58	323 127	323 185	31
	100	17	52	22 212	122	549 877	549 999	23
	200	33	98	†	116	539 171	539 287	12
	500	83	508	†	184	154	338	2
	1000	165	2465	†	185	360	545	2
3	20	2	18	20 980	304	113 962	114 266	43
	50	6	36	1737	278	372 511	372 789	37
	100	12	64	176	395	227 744	228 139	28
	200	25	142	†	371	350 985	351 356	20
	500	62	841	†	501	97 710	98 211	13
	1000	124	5330	†	481	496 545	497 026	12
5	20	2	196	17 151	794	130 191	130 985	42
	50	4	220	4006	729	584 713	585 442	39
	100	8	261	56 890	1045	601 360	602 405	38
	200	17	402	†	975	503 696	504 671	19
	500	41	5800	†	1315	434 145	435 460	15
	1000	83	†	†	1239	534 326	535 565	13
10	20	1	49	†	3090	203 039	206 129	37
	50	2	97	†	2774	473 233	476 007	15
	100	5	179	†	4566	380 178	384 744	18
	200	9	471	†	4270	498 801	503 071	16
	500	23	7253	†	5962	489 085	495 047	13
	1000	45	†	†	5704	312 874	318 578	11
20	20	1	150	†	19 263	544 595	563 858	19
	50	2	327	†	19 426	5208	24 634	1
	100	2	507	†	31 175	5317	36 492	2
	200	5	984	†	30 654	203 212	233 866	6
	500	12	†	†	41 782	18 424	60 206	2
	1000	24	†	†	40 914	359 286	400 200	5
			Bus instances					
Simple (SBL)			43	532	43	290 389	290 432	41
Consecutive (CBL)			14	218	18	552 779	552 797	60
Realistic (RBL)			3635	†	19	57 978	57 997	16

Table 1: CPU times in seconds of SPLA, NSEP, AA, and H-2PIAlg with AA baseline for the hub-and-spoke and bus networks. For H-2PIAlg, we display both the times taken to merely enhance AA, and the times including the computation of the AA baseline (H-2PIAlg+AA column). The last set of columns display the optimality gap (UB vs LB) of H-2PIAlg and the number of exponential ridge basis functions for which the best policy was obtained.

To address the apparent inconsistency in instance (2,100,17), where NSEP's estimated LB slightly exceeds its reported UB, we provide statistical evidence that this difference is plausibly due to sampling variability. Specifically, we test whether the policy's true expected reward μ exceeds the reported UB of 4744.31. The estimated mean reward is $\bar{R} = 4745.98$, based on 15,892 sample paths, with a standard error of 4.7458. Since the true variance is unknown and the underlying distribution may not be normal, we use a one-sided t-test to test the hypotheses:

$$H_0 : \mu \leq 4744.31$$
$$H_1 : \mu > 4744.31$$

The test statistic is:

$$t = \frac{4745.98 - 4744.31}{4.7458} \approx 0.3517$$

With $n = 15892$ observations, the degrees of freedom are 15,891. The p-value associated with this test statistic is:

$$\text{p-value} = P(T > 0.3517) \approx 0.3623$$

This means the smallest significance level at which the null hypothesis would be rejected is $\alpha = 0.3623$. As this is well above conventional thresholds (e.g., 0.05), the observed difference is not statistically significant.

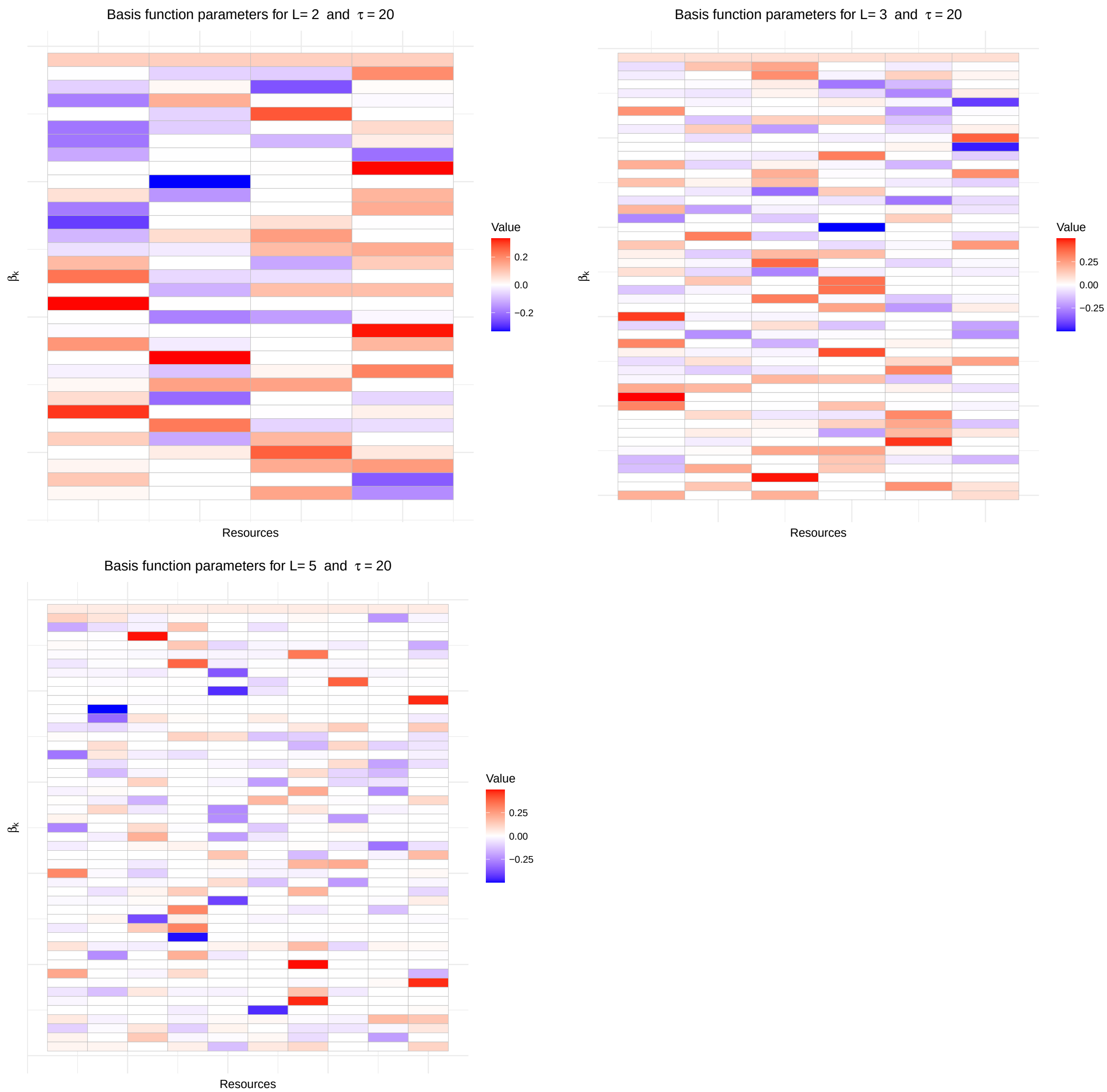

Figure 1: Heatmaps of the basis function parameters for instances in which H-2PIAlg outperforms NSEP in policy performance. Each plot has K rows, corresponding to the coefficient vector $\boldsymbol{\beta}_k$ of the k-th basis function, and I columns, each representing a resource. Red indicates a positive coefficient, while blue indicates a negative one. Color intensity reflects the magnitude of the coefficient.