

Online Supplementary Material

A Low-Rank ADMM Splitting Approach for Semidefinite Programming

Appendix A: Complete Theoretical Analysis

A.1. Proof of Theorem 1

Proof We prove Theorem 1 in three parts. In Part 1, we establish an upper bound for $\|\bar{U} - \bar{V}\|_F$. Part 2 focuses on deriving an upper bound for $\left\|2\nabla f(\hat{U}\hat{U}^\top)\hat{U} + 2\sum_{i=1}^m \bar{\lambda}_i A_i \hat{U}\right\|_F$ by leveraging the result obtained in Part 1. Finally, in Part 3, we establish the upper bound for the primal residual of (4). For simplicity, throughout the proof, we omit the superscript σ from the constants L_f^σ , L^σ , and L_0^σ .

(i) First, using the ϵ -KKT condition in (8) for problem (4), we derive the following inequality:

$$\gamma\|\bar{U} - \bar{V}\|_F \leq \epsilon + \left\| \nabla f(\bar{V}\bar{U}^\top)\bar{U} + \sum_{i=1}^m \bar{\lambda}_i A_i \bar{U} \right\|_F \leq \epsilon + L_f \|\bar{U}\|_F + \left\| \sum_{i=1}^m \bar{\lambda}_i A_i \bar{U} \right\|_F.$$

Here, the inequalities hold because $(\bar{U}, \bar{V}, \bar{\lambda})$ belongs to the bounded set $\mathcal{B}_\sigma(0^{n \times r}, 0^{n \times r}, 0^m)$, and $\|\nabla f\|_F$ is bounded by L_f on $B_{\sigma^2}(0^{n \times n})$. Moreover, we have:

$$\left\| \sum_{i=1}^m \bar{\lambda}_i A_i \bar{U} \right\|_F \leq \sum_{i=1}^m |\bar{\lambda}_i| \cdot \|A_i\|_2 \cdot \|\bar{U}\|_F \leq \sqrt{s_A} \|\bar{\lambda}\|_2 \cdot \|\bar{U}\|_F,$$

where the first inequality follows from the triangle inequality and the second from the CauchySchwarz inequality.

Combining the previous results and using the boundedness of \bar{U} and $\bar{\lambda}$, we obtain:

$$\|\bar{U} - \bar{V}\|_F \leq \frac{\epsilon + L_f \sigma + \sqrt{s_A} \sigma^2}{\gamma} \quad (\text{EC.1})$$

(ii) Next, adding up the first two equations in (8) and using the triangle inequality, we have

$$\left\| \nabla f(\bar{U}\bar{V}^\top)\bar{V} + \nabla f(\bar{V}\bar{U}^\top)\bar{U} + 2\sum_{i=1}^m \bar{\lambda}_i A_i \hat{U} \right\|_F \leq 2\epsilon. \quad (\text{EC.2})$$

Moreover, it is easily verified that

$$\begin{aligned} & \nabla f(\bar{U}\bar{V}^\top)\bar{V} + \nabla f(\bar{V}\bar{U}^\top)\bar{U} - 2\nabla f(\hat{U}\hat{U}^\top)\hat{U} \\ &= \left(\nabla f(\bar{U}\bar{V}^\top) + \nabla f(\bar{V}\bar{U}^\top) - 2\nabla f\left(\frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2}\right) \right) \hat{U} \\ &+ 2 \left(\nabla f\left(\frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2}\right) - \nabla f(\hat{U}\hat{U}^\top) \right) \hat{U} + \nabla f(\bar{U}\bar{V}^\top)(\bar{V} - \hat{U}) + \nabla f(\bar{V}\bar{U}^\top)(\bar{U} - \hat{U}) \end{aligned} \quad (\text{EC.3})$$

Using (Ortega and Rheinboldt 2000, Proposition 3.2.12) and the Lipschitz continuity of $\nabla^2 f$, we know

$$\|\nabla f(X) - \nabla f(Y) - \nabla^2 f(Y)(X - Y)\|_F \leq \frac{L_0}{2} \|X - Y\|_F^2,$$

for $X, Y \in \mathcal{B}_{\sigma^2}(0^{n \times n})$. Then, by applying this inequality to the pairs $(\bar{U}\bar{V}^\top, \frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2})$ and $(\bar{V}\bar{U}^\top, \frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2})$, and then summing up these resulting inequalities, one can easily show that

$$\begin{aligned} & \left\| \nabla f(\bar{U}\bar{V}^\top) + \nabla f(\bar{V}\bar{U}^\top) - 2\nabla f\left(\frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2}\right) \right\|_F \\ & \leq \frac{L_0}{4} \|\bar{U}\bar{V}^\top - \bar{V}\bar{U}^\top\|_F^2 = \frac{L_0}{4} \|\bar{U}(\bar{V}^\top - \bar{U}^\top) + (\bar{U} - \bar{V})\bar{U}^\top\|_F^2 \leq L_0 \sigma^2 \|\bar{U} - \bar{V}\|_F^2, \end{aligned} \quad (\text{EC.4})$$

where the last inequality holds because \bar{U} and \bar{V} are bounded by σ . Besides, since ∇f is Lipschitz continuous, we have

$$\left\| \nabla f \left(\frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2} \right) - \nabla f(\hat{U}\hat{U}^\top) \right\|_F \leq L \left\| \frac{\bar{U}\bar{V}^\top + \bar{V}\bar{U}^\top}{2} - \hat{U}\hat{U}^\top \right\|_F = \frac{L}{4} \|(\bar{U} - \bar{V})(\bar{U} - \bar{V})^\top\|_F \leq \frac{L}{4} \|\bar{U} - \bar{V}\|_F^2. \quad (\text{EC.5})$$

For the third term in (EC.3), we can derive that

$$\nabla f(\bar{U}\bar{V}^\top)(\bar{V} - \hat{U}) + \nabla f(\bar{V}\bar{U}^\top)(\bar{U} - \hat{U}) = \frac{1}{2} \left(\nabla f(\bar{U}\bar{V}^\top) - \nabla f(\bar{V}\bar{U}^\top) \right) (\bar{V} - \bar{U}),$$

which implies that

$$\begin{aligned} \|\nabla f(\bar{U}\bar{V}^\top)(\bar{V} - \hat{U}) + \nabla f(\bar{V}\bar{U}^\top)(\bar{U} - \hat{U})\|_F &\leq \frac{1}{2} \|\nabla f(\bar{U}\bar{V}^\top) - \nabla f(\bar{V}\bar{U}^\top)\|_F \|\bar{V} - \bar{U}\|_F \\ &\leq \frac{L}{2} \|\bar{U}\bar{V}^\top - \bar{V}\bar{U}^\top\|_F \|\bar{V} - \bar{U}\|_F \leq L\sigma \|\bar{V} - \bar{U}\|_F^2. \end{aligned} \quad (\text{EC.6})$$

Substituting (EC.4), (EC.5) and (EC.6) into (EC.3), we get

$$\|\nabla f(\bar{U}\bar{V}^\top)\bar{V} + \nabla f(\bar{V}\bar{U}^\top)\bar{U} - 2\nabla f(\hat{U}\hat{U}^\top)\hat{U}\|_F \leq (L_0\sigma^3 + \frac{3L\sigma}{2}) \|\bar{U} - \bar{V}\|_F^2,$$

which together with (EC.1) and (EC.2) yields

$$\left\| 2\nabla f(\hat{U}\hat{U}^\top)\hat{U} + 2 \sum_{i=1}^m \bar{\lambda}_i A_i \hat{U} \right\|_F \leq 2\epsilon + \frac{(2L_0\sigma^3 + 3L\sigma)(\epsilon + L_f\sigma + \sqrt{s_A}\sigma^2)^2}{2\gamma^2}. \quad (\text{EC.7})$$

(iii) Similarly, since $\|\mathcal{A}(\bar{U}\bar{V}^\top) - b\|_2 = \|\mathcal{A}(\bar{V}\bar{U}^\top) - b\|_2 \leq \epsilon$ and thus it holds

$$\begin{aligned} \|\mathcal{A}(\hat{U}\hat{U}^\top) - b\|_2 &\leq \epsilon + \frac{1}{2} \left\| 2\mathcal{A}(\hat{U}\hat{U}^\top) - \mathcal{A}(\bar{U}\bar{V}^\top) - \mathcal{A}(\bar{V}\bar{U}^\top) \right\|_2 \\ &\leq \epsilon + \frac{\|\mathcal{A}\|_2}{2} \cdot \left\| 2\hat{U}\hat{U}^\top - \bar{U}\bar{V}^\top - \bar{V}\bar{U}^\top \right\|_F \leq \epsilon + \frac{\|\mathcal{A}\|_2(\epsilon + L_f\sigma + \sqrt{s_A}\sigma^2)^2}{4\gamma^2}, \end{aligned}$$

which together with (EC.7) completes the proof. \square

A.2. Proof of Proposition 1

Proof (i) In view of the definition of the augmented Lagrangian function and the update rule for λ^{k+1} , we derive the following sequence of equalities:

$$\begin{aligned} &\mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^{k+1}) - \mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^k) \\ &= (\lambda^{k+1} - \lambda^k)^\top (\mathcal{A}(U^{k+1}(V^{k+1})^\top) - b) = \rho \|\mathcal{A}(U^{k+1}(V^{k+1})^\top) - b\|_2^2 = \frac{1}{\rho} \|\lambda^{k+1} - \lambda^k\|_2^2. \end{aligned}$$

This proves the first inequality in Proposition 1.

(ii) Moreover, due to the strong convexity of the subproblems in Algorithm 1, we have the following inequality:

$$\mathcal{L}_\rho(U^{k+1}, V^k, \lambda^k) + \frac{\gamma}{2} \|U^{k+1} - U^k\|_F^2 \leq \mathcal{L}_\rho(U^k, V^k, \lambda^k), \quad (\text{EC.8})$$

and similarly,

$$\mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^k) + \frac{\gamma}{2} \|V^{k+1} - V^k\|_F^2 \leq \mathcal{L}_\rho(U^{k+1}, V^k, \lambda^k). \quad (\text{EC.9})$$

By combining inequalities (EC.8) and (EC.9), we obtain:

$$\mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^k) - \mathcal{L}_\rho(U^k, V^k, \lambda^k) \leq -\frac{\gamma}{2} \|V^{k+1} - V^k\|_F^2 - \frac{\gamma}{2} \|U^{k+1} - U^k\|_F^2,$$

which completes the proof. \square

A.3. Proposition EC.1

PROPOSITION EC.1. Assumption 2 holds for some $\rho_0 > 0$ if either of the following conditions is satisfied:

- (i) $f(X) + \frac{\rho_0}{2} \|\mathcal{A}(X) - b\|_2^2$ is strongly convex with respect to $\|\cdot\|_F$.
- (ii) $f(X) = \langle C, X \rangle$ with $C \in \mathbb{S}^n$, and the dual problem of (2) admits a strictly feasible solution y^* . Let λ_{\min} and λ_{\max} be the smallest and largest eigenvalues of $C - \sum_{i=1}^m y_i^* A_i$, respectively. Assume $\rho_0 \geq 1 + \lambda_{\max} \kappa_c$, where $\kappa_c = \frac{\lambda_{\max}}{\lambda_{\min}}$ is the condition number.

Proof For any given $\beta > 0$, S_β is closed because f is a smooth convex function. Suppose S_β is not empty.

- (i) Let μ be the strongly convex modulus of $f(X) + \frac{\rho_0}{2} \|\mathcal{A}(X) - b\|_2^2$ and X^* be its minimum point. It follows from the strong convexity that

$$f(UV^\top) + \frac{\rho_0}{2} \|\mathcal{A}(UV^\top) - b\|_2^2 \geq f(X^*) + \frac{\rho_0}{2} \|\mathcal{A}(X^*) - b\|_2^2 + \frac{\mu}{2} \|UV^\top - X^*\|_F^2.$$

Moreover, if $(U, V) \in S_\beta$, then we have

$$\begin{aligned} \beta &\geq f(UV^\top) + \frac{\rho_0}{2} \|U - V\|_F^2 + \frac{\rho_0}{2} \|\mathcal{A}(UV^\top) - b\|_2^2 \\ &\geq f(X^*) + \frac{\rho_0}{2} \|\mathcal{A}(X^*) - b\|_2^2 + \frac{\mu}{2} \|UV^\top - X^*\|_F^2 + \frac{\rho_0}{2} \|U - V\|_F^2, \end{aligned}$$

which implies that both UV^\top and $\|U - V\|_F$ are bounded for any $(U, V) \in S_\beta$. Note that

$$\|U\|_F^2 + \|V\|_F^2 = \|U - V\|_F^2 + 2\langle U, V \rangle.$$

From this, we can deduce that the set S_β is bounded. This completes the proof of the first part.

- (ii) Given the strictly feasible solution y^* , we have

$$\langle C, UV^\top \rangle = \langle C - \sum_{i=1}^m y_i^* A_i, UV^\top \rangle + \sum_{i=1}^m y_i^* \langle A_i, UV^\top \rangle = \langle C - \sum_{i=1}^m y_i^* A_i, UV^\top \rangle + \langle y^*, \mathcal{A}(UV^\top) \rangle. \quad (\text{EC.10})$$

For the first term in (EC.10), we have

$$\begin{aligned} \langle C - \sum_{i=1}^m y_i^* A_i, UV^\top \rangle &= \langle C - \sum_{i=1}^m y_i^* A_i, UU^\top \rangle + \langle C - \sum_{i=1}^m y_i^* A_i, U(V - U)^\top \rangle \\ &\geq \lambda_{\min} \|U\|_F^2 + \langle (C - \sum_{i=1}^m y_i^* A_i)U, V - U \rangle \\ &\geq \lambda_{\min} \|U\|_F^2 - \lambda_{\max} \|U\|_F \|V - U\|_F \\ &\geq \lambda_{\min} \|U\|_F^2 - \lambda_{\max} \left(\frac{\lambda_{\min}}{2\lambda_{\max}} \|U\|_F^2 + \frac{\lambda_{\max}}{2\lambda_{\min}} \|V - U\|_F^2 \right) \\ &\geq \frac{\lambda_{\min}}{2} \|U\|_F^2 - \frac{\lambda_{\max} \kappa_c}{2} \|V - U\|_F^2, \end{aligned}$$

where $\kappa_c = \frac{\lambda_{\max}}{\lambda_{\min}}$. For the second term in (EC.10), we have

$$\langle y^*, \mathcal{A}(UV^\top) \rangle = \langle y^*, \mathcal{A}(UV^\top) - b \rangle + \langle y^*, b \rangle \geq -\frac{1}{2} \|y^*\|_2^2 - \frac{1}{2} \|\mathcal{A}(UV^\top) - b\|_2^2 + \langle y^*, b \rangle.$$

By choosing $\rho_0 \geq 1 + \lambda_{\max} \kappa_c$, we have

$$\begin{aligned} &f(UV^\top) + \frac{\rho_0}{2} \|U - V\|_F^2 + \frac{\rho_0}{2} \|\mathcal{A}(UV^\top) - b\|_2^2 \\ &\geq \frac{\lambda_{\min}}{2} \|U\|_F^2 + \left(\frac{\rho_0}{2} - \frac{\lambda_{\max} \kappa_c}{2} \right) \|V - U\|_F^2 + \left(\frac{\rho_0}{2} - \frac{1}{2} \right) \|\mathcal{A}(UV^\top) - b\|_2^2 - \frac{1}{2} \|y^*\|_2^2 + \langle y^*, b \rangle \\ &\geq \frac{\lambda_{\min}}{2} \|U\|_F^2 - \frac{1}{2} \|y^*\|_2^2 + \langle y^*, b \rangle. \end{aligned}$$

We can deduce from the above inequality that

$$\|U\|_F^2 \leq \frac{1}{\lambda_{\min}} (\|y^*\|_2^2 - 2\langle y^*, b \rangle + 2\beta). \quad (\text{EC.11})$$

By exchanging U and V , it is easy to show that (EC.11) holds for V . Hence, S_{β} is bounded and thus compact. \square

A.4. Proof of Lemma 1

Proof We establish part (i) of this lemma by mathematical induction, while parts (ii)(iv) are proven concurrently throughout the process. The proof is divided into four steps:

- **Step 1:** Prove that part (i) holds for $k = 0$;
- **Step 2:** Assume part (i) holds for the k -th iteration, and show that $(U^k, V^k) \in S_{\hat{\beta}}$ (part (ii)) and $(U^{k+1}, V^{k+1}) \in S_{\hat{\beta}}$, which implies that (U^k, V^k) and (U^{k+1}, V^{k+1}) are bounded, assuming $S_{\hat{\beta}}$ is bounded;
- **Step 3:** Utilizing this boundedness condition, together with the optimality conditions of the subproblems, bound the change in the dual variable, corresponding to part (iii);
- **Step 4:** Demonstrate that, provided ρ and γ are chosen appropriately, the value of the augmented Lagrangian function is non-increasing (part (iv)), thereby ensuring that part (i) holds for the next iteration.

Step 1. Consider the case that $k = 0$. As (U^0, V^0, λ^0) is selected according to (20), part (i) is naturally true for $k = 0$.

Step 2. Suppose that part (i) holds for the k -th iteration. Then, it follows that

$$\begin{aligned} \hat{\beta} &> L_{\rho}(U^k, V^k, \lambda^k) + \frac{1}{\rho} \|\lambda^k\|_2^2 \\ &= f(U^k(V^k)^T) + \frac{\gamma}{2} \|U^k - V^k\|_F^2 + \langle \lambda^k, \mathcal{A}(U^k V^{kT}) - b \rangle + \frac{\rho}{2} \|\mathcal{A}(U^k V^{kT}) - b\|_2^2 + \frac{1}{\rho} \|\lambda^k\|_2^2, \end{aligned}$$

where the equality follows from the definition of the augmented Lagrangian function. Applying the Cauchy-Schwarz inequality and the AM-GM inequality, we further derive

$$\begin{aligned} \hat{\beta} &> f(U^k(V^k)^T) + \frac{\gamma}{2} \|U^k - V^k\|_F^2 - \frac{\|\lambda^k\|_2^2}{2(\rho - \rho_0)} - \frac{(\rho - \rho_0)}{2} \|\mathcal{A}(U^k V^{kT}) - b\|_2^2 + \frac{1}{\rho} \|\lambda^k\|_2^2 \\ &\quad + \frac{\rho}{2} \|\mathcal{A}(U^k V^{kT}) - b\|_2^2 \\ &\geq f(U^k(V^k)^T) + \frac{\rho_0}{2} \|U^k - V^k\|_F^2 + \frac{\rho_0}{2} \|\mathcal{A}(U^k V^{kT}) - b\|_2^2 + \left(\frac{1}{\rho} - \frac{1}{2}(\rho - \rho_0) \right) \|\lambda^k\|_2^2 \\ &\geq f(U^k(V^k)^T) + \frac{\rho_0}{2} \|U^k - V^{k+1}\|_F^2 + \frac{\rho_0}{2} \|\mathcal{A}(U^k V^{kT}) - b\|_2^2, \end{aligned}$$

where the last two inequalities hold due to $\gamma > \rho_0$ and $\rho > 2\rho_0$. This implies that $(U^k, V^k) \in S_{\hat{\beta}}$, and thus, by Assumption 2, we have $\|U^k\|_F \leq \delta$ and $\|V^k\|_F \leq \delta$.

Furthermore, from Proposition 1, we have

$$\begin{aligned} \hat{\beta} &> L_{\rho}(U^k, V^k, \lambda^k) + \frac{1}{\rho} \|\lambda^k\|_2^2 \\ &\geq L_{\rho}(U^{k+1}, V^{k+1}, \lambda^k) + \frac{1}{\rho} \|\lambda^k\|_2^2. \end{aligned}$$

Following an argument similar to the proof that $(U^k, V^k) \in S_{\hat{\beta}}$, we can easily show that $(U^{k+1}, V^{k+1}) \in S_{\hat{\beta}}$, and thus $\|U^{k+1}\|_F \leq \delta$ and $\|V^{k+1}\|_F \leq \delta$.

Step 3. Next, we bound the change in dual update, that is, $\lambda^{k+1} - \lambda^k$. To simplify the notation, we will omit the superscript δ from the constants L_f^{δ} and L^{δ} throughout the proof. According to the optimality condition (12) and the update rule of λ^{k+1} , we have

$$\nabla f(V^{k+1}(U^{k+1})^T)U^{k+1} + \sum_{i=1}^m \lambda_i^{k+1} A_i U^{k+1} - \gamma(U^{k+1} - V^{k+1}) = 0. \quad (\text{EC.12})$$

Note that $(U^{k+1}, V^{k+1}) \in S_{\beta}$, then it follows from Assumption 3 that

$$\sigma_{\min}(\mathcal{C}(U^{k+1})) \geq \eta. \quad (\text{EC.13})$$

Let $B_{k+1} = \nabla f(V^{k+1}(U^{k+1})^\top)U^{k+1} - \gamma(U^{k+1} - V^{k+1})$, and define $b_{k+1} = \text{vec}(B_{k+1}) \in \mathbb{R}^{nr}$. Additionally, let $C^{k+1} = \mathcal{C}(U^{k+1}) = [\text{vec}(A_1 U^{k+1}), \dots, \text{vec}(A_m U^{k+1})] \in \mathbb{R}^{nr \times m}$. Then we can rewrite equation (EC.12) as

$$C^{k+1} \lambda^{k+1} = -b^{k+1}. \quad (\text{EC.14})$$

Combining (EC.13) and (EC.14), one can easily show

$$\begin{aligned} \eta \|\lambda^{k+1}\|_2 &\leq \|b^{k+1}\|_2 = \|B^{k+1}\|_F \\ &\leq \|\nabla f(V^{k+1}(U^{k+1})^\top)U^{k+1}\|_F + \gamma \|U^{k+1} - V^{k+1}\|_F \\ &\leq L_f \delta + 2\gamma \delta, \end{aligned} \quad (\text{EC.15})$$

and similar inequality also holds for λ^k because $\|U^k\|_F \leq \delta$ and $\|V^k\|_F \leq \delta$. Note that $C^k \lambda^k = -b^k$, and it therefore holds $C^{k+1} \lambda^{k+1} - C^k \lambda^k = C^k \Delta \lambda^{k+1} + (C^{k+1} - C^k) \lambda^{k+1} = -(b^{k+1} - b^k)$. Then we can bound $\|\Delta \lambda^{k+1}\|_2$ by the following inequality

$$\eta \|\Delta \lambda^{k+1}\|_2 \leq \|C^k \Delta \lambda^{k+1}\|_2 \leq \|b^{k+1} - b^k\|_2 + \|C^{k+1} - C^k\|_F \|\lambda^{k+1}\|_2. \quad (\text{EC.16})$$

In what follows, we bound $\|b^{k+1} - b^k\|_2$ and $\|C^{k+1} - C^k\|_F$ separately. It is easily seen that

$$\begin{aligned} \|b^{k+1} - b^k\|_2 &= \|B^{k+1} - B^k\|_F \\ &\leq \|\nabla f(V^{k+1}(U^{k+1})^\top)U^{k+1} - \nabla f(V^k(U^k)^\top)U^k\|_F + \gamma \|\Delta U^{k+1}\|_F + \gamma \|\Delta V^{k+1}\|_F, \end{aligned} \quad (\text{EC.17})$$

where the inequality follows from the definition of B^k and B^{k+1} . Note that

$$\begin{aligned} &\|\nabla f(V^{k+1}(U^{k+1})^\top)U^{k+1} - \nabla f(V^k(U^k)^\top)U^k\|_F \\ &\leq \|\nabla f(V^{k+1}(U^{k+1})^\top)U^{k+1} - \nabla f(V^{k+1}(U^{k+1})^\top)U^k\|_F + \|\nabla f(V^{k+1}(U^{k+1})^\top)U^k - \nabla f(V^k(U^k)^\top)U^k\|_F \\ &\leq L_f \|\Delta U^{k+1}\|_F + L\delta \|V^{k+1}(U^{k+1})^\top - V^k(U^k)^\top\|_F \\ &\leq L_f \|\Delta U^{k+1}\|_F + L\delta \left(\|V^{k+1}(U^{k+1})^\top - V^{k+1}(U^k)^\top\|_F + \|V^{k+1}(U^k)^\top - V^k(U^k)^\top\|_F \right) \\ &\leq L_f \|\Delta U^{k+1}\|_F + L\delta^2 \left(\|\Delta U^{k+1}\|_F + \|\Delta V^{k+1}\|_F \right), \end{aligned} \quad (\text{EC.18})$$

where the first inequality follows from the triangle inequality, and the second inequality is due to the Lipschitz continuity and smoothness of f assumed in Assumption 1. Substituting (EC.18) into (EC.17), we establish that

$$\|b^{k+1} - b^k\|_2 \leq (L\delta^2 + L_f + \gamma) \|\Delta U^{k+1}\|_F + (L\delta^2 + \gamma) \|\Delta V^{k+1}\|_F. \quad (\text{EC.19})$$

Moreover, according to the definition of C^k , we have

$$\|C^{k+1} - C^k\|_F^2 = \sum_{i=1}^m \|A_i(U^{k+1} - U^k)\|_F^2 \leq \sum_{i=1}^m \|A_i\|_F^2 \|U^{k+1} - U^k\|_F^2 = s_A \|\Delta U^{k+1}\|_F^2, \quad (\text{EC.20})$$

which together with (EC.15) yields

$$\|C^{k+1} - C^k\|_F \|\lambda^k\|_2 \leq \frac{1}{\eta} (L_f \delta \sqrt{s_A} + 2\gamma \delta \sqrt{s_A}) \|\Delta U^{k+1}\|_F. \quad (\text{EC.21})$$

Combining (EC.16), (EC.19) and (EC.21), we have

$$\eta \|\lambda^{k+1} - \lambda^k\|_2 \leq (L\delta^2 + L_f + \gamma + \frac{1}{\eta}(L_f\delta\sqrt{s_A} + 2\gamma\delta\sqrt{s_A})) \|\Delta U^{k+1}\|_F + (L\delta^2 + \gamma) \|\Delta V^{k+1}\|_F.$$

Therefore, we have

$$\begin{aligned} \|\lambda^{k+1} - \lambda^k\|_2^2 &\leq \frac{2}{\eta^2} (L\delta^2 + L_f + \gamma + \frac{1}{\eta}L_f\delta\sqrt{s_A} + \frac{2}{\eta}\gamma\eta\sqrt{s_A})^2 \|\Delta U^{k+1}\|_F^2 + \frac{2}{\eta^2} (L\delta^2 + \gamma)^2 \|\Delta V^{k+1}\|_F^2 \\ &\triangleq c_1 \|\Delta U^{k+1}\|_F^2 + c_2 \|\Delta V^{k+1}\|_F^2. \end{aligned} \quad (\text{EC.22})$$

Step 4. Finally, we establish the descent of the augmented Lagrangian function and justify the validation of part (i) for iterate $k + 1$. By invoking Proposition 1 and using (EC.22), it holds that

$$\begin{aligned} &\mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^{k+1}) - \mathcal{L}_\rho(U^k, V^k, \lambda^k) \\ &\leq \frac{1}{\rho} \|\Delta \lambda^{k+1}\|_2^2 - \frac{\gamma}{2} (\|\Delta U^{k+1}\|_F^2 + \|\Delta V^{k+1}\|_F^2) \\ &\leq -(\frac{\gamma}{2} - \frac{c_1}{\rho}) \|\Delta U^{k+1}\|_F^2 - (\frac{\gamma}{2} - \frac{c_2}{\rho}) \|\Delta V^{k+1}\|_F^2 \leq 0, \end{aligned}$$

where the last inequality holds because $\rho > \frac{2c_1}{\gamma}$ and $\rho > \frac{2c_2}{\gamma}$. Since $\rho > \frac{1}{\beta - \beta^0} \left(\frac{L_f\delta + 2\gamma\delta}{\eta} \right)^2$, we know

$$L_\rho(U^{k+1}, V^{k+1}, \lambda^{k+1}) + \frac{1}{\rho} \|\lambda^{k+1}\|_2^2 \leq L_\rho(U^k, V^k, \lambda^k) + \frac{1}{\rho} \|\lambda^{k+1}\|_2^2 \leq \beta^0 + \frac{1}{\rho} \left(\frac{L_f\delta + 2\gamma\delta}{\eta} \right)^2 < \hat{\beta}.$$

This completes the proof. □

A.5. Proof of Theorem 2

Proof Based on the developments outlined in Lemma 1, we deduce the existence of a positive constant κ such that

$$\mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^{k+1}) - \mathcal{L}_\rho(U^k, V^k, \lambda^k) \leq -\kappa \|\Delta U^{k+1}\|_F^2 - \kappa \|\Delta V^{k+1}\|_F^2. \quad (\text{EC.23})$$

On the other hand, we know from (15) and (16) that

$$\|\nabla \mathcal{L}(U^{k+1}, V^{k+1}, \lambda^{k+1})\|_F^2 \leq (L_f^\delta + L^\delta \delta^2 + \rho s_A \delta^2 + \|\mathcal{A}\|_2 \delta + \gamma)^2 \|\Delta V^{k+1}\|_F^2 + \frac{1}{\rho^2} \|\Delta \lambda^{k+1}\|_2^2,$$

where \mathcal{L} is the Lagrangian function of (6). Note that

$$\mathcal{L}_\rho(U, V, \lambda) = \mathcal{L}(U, V, \lambda) + \frac{\rho}{2} \|\mathcal{A}(UV^T) - b\|_2^2.$$

Then, there must exist a constant $\zeta > 0$ such that

$$\|\nabla \mathcal{L}_\rho(U^{k+1}, V^{k+1}, \lambda^{k+1})\|_F \leq \zeta (\|\Delta U^{k+1}\|_F + \|\Delta V^{k+1}\|_F + \|\Delta \lambda^{k+1}\|_2). \quad (\text{EC.24})$$

Therefore, by using (EC.24), (EC.23) and Assumption 4, we can invoke (Attouch et al. 2013, Theorem 2.9) that the sequence $\{(U^k, V^k, \lambda^k)\}_{k=0}^\infty$ converges to a critical point of $L_\rho(U, V, \lambda)$, which corresponds to a KKT point of problem (6). We further have the sequence $\{(U^k, V^k, \lambda^k)\}_{k=0}^\infty$ has a finite length, i.e.

$$\sum_{j=1}^{\infty} (\|\Delta U^{j+1}\|_F + \|\Delta V^{j+1}\|_F + \|\Delta \lambda^{j+1}\|_2) < \infty.$$

By elementary calculus, one can show that

$$\min_{1 \leq j \leq K} \|\Delta U^{j+1}\|_F + \|\Delta V^{j+1}\|_F + \|\Delta \lambda^{j+1}\|_2 = O\left(\frac{1}{K}\right).$$

This, together with (15) and (16), implies that

$$\begin{aligned} \min_{1 \leq j \leq K} & \left\{ \|\nabla f(U^{k+1}(V^{k+1})^\top)V^{k+1} + \gamma(U^{k+1} - V^{k+1}) + \sum_{i=1}^m \lambda_i^{k+1} A_i V^{k+1}\|_F \right. \\ & + \|\nabla f(V^{k+1}(U^{k+1})^\top)V^{k+1} + \gamma(V^{k+1} - U^{k+1}) + \sum_{i=1}^m \lambda_i^{k+1} A_i V^{k+1}\|_F \\ & \left. + \|\mathcal{A}(U^{k+1}(V^{k+1})^T) - b\|_2 \right\} = O\left(\frac{1}{K}\right). \end{aligned}$$

In other words, Algorithm 1 requires $O(1/\epsilon)$ iterations to produce an ϵ -KKT point of (6). \square

A.6. Proof of Theorem 3

Proof Given the convergence of the sequence $(U^k, V^k, \lambda^k)_{k=0}^\infty$, the iterates eventually enter a local neighborhood of its limit point $(\bar{U}, \bar{V}, \bar{\lambda})$, and after a finite number of iterations, satisfy the Łojasiewicz inequality. For ease of exposition in the subsequent analysis, we assume that $(U^k, V^k, \lambda^k) \in \mathcal{B}_{\delta_c}(\bar{U}, \bar{V}, \bar{\lambda})$ holds for all $k \geq 0$.

(i) First, consider the case where $\alpha = 0$. Assume that the sequence does not converge in finitely many steps and hence $\mathcal{L}_\rho(U^k, V^k, \lambda^k) - \mathcal{L}_\rho(\bar{U}, \bar{V}, \bar{\lambda}) > 0$ holds for any $k \geq 0$. According to the Łojasiewicz inequality, we then have $c\|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq 1$. This contradicts the result that the sequence converges to a KKT point. Hence, the sequence must converge in a finite number of steps when $\alpha = 0$.

(ii) Next, we consider the remaining case where $\alpha \in (0, 1)$. Let $S^k = \sum_{t=k}^\infty (\|\Delta U^{t+1}\|_F + \|\Delta V^{t+1}\|_F + \|\Delta \lambda^{t+1}\|_2)$. Observing that $\|U^{k+1} - \bar{U}\|_F + \|V^{k+1} - \bar{V}\|_F + \|\lambda^{k+1} - \bar{\lambda}\|_2 \leq S^k$, it is sufficient to establish the convergence rate of S^k . We proceed to prove this lemma in two steps. Step 1 focuses on deriving an inequality involving $\|\Delta U^{k+1}\|_F + \|\Delta V^{k+1}\|_F + \|\Delta \lambda^{k+1}\|_2$ and $\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2$. Based on the result of Step 1, Step 2 establishes an inequality relating S^k and S^{k+1} , which will directly lead to the desired conclusion.

Step 1. Define $\phi(t) = \frac{c}{1-\alpha} t^{1-\alpha}$. We can then rewrite the Łojasiewicz inequality (22) as

$$\phi'(\mathcal{L}_\rho(U, V, \lambda) - \mathcal{L}_\rho(\bar{U}, \bar{V}, \bar{\lambda})) \cdot \|\nabla \mathcal{L}_\rho(U, V, \lambda)\|_F \geq 1. \quad (\text{EC.25})$$

Let $l^k = \mathcal{L}_\rho(U^k, V^k, \lambda^k) - \mathcal{L}_\rho(\bar{U}, \bar{V}, \bar{\lambda})$. Then according to the Łojasiewicz inequality (EC.25), we have $\phi'(l^k) \cdot \|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq 1$. Because $l^k - l^{k+1} \geq 0$, by multiplying it to both side we get the following inequality, we have $\phi'(l^k)(l^k - l^{k+1}) \cdot \|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq l^k - l^{k+1}$. Note that ϕ is concave, we further have

$$(\phi(l^k) - \phi(l^{k+1})) \cdot \|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq l^k - l^{k+1}. \quad (\text{EC.26})$$

On the other hand, according to Lemma 1 and (EC.22), there exists some constant $\tilde{\kappa} > 0$ such that

$$l^k - l^{k+1} \geq \tilde{\kappa} (\|\Delta U^{k+1}\|_F^2 + \|\Delta V^{k+1}\|_F^2 + \|\Delta \lambda^{k+1}\|_2^2). \quad (\text{EC.27})$$

Substituting (EC.24) and (EC.27) into (EC.26), we obtain

$$\begin{aligned} & \frac{\zeta}{\tilde{\kappa}} (\phi(l^k) - \phi(l^{k+1})) (\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2) \\ & \geq \frac{1}{\tilde{\kappa}} (\phi(l^k) - \phi(l^{k+1})) \|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq \frac{1}{\tilde{\kappa}} (l^k - l^{k+1}) \geq \|\Delta U^{k+1}\|_F^2 + \|\Delta V^{k+1}\|_F^2 + \|\Delta \lambda^{k+1}\|_2^2. \end{aligned} \quad (\text{EC.28})$$

Recall the inequality that $2\sqrt{ab} \leq a + b$ and using (EC.28), we can further get

$$\begin{aligned} & \frac{3\zeta}{\tilde{\kappa}}(\phi(l^k) - \phi(l^{k+1})) + (\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2) \\ & \geq 2\sqrt{\frac{3\zeta}{\tilde{\kappa}}(\phi(l^k) - \phi(l^{k+1}))(\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2)} \\ & \geq 2\sqrt{3(\|\Delta U^{k+1}\|_F^2 + \|\Delta V^{k+1}\|_F^2 + \|\Delta \lambda^{k+1}\|_2^2)} \geq 2(\|\Delta U^{k+1}\|_F + \|\Delta V^{k+1}\|_F + \|\Delta \lambda^{k+1}\|_2). \end{aligned}$$

Step 2. By adding the above inequality from k to K , we have

$$\begin{aligned} & \frac{3\zeta}{\tilde{\kappa}}(\phi(l^k) - \phi(l^{K+1})) + (\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2) \\ & \geq \sum_{t=k}^K (\|\Delta U^{t+1}\|_F + \|\Delta V^{t+1}\|_F + \|\Delta \lambda^{t+1}\|_2) + (\|\Delta U^{K+1}\|_F + \|\Delta V^{K+1}\|_F + \|\Delta \lambda^{K+1}\|_2). \end{aligned}$$

Because $\mathcal{L}_\rho(U^k, V^k, \lambda^k)$ is non-increasing, we have $l^{K+1} > 0$ and hence $\phi(l^k) - \phi(l^{K+1}) \leq \phi(l^k)$. By letting $K \rightarrow \infty$, we have the following inequality

$$\frac{3\zeta}{\tilde{\kappa}}\phi(l^k) + (\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2) \geq \sum_{t=k}^{\infty} (\|\Delta U^{t+1}\|_F + \|\Delta V^{t+1}\|_F + \|\Delta \lambda^{t+1}\|_2).$$

Then we can rewrite the above inequality as follows

$$\frac{3\zeta}{\tilde{\kappa}}\phi(l^k) + S^{k-1} - S^k \geq S^k. \quad (\text{EC.29})$$

According to the definition of ϕ and the Łojasiewicz inequality, we have

$$c(l^k)^{-\alpha} \cdot \|\nabla \mathcal{L}_\rho(U^k, V^k, \lambda^k)\|_F \geq 1, \quad (\text{EC.30})$$

and by using (EC.30), it is easily verified that there exists a constant $\tilde{c} > 0$ such that

$$\phi(l^k) = \frac{c}{1-\alpha}(l^k)^{1-\alpha} \leq \tilde{c}(\|\Delta U^k\|_F + \|\Delta V^k\|_F + \|\Delta \lambda^k\|_2)^{\frac{1-\alpha}{\alpha}} = \tilde{c}(S^{k-1} - S^k)^{\frac{1-\alpha}{\alpha}}, \quad (\text{EC.31})$$

where the last equation follows from the definition of S^k . Combining (EC.31) with (EC.29), we have $\frac{3\zeta\tilde{c}}{\tilde{\kappa}}(S^{k-1} - S^k)^{\frac{1-\alpha}{\alpha}} + S^{k-1} - S^k \geq S^k$. The convergence rate follows directly from Theorem 2 of Attouch and Bolte (2009). \square

Appendix B: Additional Solver and Experiment Details

B.1. Additional Solver Description

Our solver incorporates a combination of the LRALM (Burer and Monteiro 2003b, Burer and Choi 2006) and LRADMM algorithms. Due to the solver's complexity, this section will focus on providing a detailed explanation of key implementation aspects. Specifically, we will describe three main components: the utilization of LRALM subproblems, the method for solving LRADMM subproblems, and the dynamic adjustment of the matrix rank.

LRALM as a warm start. In this phase, we primarily use the LBFGS method to solve the subproblems of the augmented Lagrangian method. After determining the optimization direction using LBFGS, the optimal step size is computed through an exact line search method, which can be found in [1].

LRADMM for accelerated solving. In this context, the LRADMM subproblems, specifically the solution of the linear system $\rho \mathcal{A}^*(\mathcal{A}(UV^\top))V + \rho U = -CV - \mathcal{A}^*(\lambda)V + \rho \mathcal{A}^*(b)V + \rho V$, are solved using the conjugate gradient (CG) method.

The dynamic rank adjustment strategy. To adjust the matrix rank dynamically, we adopt a strategy where adjustments are only made after completing the ALM subproblems. We evaluate the difficulty of each subproblem by monitoring the number of sub-iterations required, using a heuristic approach with thresholds set at 20, 100, and 400. These thresholds classify the problems hardness, and when a threshold is exceeded, we increment the rank update counter. The increments become progressively larger as the threshold increases, reflecting an increase in problem difficulty. If the rank update counter exceeds certain predefined thresholds, we determine that the sub-problem becomes particularly challenging, and we increase the rank. Following any rank adjustment, we reset the rank update counter to zero for the next evaluation cycle. The specific operation to add the rank is to increase the column number of variable U or V . During the experiments, we consider adding a diagonal matrix with value $1/\sqrt{r}$ on the main diagonal, where r is the column number of the variable. Among the 97 problems tested in Section 5, 78 achieved satisfactory performance with the initial logarithmic rank. For the remaining 19 problems, dynamic rank adjustment was triggered, and only two problems finally hit the rank of $\lceil\sqrt{2m}\rceil$.

B.2. The Evaluation of Final Errors

In this section, we discuss the time taken to evaluate final errors by LoRADS for the instances examined in Section 5. LoRADS computes three specific errors Primal Infeasibility, Dual Infeasibility, and Primal-Dual Gaps defined in Section 5.2, to certify the obtained solutions. Computing the dual infeasibility is the most time-consuming part of this error evaluation, as it involves finding the minimum eigenvalue of an $n \times n$ matrix. We utilize ARPACK for this computation, which depends on the Arnoldi iteration a first-order method known for its efficiency in large-scale problems. However, this computation can still be particularly costly for large-scale problems. Additionally, the stopping criteria for both LoRADs and SDPLR do not rely on dual residuals, nor do they compute dual residuals during iterations. This is due to the matrix factorization formulation (4) of the original SDP, which is highly nonconvex and, therefore, does not guarantee a global solution. Achieving dual feasibility would obtain both the solution to the original SDP (1) and the global solution to (4), which is not guaranteed. As a result, it is practical to evaluate dual infeasibility only after the solver has stopped, to provide a certification.

Moreover, the method of evaluating dual residuals varies among solvers, leading to divergent performance for final error evaluations. To ensure fair comparisons and clearly benchmark the solvers' capabilities in their algorithms for solving SDP problems, we have chosen to exclude the time taken to evaluate final errors from the solving time reported in Section 5¹⁰. Instead, we report the final errors evaluation time for LoRADS here in Tables EC.1, EC.3 and EC.4, providing a sense of the time required to get a certification for the solution. We also include the final errors evaluation time of COPT, which has optimized its process for final error computations as a commercial solver¹¹.

¹⁰ Both LoRADS and COPT consistently evaluate all three final errors after stopping. In contrast, SDPLR often omits the evaluation of the final dual residual. We exclude the final errors evaluation time for the above three solvers in Section 5. SDPNAL+ evaluates the dual residual intermittently during the iterations, making it impractical to exclude this time from the solving times.

¹¹ On large-scale problems, COPT demonstrates less efficiency in evaluating the final errors compared to LoRADS. This discrepancy arises because LoRADS focuses on large-scale problems and employs first-order methods to compute the minimum eigenvalue in dual infeasibility evaluations.

Table EC.1 Final Errors Evaluation Time of LoRADS and COPT in Table 4

Problem	n, m	Errors Eval Time		Problem	n, m	Errors Eval Time	
		LoRADS	COPT			LoRADS	COPT
G1	800	0.04	0.05	G54	1000	0.02	0.07
G5	800	0.06	0.05	G55	5000	0.07	11.98
G9	800	0.06	0.05	G56	5000	0.08	11.96
G13	800	0.03	0.04	G57	5000	1.21	11.84
G17	800	0.04	0.04	G58	5000	0.28	12.07
G21	800	0.20	0.04	G59	5000	0.59	12.27
G25	2000	0.05	0.43	G60	7000	0.08	37.49
G29	2000	0.03	0.40	G61	7000	0.10	37.16
G33	2000	0.50	0.36	G62	7000	3.47	37.49
G37	2000	0.09	0.38	G63	7000	0.56	37.43
G41	2000	0.19	0.36	G64	7000	0.65	38.22
G45	1000	0.04	0.08	G65	8000	1.62	48.02
G49	3000	0.02	1.69	G66	9000	2.27	79.01
G53	1000	0.04	0.07	G67	10000	2.09	109.22

Table EC.2 Final Errors Evaluation Time of LoRADS in Table 5

Name	n	LoRADS	Name	n, m	LoRADS
p2p-Gnutella04	10879	0.02	delaunay_n16	65536	0.17
vsp_befref_fxm_2_4_air02	14109	0.40	fe_tooth	78136	0.50
delaunay_n14	16384	0.05	598a	110971	0.65
cs4	22499	0.02	fe_ocean	143437	5.17
cit-HepTh	27770	0.44	amazon0302	262111	0.67
delaunay_n15	32768	0.06	amazon0505	410236	1.88
cit-HepPh	34546	0.17	delaunay_n19	524288	2.85
p2p-Gnutella30	36682	0.09	rgg_n_2_19_s0	524288	4.52
vsp_sctap1-2b_and_seymour1	40174	0.77	delaunay_n20	1048576	3.51
vsp_bump2_e18_aa01_model1_crew1	56438	3.32	rgg_n_2_20_s0	1048576	14.23

Table EC.3 Final Errors Evaluation Time of LoRADS and COPT in Table 6

Problem	n	m	Errors Eval Time		Problem	n	m	Errors Eval Time	
			LoRADS	COPT				LoRADS	COPT
MC_1000	1000	199424	0.04	0.23	MC_18000	18000	6889768	4.11	-
MC_2000	2000	550536	0.18	1.60	MC_20000	20000	7688309	3.07	-
MC_3000	3000	930328	0.59	7.71	MC_40000	40000	15684167	10.90	-
MC_4000	4000	1318563	1.09	20.64	MC_60000	60000	23683563	25.87	-
MC_5000	5000	1711980	2.85	50.09	MC_80000	80000	31682246	22.48	-
MC_6000	6000	2107303	3.33	90.28	MC_100000	100000	39682090	28.32	-
MC_8000	8000	2900179	1.47	-	MC_120000	120000	47682387	50.95	-
MC_10000	10000	3695929	3.90	-	MC_140000	140000	55682003	38.20	-
MC_12000	12000	4493420	1.91	-	MC_160000	160000	63681433	60.53	-
MC_14000	14000	5291481	2.07	-	MC_180000	180000	71681424	56.49	-
MC_16000	16000	6089963	4.23	-					

Table EC.4 Final Errors Evaluation Time of LoRADS and COPT in Table 7

Problem	n	m	Errors Eval Time		Problem	n	m	Errors Eval Time	
			LoRADS	COPT				LoRADS	COPT
Izc.1024	1024	16641	0.53	0.19	H30	3162	2964	0.55	0.14
AIH	5990	7230	1.19	1.25	hand	1296	1297	8.89	0.17
Bex2	8096	3002	0.06	0.03	ice_2.0	8113	8113	0.04	62.24
BH2	2166	1743	0.28	0.06	neosfbr25	577	14376	0.22	0.12
cancer_100	569	10470	0.88	0.08	neosfbr30e8	842	25201	0.03	1.11
CH2	2166	1743	0.14	0.06	NH2	2046	1743	0.07	0.06
checker_1.5	3970	3971	4.12	5.78	NH3	3162	2964	0.57	0.14
ephil12	363	12376	0.005	0.05	NH4	4426	4743	0.43	0.55
G40_mb	2000	2001	5.43	0.49	p_auss2_3.0	9115	9115	0.52	90.06
G48_mb	3000	3001	25.24	2.02	rend11_2000	2000	2001	0.99	0.51
G48mc	3000	3000	0.03	1.72	shmup4	799	4962	4.10	1.43
G55mc	5000	5000	0.06	11.94	theta102	500	37467	0.01	0.04
G59mc	5000	5000	0.39	12.06	theta12	600	17979	0.03	0.07
G60_mb	7000	7001	12.64	40.31	theta123	600	90020	0.07	0.07