Online Supplement for "A Distribution-Free Sequential Selection Strategy: Data-Driven Optimal Allocation via Balancing Empirical Large Deviations"

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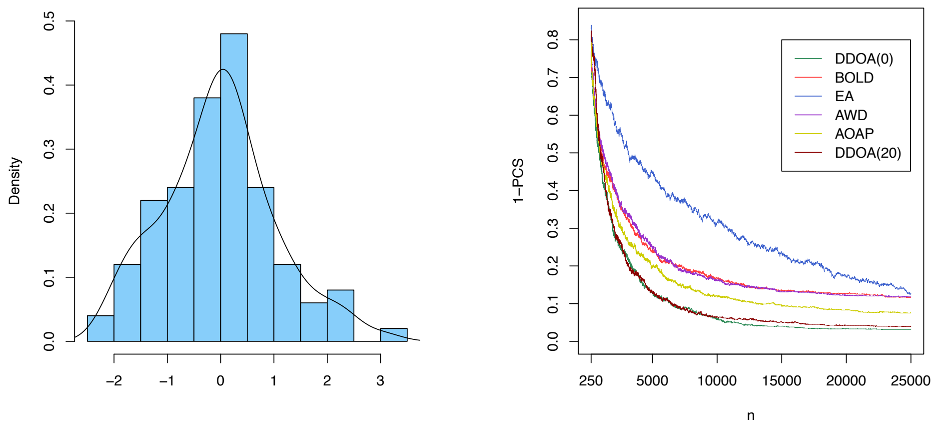
A Complex Sampling Distributions and Input Distribution Analysis

For unknown F_x, some recent work (e.g., Li et al. 2018; Shen et al. 2021) in the R&S literature suggests estimating the sampling distribution through the input modeling process (Law and Kelton 2000). This process typically hypothesizes F_x based on the similarity between the shape of the histogram of collected samples and the density plots of some general distribution families (e.g., normal, exponential, Poisson), and then perform goodness-of-fit hypothesis testing (e.g., χ^2 test) to determine whether the hypothesized distribution is a good fit. However, identifying F_x correctly through such a process in practice can be quite hard if not completely impossible, especially when F_x is complex. In Figure 3, we consider a mixture distribution family $F_x \sim B \cdot \mathcal{N}\left(\mu_{x,1}, \sigma_x^2\right) + (1-B) \cdot \mathrm{Exp}\left(1/\mu_{x,2}\right)$, where $B \sim \mathrm{Bernoulli}\,(0.4)$, $\mu_{x,1}$ is generated from $\mathcal{U}[0.6, 1.6]$, $\mu_{x,2}$ is generated from $\mathcal{U}[0.4, 1.4]$, σ_x is generated from $\mathcal{U}[0.75, 1.25]$, and $B, \mathcal{N}\left(\mu_{x,1}, \sigma_x^2\right)$ and $\mathrm{Exp}\left(1/\mu_{x,2}\right)$ are independent. There are 25 alternatives. The goal is to find the optimal alternative that has the minimum population mean. The allocation strategies are implemented the same way as in Section 5.2. Figure 3a gives an example of the histogram of 100 i.i.d. samples drawn from alternative 8, of which the shape is quite similar to the density of a normal distribution. More interestingly, in this case, a χ^2 goodness-of-fit test will even fail to reject the null hypothesis (at significance level 5%) that the samples are from a normal distribution. However, as shown in Figure 3b (as well as in Figure 2), if F_x is misspecified as a normal distribution, then even a well-designed allocation strategy under the (incorrecly) assumed distribution, such as BOLD and AOAP, may deteriorate substantially and can be outperformed by a naive one like EA when the budget gets large. This demonstrates the importance of developing an efficient allocation strategy capable of handling complex non-Gaussian distributions, or more broadly, unknown sampling distributions.



(a) (b)

Figure 3: Mixture distribution.

B Proof of Lemma 2

The convexity of $\Psi_x(\gamma)$ can be shown by Hölder's inequality. Since $\frac{\partial}{\partial\gamma} H_x = \frac{\partial^2}{\partial\gamma^2}\Psi_x$, we have that $H_x(\gamma)$ is increasing in γ. For the LHS of (12), we can see that, for any $x \neq x^*$, at $\gamma = 0$, $H_{x^*}(0) - H_x(0) = \mu_{x^*} - \mu_x > 0$. Furthermore, $\frac{\partial}{\partial\gamma}\left[H_{x^*}(\gamma) - H_x\left(-\frac{N^n_{x^*}}{N^n_x}\gamma\right)\right] = \frac{\partial}{\partial\gamma}H_{x^*}(\gamma) + \frac{\alpha_{x^*}}{\alpha_x}\frac{\partial}{\partial\gamma}H_x\left(-\frac{\alpha_{x^*}}{\alpha_x}\gamma\right) > 0$. Thus, the LHS of (12) is an increasing function of γ. Therefore, (12) has a unique negative root. Note that $\tilde{I}_x(\gamma) = 0$ if $\gamma = 0$. In addition, $\frac{\partial}{\partial\gamma}\tilde{I}_x(\gamma) = \gamma\frac{\partial}{\partial\gamma}H_x(\gamma) + H_x(\gamma) - H_x(\gamma) = \gamma\frac{\partial}{\partial\gamma}H_x(\gamma)$. Thus, $\tilde{I}_x(\gamma)$ is decreasing in γ when $\gamma < 0$ and increasing in γ when $\gamma > 0$. Consequently, $\tilde{I}_x(\gamma) \geq 0$ for all γ, and $\tilde{I}_x(\gamma) = 0$ if and only if $\gamma = 0$.

C Proof of Lemma 3

First, for any x, by the Cauchy-Schwarz inequality, we can see that

$$\frac{\partial}{\partial\gamma}H^n_x(\gamma) = \frac{\sum_{m=0}^{n-1} 1^x_m \left(W^{m+1}_x\right)^2 e^{\gamma W^{m+1}_x} \sum_{m=0}^{n-1} 1^x_m e^{\gamma W^{m+1}_x} - \left(\sum_{m=0}^{n-1} 1^x_m W^{m+1}_x e^{\gamma W^{m+1}_x}\right)^2}{\left(\sum_{m=0}^{n-1} 1^x_m e^{\gamma W^{m+1}_x}\right)^2} > 0, \tag{34}$$

where the inequality holds strictly since $\rho \neq \nu$. Second, since $\frac{\partial^2}{\partial \gamma^2}\Psi_x^n = \frac{\partial}{\partial \gamma}H_x^n$, we obtain the convexity of $\Psi_x^n(\gamma)$. For the LHS of (16), we can see that, at $\gamma = 0$, $H_{x^{*,n}}^n(0) - H_x^n(0) = \theta_{x^{*,n}}^n - \theta_x^n > 0$, where the equality holds by (5), and the inequality holds by the definition of $x^{*,n}$. Furthermore, $\frac{\partial}{\partial \gamma}\left[H_{x^{*,n}}^n(\gamma) - H_x^n\left(-\frac{N_{x^{*,n}}^n}{N_x^n}\gamma\right)\right] = \frac{\partial}{\partial \gamma}H_{x^{*,n}}^n(\gamma) + \frac{N_{x^{*,n}}^n}{N_x^n}\frac{\partial}{\partial \gamma}H_x^n\left(-\frac{N_{x^{*,n}}^n}{N_x^n}\gamma\right) > 0$. Thus, the LHS of (16) is an increasing function of γ. Therefore, (16) has a unique negative root. Third, we have

$$\frac{\partial}{\partial \gamma}I_x^n(\gamma) = \gamma\frac{\partial}{\partial \gamma}H_x^n(\gamma) + H_x^n(\gamma) - H_x^n(\gamma) = \gamma\frac{\partial}{\partial \gamma}H_x^n(\gamma), \tag{35}$$

where $\frac{\partial}{\partial \gamma}H_x^n(\gamma) > 0$ by (34). Thus, $I_x^n(\gamma)$ is decreasing in γ when $\gamma < 0$ and increasing in γ when $\gamma > 0$. Finally, we can see that $I_x^n(0) = 0$. From above, we have $I_x^n(\gamma) \geq 0$ for all γ, and $\gamma = 0$ is the only root of $I_x^n(\gamma) = 0$.

D Proof of Theorem 1

To establish the proof, we introduce two technical lemmas first:

Lemma 4. *Let Assumption 1 hold. For any $z \in \{1, ..., M\}$, if alternative z is only sampled finitely often as $n \to \infty$, i.e., $\lim_{n\to\infty} N_z^n < \infty$, then $x^{*,n} \neq z$ for all large enough n.*

Lemma 5. *Let Assumption 1 hold. For all large enough n, condition (22) will be checked. Furthermore, there are infinitely many times n at which condition (22) does not hold.*

In short, Lemma 4 states that, under Algorithm 1, for all large n, the alternative that has the largest sample mean cannot be an alternative that is only sampled finitely many times, thus must be an alternative that is sampled infinitely often. By the law of large numbers, this further implies that $\theta_{x^{*,n}}^n - \mu_{x^{*,n}} \to 0$ as $n \to \infty$. Then, Lemma 5 shows two facts of DDOA. First, condition (22) being checked for all large n suggests that $\arg\max_x \theta_x^n$ is unique for all large n, thus guarantees that DDOA will not malfunction in situations such as discrete F_x. Second, condition (23) must be invoked infinitely often when (22) does not hold. Thus, x^n is not always equal to $x^{*,n}$, yielding that alternatives with smaller sample means are also sampled infinitely often.

Now we establish the proof for Theorem 1 by contradiction. Assume there exists some alternative z such that $\lim_{n\to\infty} N_z^n < \infty$. First, from Lemma 4, we know that $x^{*,n} \neq z$ for all large enough n. More specifically, let $\bar{x} = \arg\max_{x\in\mathcal{A}} \mu_x$, where $\mathcal{A} = \{x | N_x^n \to \infty\}$. Then, for all large enough n, we always have $x^{*,n} = \bar{x}$. Second, from Lemma 5, condition (22) will be checked for all large enough n, and there are infinitely many times n at which condition (22) does not hold. Then, for all large enough n, there must exist some alternative $y \neq x^{*,n}$ that is sampled when condition (22) does not hold. Obviously, $N_y^n \to \infty$ and $N_{x^{*,n}}^n \to \infty$.

On the one hand, from the proof of Lemma 4, we can see that $\gamma_{x^{*,n},z}^n \to 0$ and $\limsup_{n\to\infty}\left|\gamma_{z,x^{*,n}}^n\right| < \infty$. Moreover, since $N_{x^{*,n}}^n / N_z^n \to \infty$, we have $\left|\sqrt{\frac{N_{x^{*,n}}^n}{N_z^n}}\gamma_{x^{*,n},z}^n\right| = \left|\frac{\gamma_{z,x^{*,n}}^n}{\sqrt{N_{x^{*,n}}^n / N_z^n}}\right| \to 0$. Then, for all large enough n,

$$\begin{aligned}
\frac{I_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right)}{I_z^n\left(\gamma_{z,x^{*,n}}^n\right)} &\leq \frac{I_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right)}{I_z^n\left(\frac{\gamma_{z,x^{*,n}}^n}{\sqrt{N_{x^{*,n}}^n / N_z^n}}\right)} = \frac{I_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right)}{I_z^n\left(-\sqrt{\frac{N_{x^{*,n}}^n}{N_z^n}}\gamma_{x^{*,n},z}^n\right)} = -\sqrt{\frac{N_z^n}{N_{x^{*,n}}^n}} \cdot \frac{\frac{\partial}{\partial \gamma}I_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right)}{\frac{\partial}{\partial \gamma}I_z^n\left(-\sqrt{\frac{N_{x^{*,n}}^n}{N_z^n}}\gamma_{x^{*,n},z}^n\right)} \\
&= \frac{N_z^n}{N_{x^{*,n}}} \cdot \frac{\frac{\partial}{\partial \gamma}H_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right)}{\frac{\partial}{\partial \gamma}H_z^n\left(\frac{\gamma_{z,x^{*,n}}^n}{\sqrt{N_{x^{*,n}}^n / N_z^n}}\right)} = \frac{N_z^n}{N_{x^{*,n}}} \cdot \Theta(1) = \Theta\left(\frac{N_z^n}{N_{x^{*,n}}}\right),
\end{aligned} \tag{36}$$

where the inequality holds by Lemma 3, the first equality holds due to (17), the second equality holds by Taylor's theorem, the third equality holds by (17) and (35), and the fourth equality holds by Lemmas 1-3. Then, (21) implies

$$\begin{aligned}
\Gamma_{x^{*,n},z}^n &= N_{x^{*,n}}^n I_{x^{*,n}}^n\left(\gamma_{x^{*,n},z}^n\right) + N_z^n I_z^n\left(\gamma_{z,x^{*,n}}^n\right) = N_{x^{*,n}}^n I_z^n\left(\gamma_{z,x^{*,n}}^n\right) \cdot O\left(N_z^n / N_{x^{*,n}}\right) + N_z^n I_z^n\left(\gamma_{z,x^{*,n}}^n\right) \\
&= \Theta\left(N_z^n I_z^n\left(\gamma_{z,x^{*,n}}^n\right)\right) = \Theta\left(N_z^n\right),
\end{aligned} \tag{37}$$

where the second equality holds by (36), the third equality holds since $\lim_{n\to\infty} N_z^n < \infty$ and $\lim_{n\to\infty} N_{x^{*,n}}^n = \infty$, and the last equality holds due to Lemma 3 and $\limsup_{n\to\infty}\left|\gamma_{z,x^{*,n}}^n\right| < \infty$. Since $\lim_{n\to\infty} N_z^n < \infty$, (37) implies that $\Gamma_{x^{*,n},z}^n$ must be bounded by some fixed constant.

On the other hand, let (n_k) denote the sequence of stages at which (22) does not hold and y is sampled. Then, along the sequence (n_k), we cannot have both $\lim_{k\to\infty}\gamma^{n_k}_{x^{*,n_k},y}=0$ and $\lim_{k\to\infty}\gamma^{n_k}_{y,x^{*,n_k}}=0$ simultaneously. We show this by contradiction. Assume that both $\gamma^{n_k}_{x^{*,n_k},y}$ and $\gamma^{n_k}_{y,x^{*,n_k}}$ converge to 0. Note that $N^n_y\to\infty$ and $N^n_{x^{*,n}}\to\infty$. Then, by Lemmas 1-3, $H^{n_k}_{x^{*,n_k}}(\gamma^{n_k}_{x^{*,n_k},y})-H^{n_k}_y\left(\gamma^{n_k}_{y,x^{*,n_k}}\right)\to H_{\bar{x}}(0)-H_y(0)$, where $H_{\bar{x}}(0)-H_y(0)=\mu_{\bar{x}}-\mu_y>0$. Thus, for all large enough k, we have $H^{n_k}_{x^{*,n_k}}(\gamma^{n_k}_{x^{*,n_k},y})-H^{n_k}_y\left(\gamma^{n_k}_{y,x^{*,n_k}}\right)>0$. However, this contradicts the fact that $\gamma^{n_k}_{x^{*,n_k},y}$ is the root of (16). Therefore, we cannot have both $\gamma^{n_k}_{x^{*,n_k},y}$ and $\gamma^{n_k}_{y,x^{*,n_k}}$ converge to 0 simultaneously.

We consider the case where $\liminf_{k\to\infty}\left|\gamma^{n_k}_{x^{*,n_k},y}\right|>0$. A symmetric argument can be established when $\liminf_{k\to\infty}\left|\gamma^{n_k}_{y,x^{*,n_k}}\right|>0$. Then, for all large enough k,

$$\Gamma^{n_k}_{x^{*,n_k},y}=N^{n_k}_{x^{*,n_k}}I^{n_k}_{x^{*,n_k}}\left(\gamma^{n_k}_{x^{*,n_k},y}\right)+N^{n_k}_yI^{n_k}_y\left(\gamma^{n_k}_{y,x^{*,n_k}}\right)\geq N^{n_k}_{x^{*,n_k}}I^{n_k}_{x^{*,n_k}}\left(\gamma^{n_k}_{x^{*,n_k},y}\right)=\Theta\left(N^{n_k}_{x^{*,n_k}}\right),\tag{38}$$

where the inequality holds by Lemma 3, and the last equality holds due to $\liminf_{k\to\infty}\left|\gamma^{n_k}_{x^{*,n_k},y}\right|>0$ and Lemmas 1-3. Since $\lim_{k\to\infty}N^{n_k}_{x^{*,n_k}}=\infty$, (38) implies that $\Gamma^{n_k}_{x^{*,n_k},y}\to\infty$ as $k\to\infty$. Then, for all large enough k, we have $\Gamma^{n_k}_{x^{*,n_k},y}>\Gamma^{n_k}_{x^{*,n_k},z}$. From Step 4 in Algorithm 1, y will not be sampled at stage n_k. However, this contradicts the definition of the sequence (n_k). Therefore, it must be the case that all alternatives are sampled infinitely many times.

E Proof of Lemma 4

Let $\bar{\theta}_z=\lim_{n\to\infty}\theta^n_z$, the final value of the sample mean for alternative z since $\lim_{n\to\infty}N^n_z<\infty$. We will show that $x^{*,n}$ cannot be z for all large enough n by contradiction.

Assume $x^{*,n}=z$ infinitely often. First, there must exist some suboptimal alternative $y\neq z$ that is sampled infinitely often, i.e., $\lim_{n\to\infty}N^n_y=\infty$. Obviously, if $\mu_y>\bar{\theta}_z$, we will have $\theta^n_y>\theta^n_z=\bar{\theta}_z$ for all large n by the law of large numbers. Then, $x^{*,n}$ can never be z. So, we must have $\mu_y\leq\bar{\theta}_z$.

Now, assume $\mu_y<\bar{\theta}_z$. By the law of large numbers, for all large enough n, we have $\theta^n_z=\bar{\theta}_z>\theta^n_y$. Note that by assumption, there are infinitely many time stages at which $x^{*,n}=z$. Then, at these stages, from (17), as $n\to\infty$, we have $\frac{\gamma^n_{y,z}}{\gamma^n_{z,y}}=-\frac{N^n_z}{N^n_y}\to 0$. Note that $\gamma^n_{z,y}$ is the root of (16). Then, since $\lim_{n\to\infty}N^n_y=\infty$, by Lemmas 1-3, we must have $\limsup_{n\to\infty}\left|\gamma^n_{z,y}\right|<\infty$. Consequently, $\gamma^n_{y,z}=-\frac{N^n_z}{N^n_y}\gamma^n_{z,y}\to 0$. Furthermore, we have $I^n_y\left(\gamma^n_{y,z}\right)=\gamma^n_{y,z}H^n_y\left(\gamma^n_{y,z}\right)-\Psi^n_y\left(\gamma^n_{y,z}\right)\to 0$. At the same time, since $\gamma^n_{y,z}\to 0$, from (16)-(17), we also have $H^n_z\left(\gamma^n_{z,y}\right)=H^n_y\left(\gamma^n_{y,z}\right)\to\mu_y$. Then, we must have $\liminf_{n\to\infty}\left|\gamma^n_{z,y}\right|>0$, as $\mu_y<\bar{\theta}_z$ by assumption. By Lemma 3, we obtain that $\liminf_{n\to\infty}I^n_z\left(\gamma^n_{z,y}\right)>0$. Then, for all large enough n, we have $\sum_{x\neq x^{*,n}}\frac{I^n_{x^{*,n}}\left(\gamma^n_{x^{*,n},x}\right)}{I^n_x\left(\gamma^n_{x,x^{*,n}}\right)}\geq\frac{I^n_z\left(\gamma^n_{z,y}\right)}{I^n_y\left(\gamma^n_{y,z}\right)}>1$, which implies that alternative z will be sampled for any large enough n if $x^{*,n}=z$. However, this contradicts that z is only sampled finitely often. Thus, it must be the case that the assumption $\mu_y<\bar{\theta}_z$ is false, so we must have $\mu_y=\bar{\theta}_z$.

Finally, assume $\mu_y=\bar{\theta}_z$. By the law of large numbers, $\theta^n_y\to\bar{\theta}_z$. Note that there are assumed to be infinitely many times that $x^{*,n}=z$, and by the definition of $x^{*,n}$, we must have $\theta^n_y<\bar{\theta}_z$ at such stages. First, as $n\to\infty$, from (16), we have $H^n_z(0)-H^n_y(0)=\theta^n_z-\theta^n_y\to\bar{\theta}_z-\mu_y=0$. Then, by Lemmas 1-3, we must have $\lim_{n\to\infty}\gamma^n_{z,y}=0$, which also implies that $\lim_{n\to\infty}\gamma^n_{y,z}=-\lim_{n\to\infty}\frac{N^n_z}{N^n_y}\gamma^n_{z,y}=0$. Then, for all large enough n,

$$\frac{I^n_z\left(\gamma^n_{z,y}\right)}{I^n_y\left(\gamma^n_{y,z}\right)}=\frac{\gamma^n_{z,y}\frac{\partial}{\partial\gamma}I^n_z(\gamma^n_{z,y})}{\gamma^n_{y,z}\frac{\partial}{\partial\gamma}I^n_y(\gamma^n_{y,z})}=\frac{\left(\gamma^n_{z,y}\right)^2\frac{\partial}{\partial\gamma}H^n_z(\gamma^n_{z,y})}{\left(\gamma^n_{y,z}\right)^2\frac{\partial}{\partial\gamma}H^n_y(\gamma^n_{y,z})}=\left(\frac{N^n_y}{N^n_z}\right)^2\frac{\frac{\partial}{\partial\gamma}H^n_z(\gamma^n_{z,y})}{\frac{\partial}{\partial\gamma}H^n_y(\gamma^n_{y,z})}=\left(\frac{N^n_y}{N^n_z}\right)^2\Theta(1)=\Theta\left(\left(\frac{N^n_y}{N^n_z}\right)^2\right),\tag{39}$$

where the first equality holds by Taylor's theorem, the second equality holds due to (35), the third equality holds due to (17), and the fourth equality holds by Lemmas 1-3. By assumption, $N^n_y/N^n_z\to\infty$ because y is sampled infinitely often. Then, from (39), as $n\to\infty$, we have $\sum_{x\neq x^{*,n}}\frac{I^n_{x^{*,n}}\left(\gamma^n_{x^{*,n},x}\right)}{I^n_x\left(\gamma^n_{x,x^{*,n}}\right)}\geq\frac{I^n_z\left(\gamma^n_{z,y}\right)}{I^n_y\left(\gamma^n_{y,z}\right)}\to\infty$, which implies that z will be sampled infinitely often if $x^{*,n}=z$ occurs infinitely often. However, this contradicts that z is only sampled finitely often. Therefore, it must be the case that $x^{*,n}\neq z$ for all large enough n.

F Proof of Lemma 5

Let $\mathcal{A} = \{x | N_x^n \to \infty\}$ be the set of alternatives that are sampled infinitely often. Obviously, $\mathcal{A} \neq \emptyset$, because there is at least one alternative that will be sampled infinitely many times as $n \to \infty$. Note that, for any two different alternatives y and x, $\mu_y \neq \mu_x$. Then, if $\mathcal{A}$ contains more than one alternative, then we must have $\lim_{n\to\infty}\left(\theta_y^n - \theta_x^n\right) = \mu_y - \mu_x \neq 0$, for any two different alternatives $y, x \in \mathcal{A}$. Now, take n to be large enough such that the alternatives in $\mathcal{A}^c$ will not be sampled from stage n onwards and $\inf_{x\in\mathcal{A}} N_x^n > \sup_{x'\in\mathcal{A}^c} N_{x'}^n$. For such n, if condition (22) is not checked, it must be the case that there exist $x \in \mathcal{A}$ and $x' \in \mathcal{A}^c$ such that $\theta_x^n = \theta_{x'}^n$. However, from Step 1 of Algorithm 1, we will sample x' in such a situation, but this will contradict the fact that x' cannot be sampled for such n. Therefore, condition (22) will be checked for all large enough n.

We prove the second part by contradiction. First, for all $x' \in \mathcal{A}^c$, define $\bar{\theta}_{x'} = \lim_{n\to\infty}\theta_{x'}^n$. Assume there are only finitely many times that condition (22) does not hold. That is, condition (22) holds for all large enough n. Then, $\mathcal{A}$ must contain at least two alternatives. Assume this is not the case, i.e., there is only one alternative in $\mathcal{A}$. Denote this alternative by z. Then, $\mathcal{A} = \{z\}$ and $\mathcal{A}^c = \{x' | x' \neq z\}$. By assumption, condition (22) holds for all large enough n, and z is the only alternative that is sampled infinitely many times. Then, we must have $x^{*,n} = z$ for all large enough n. By the definition of $x^{*,n}$, this implies $\mu_z > \sup_{x'\in\mathcal{A}^c}\bar{\theta}_{x'}$. However, from the proof of Lemma 4, we can see that, for any $x' \neq z$, $\lim_{n\to\infty} I_z^n\left(\gamma_{z,x'}^n\right) = 0$ and $\liminf_{n\to\infty} I_{x'}^n\left(\gamma_{x',z}^n\right) > 0$, which implies that $I_z^n\left(\gamma_{z,x'}^n\right)/I_{x'}^n\left(\gamma_{x',z}^n\right) < \frac{1}{M}$ for all large enough n. Consequently, condition (22) will not hold, because $\sum_{x'\neq x^{*,n}} I_{x^{*,n}}^n\left(\gamma_{x^{*,n},x'}^n\right)/I_{x'}^n\left(\gamma_{x',x^{*,n}}^n\right) = \sum_{x'\neq z} I_z^n\left(\gamma_{z,x'}^n\right)/I_{x'}^n\left(\gamma_{x',z}^n\right) < \sum_{x'\neq z}\frac{1}{M} < 1$. However, this contradicts the assumption that condition (22) holds for all large enough n. Therefore, it must be the case that $\mathcal{A}$ contains at least two alternatives.

Let z and y be two different alternatives in $\mathcal{A}$. By the law of large numbers, $\lim_{n\to\infty}\theta_z^n = \mu_z$ and $\lim_{n\to\infty}\theta_y^n = \mu_y$. Without loss of generality, suppose $\mu_z < \mu_y$. Then, for all large enough n, we must have $\theta_z^n < \theta_y^n$, and consequently, $x^{*,n} \neq z$. At the same time, note that condition (22) holds for all large enough n by assumption, which implies that only $x^{*,n}$ is sampled for all large enough n. Then, z will not be sampled for any large n, which contradicts the fact that $z \in \mathcal{A}$. Therefore, there must be infinitely many times n at which condition (22) does not hold.

G Proof of Theorem 2

We will establish the proof by contradiction. Depending on whether x^* is one of the alternatives in the comparison, we have the following cases.

Case 1. $y = x^*$.

Assume the result does not hold, i.e., there exists some $x \neq x^*$ such that $\limsup_{n\to\infty}\frac{N_x^n}{N_{x^*}^n} = \infty$. Then, there must exist a subsequence (n_k) such that $\lim_{k\to\infty}\frac{N_x^{n_k}}{N_{x^*}^{n_k}} = \infty$. Define a subsequence (m_k), where $m_k = \sup\left\{l \leq n_k : x^l = x\right\}$. We must have $\lim_{k\to\infty}\frac{N_x^{m_k}}{N_{x^*}^{m_k}} = \infty$ for this subsequence (m_k) since $\lim_{k\to\infty} N_{x^*}^{n_k} = \infty$ and $N_{x^*}^n$ is monotonically increasing in n. Note that only x is sampled along the subsequence (m_k). From (17), as $k \to \infty$, we have $\frac{\gamma_{x,x^*}^{m_k}}{\gamma_{x^*,x}^{m_k}} = -\frac{N_{x^*}^{m_k}}{N_x^{m_k}} \to 0$. Note that $\gamma_{x^*,x}^{m_k}$ is the root of (16). Then, since $\lim_{k\to\infty} N_x^{m_k} = \infty$ and $\lim_{k\to\infty} N_{x^*}^{m_k} = \infty$, by Lemmas 1-3, we must have $\limsup_{k\to\infty}\left|\gamma_{x^*,x}^{m_k}\right| < \infty$. Consequently, $\gamma_{x,x^*}^{m_k} = -\frac{N_{x^*}^{m_k}}{N_x^{m_k}}\gamma_{x^*,x}^{m_k} \to 0$. Furthermore, we have $I_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right) = \gamma_{x,x^*}^{m_k} H_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right) - \Psi_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right) \to 0$. At the same time, since $\gamma_{x,x^*}^{m_k} \to 0$, from (16)-(17), we also have $H_{x^*}^{m_k}\left(\gamma_{x^*,x}^{m_k}\right) = H_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right) \to \mu_x$. Then, we must have $\liminf_{k\to\infty}\left|\gamma_{x^*,x}^{m_k}\right| > 0$, as $\mu_x < \mu_{x^*}$. By Lemmas 1-3, we obtain that $\liminf_{k\to\infty} I_{x^*}^{m_k}\left(\gamma_{x^*,x}^{m_k}\right) > 0$. Then, for all large enough k, we have $\Delta^{m_k} \geq \frac{I_{x^*}^{m_k}\left(\gamma_{x^*,x}^{m_k}\right)}{I_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right)} > 1$. From Step 3 in Algorithm 1, x^* will be sampled along (m_k) for all large enough k. But this contradicts the definition of (m_k) that $x^{m_k} = x$ for all k. Therefore, it must be the case that $\limsup_{n\to\infty}\frac{N_x^n}{N_{x^*}^n} < \infty$ for all $x \neq x^*$.

Case 2. $x, y \neq x^*$.

Assume the result does not hold, i.e., there exist $x, y \neq x^*$ such that $\limsup_{n\to\infty}\frac{N_x^n}{N_y^n} = \infty$. By repeating the same argument from above, there must exist a subsequence (n_k) such that $\lim_{k\to\infty}\frac{N_x^{n_k}}{N_y^{n_k}}=\infty$ and only alternative x is sampled along this subsequence.

On the one hand, we will show that $\liminf_{k\to\infty}\left|\gamma_{x,x^*}^{n_k}\right|>0$ by contradiction. Assume that $\liminf_{k\to\infty}\left|\gamma_{x,x^*}^{n_k}\right|=0$, i.e., there exists a subsequence (m_k) such that $\lim_{k\to\infty}\gamma_{x,x^*}^{m_k}=0$. Since $\limsup_{n\to\infty}\frac{N_x^n}{N_{x^*}^n}<\infty$ for all $x\neq x^*$, (17) implies that, as $k\to\infty$, $\gamma_{x^*,x}^{m_k}=-\frac{N_x^{m_k}}{N_{x^*}^{m_k}}\gamma_{x,x^*}^{m_k}\to 0$. Then, by Lemmas 1-3, $H_{x^*}^{m_k}\left(\gamma_{x^*,x}^{m_k}\right)-H_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right)\to H_{x^*}(0)-H_x(0)$, where $H_{x^*}(0)-H_x(0)=\mu_{x^*}-\mu_x>0$. Thus, for all large enough k, we have $H_{x^*}^{m_k}(\gamma_{x^*,x}^{m_k})-H_x^{m_k}\left(\gamma_{x,x^*}^{m_k}\right)>0$. However, this contradicts the fact that $\gamma_{x^*,x}^{m_k}$ is the root of (16). Therefore, it must be the case that $\liminf_{k\to\infty}\left|\gamma_{x,x^*}^{n_k}\right|>0$. At the same time, by Lemmas 1-3, for all large enough k, $\Gamma_{x^*,x}^{n_k}=N_{x^*}^{n_k}I_{x^*}^{n_k}\left(\gamma_{x^*,x}^{n_k}\right)+N_x^{n_k}I_x^{n_k}\left(\gamma_{x,x^*}^{n_k}\right)\geq N_x^{n_k}I_x^{n_k}\left(\gamma_{x,x^*}^{n_k}\right)$. Since $\liminf_{k\to\infty}\left|\gamma_{x,x^*}^{n_k}\right|>0$, by Lemmas 1-3, we have $\Gamma_{x^*,x}^{n_k}=\Omega\left(N_x^{n_k}\right)$.

On the other hand, we will show that $\Gamma_{x^*,y}^{n_k}=\Theta\left(N_y^{n_k}\right)$. There are two possible cases, depending on whether $N_{x^*}^{n_k}$ is $O\left(N_y^{n_k}\right)$. First, suppose $\liminf_{k\to\infty}\frac{N_y^{n_k}}{N_{x^*}^{n_k}}>0$. Since we know that $\limsup_{n\to\infty}\frac{N_y^n}{N_{x^*}^n}<\infty$, we have $N_{x^*}^{n_k}=\Theta\left(N_y^{n_k}\right)$. Then, Lemmas 1-3 lead to $0<\liminf_{k\to\infty}\left|\gamma_{y,x^*}^{n_k}\right|\leq\limsup_{k\to\infty}\left|\gamma_{y,x^*}^{n_k}\right|<\infty$ and $0<\liminf_{k\to\infty}\left|\gamma_{x^*,y}^{n_k}\right|\leq\limsup_{k\to\infty}\left|\gamma_{x^*,y}^{n_k}\right|<\infty$. Consequently, we have $\Gamma_{x^*,y}^{n_k}=N_{x^*}^{n_k}I_{x^*}^{n_k}\left(\gamma_{x^*,y}^{n_k}\right)+N_y^{n_k}I_y^{n_k}\left(\gamma_{y,x^*}^{n_k}\right)=\Theta\left(N_y^{n_k}\right)$. Second, suppose $\liminf_{k\to\infty}\frac{N_y^{n_k}}{N_{x^*}^{n_k}}=0$. Let $\left(n_{k_j}\right)$ denote the subsequence of (n_k) such that $\lim_{j\to\infty}\frac{N_y^{n_{k_j}}}{N_{x^*}^{n_{k_j}}}=0$. From the proof of Theorem 1, we can see that, $\lim_{j\to\infty}\gamma_{x^*,y}^{n_{k_j}}\to 0$, $\limsup_{j\to\infty}\left|\gamma_{y,x^*}^{n_{k_j}}\right|<\infty$, and, for all large enough j, $I_{x^*}^{n_{k_j}}\left(\gamma_{x^*,y}^{n_{k_j}}\right)/I_y^{n_{k_j}}\left(\gamma_{y,x^*}^{n_{k_j}}\right)=\Theta\left(\left(N_y^{n_{k_j}}/N_{x^*}^{n_{k_j}}\right)^2\right)$. Consequently, we have $\Gamma_{x^*,y}^{n_{k_j}}=N_{x^*}^{n_{k_j}}I_{x^*}^{n_{k_j}}\left(\gamma_{x^*,y}^{n_{k_j}}\right)+N_y^{n_{k_j}}I_y^{n_{k_j}}\left(\gamma_{y,x^*}^{n_{k_j}}\right)=\Theta\left(N_y^{n_{k_j}}\right)$.

Finally, since $\lim_{k\to\infty}\frac{N_x^{n_k}}{N_y^{n_k}}=\infty$, $\Gamma_{x^*,x}^{n_k}=\Omega\left(N_x^{n_k}\right)$ and $\Gamma_{x^*,y}^{n_k}=\Theta\left(N_y^{n_k}\right)$, we have that $\Gamma_{x^*,x}^{n_k}>\Gamma_{x^*,y}^{n_k}$ for all large enough k. From Step 3 of Algorithm 1, x will not be sampled along (n_k) for any large enough k. However, this contradicts the definition of (n_k) that $x^{n_k}=x$ for all k. Therefore, it must be the case that $\limsup_{n\to\infty}\frac{N_x^n}{N_y^n}<\infty$ for all $x,y\neq x^*$.

Case 3. $x=x^*$.

Assume the result does not hold, i.e., there exists some $x\neq x^*$ such that $\limsup_{n\to\infty}\frac{N_{x^*}^n}{N_x^n}=\infty$. By repeating the same argument from above, there must exist a subsequence (n_k) such that $\lim_{k\to\infty}\frac{N_{x^*}^{n_k}}{N_x^{n_k}}=\infty$ and only x^* is sampled along this subsequence. Furthermore, since $0<\liminf_{n\to\infty}\frac{N_x^n}{N_y^n}\leq\limsup_{n\to\infty}\frac{N_x^n}{N_y^n}<\infty$ for all $y\neq x^*$, we can see that $\lim_{k\to\infty}\frac{N_{x^*}^{n_k}}{N_y^{n_k}}=\infty$ for all $y\neq x^*$. Repeating the argument in Case 1, we have that, for all $y\neq x^*$, $\lim_{k\to\infty}\gamma_{x^*,y}^{n_k}=0$ and $\liminf_{k\to\infty}\left|\gamma_{y,x^*}^{n_k}\right|>0$. Consequently, Lemmas 1-3 imply that $\lim_{k\to\infty}I_{x^*}^{n_k}\left(\gamma_{x^*,y}^{n_k}\right)=0$ and $\liminf_{k\to\infty}I_y^{n_k}\left(\gamma_{y,x^*}^{n_k}\right)>0$. Then, for all large enough k, we have $\Delta^{n_k}=\sum_{y\neq x^*}\frac{I_{x^*}^{n_k}\left(\gamma_{x^*,y}^{n_k}\right)}{I_y^{n_k}\left(\gamma_{y,x^*}^{n_k}\right)}<1$. From Step 3 in Algorithm 1, x^* will not be sampled along (n_k) for any large enough k. But this contradicts the definition of (n_k) that $x^{n_k}=x^*$ for all k. Therefore, it must be the case that $\limsup_{n\to\infty}\frac{N_{x^*}^n}{N_x^n}<\infty$ for all $x\neq x^*$.

H Proof of Proposition 1

Note that $\gamma_{x^*,x}^n$ is the root of (16). By Lemma 3, $\gamma_{x^*,x}^n$ is always negative. By Theorem 2, we also have $N_{x^*}^n/N_x^n=\Theta(1)$. Thus, by Lemmas 1-3, (24) holds. Then, (25) follows due to (17) and (24). Consequently, by Lemmas 1-3, (24) and (25) lead to (26) and (27), respectively.

I Proof of Proposition 2

First, for any x and positive integers n and m, let $L_x^{n,n+m}\triangleq N_x^{n+m}-N_x^n$ denote the number of samples allocated to x from stage n to $n+m-1$. For tidiness, we abbreviate $L_x^{n,n+m}$ to L_x^m whenever it causes no misunderstanding.

Note that $L_x^m \le m$ for all x. Thus, we have $L_x^m = O(n\beta^n)$ for all x. First, we can see that

$$\begin{aligned}
\gamma_{x,x^*}^n - \gamma_{x,x^*}^{n+m} &= \frac{N_{x^*}^{n+m}}{N_x^{n+m}}\gamma_{x^*,x}^{n+m} - \frac{N_{x^*}^n}{N_x^n}\gamma_{x^*,x}^n = \frac{N_{x^*}^n + L_{x^*}^m}{N_x^n + L_x^m}\gamma_{x^*,x}^{n+m} - \frac{N_{x^*}^n}{N_x^n}\gamma_{x^*,x}^n \qquad (40)\\
&= \left(\frac{N_{x^*}^n + L_{x^*}^m}{N_x^n + L_x^m} - \frac{N_{x^*}^n}{N_x^n}\right)\gamma_{x^*,x}^{n+m} + \frac{N_{x^*}^n}{N_x^n}\gamma_{x^*,x}^{n+m} - \frac{N_{x^*}^n}{N_x^n}\gamma_{x^*,x}^n\\
&= \frac{L_{x^*}^m N_x^n - L_x^m N_{x^*}^n}{\left(N_x^n + L_x^m\right)N_x^n}\gamma_{x^*,x}^{n+m} + \frac{N_{x^*}^n}{N_x^n}\left(\gamma_{x^*,x}^{n+m} - \gamma_{x^*,x}^n\right)\\
&= O(\beta^n) + \Theta(1)\cdot\left(\gamma_{x^*,x}^{n+m} - \gamma_{x^*,x}^n\right), \qquad (41)
\end{aligned}$$

where (40) holds due to (17), and (41) holds by Theorem 2 and Proposition 1. Note that $\gamma_{x^*,x}^n$ and $\gamma_{x^*,x}^{n+m}$ are the root of (16) at time stages n and $n+m$, respectively. Then, for all large enough n, $0 = H_{x^*}^n(\gamma_{x^*,x}^n) - H_x^n(\gamma_{x,x^*}^n) - [H_{x^*}^{n+m}(\gamma_{x^*,x}^{n+m}) - H_x^{n+m}(\gamma_{x,x^*}^{n+m})] = H_{x^*}^n(\gamma_{x^*,x}^n) - H_{x^*}^n(\gamma_{x^*,x}^{n+m}) + H_{x^*}^n(\gamma_{x^*,x}^{n+m}) - H_{x^*}^{n+m}(\gamma_{x^*,x}^{n+m}) = [H_x^n(\gamma_{x,x^*}^{n+m}) - H_x^n(\gamma_{x,x^*}^n)] + [H_x^{n+m}(\gamma_{x,x^*}^{n+m}) - H_x^n(\gamma_{x,x^*}^{n+m})] = H_{x^*}^n(\gamma_{x^*,x}^n) - H_{x^*}^n(\gamma_{x^*,x}^{n+m}) + O(\beta^n) + [H_x^n(\gamma_{x,x^*}^{n+m}) - H_x^n(\gamma_{x,x^*}^n)] + O(\beta^n) = \Theta(1)\cdot(\gamma_{x^*,x}^n - \gamma_{x^*,x}^{n+m}) + \Theta(1)\cdot(\gamma_{x,x^*}^{n+m} - \gamma_{x,x^*}^n) + O(\beta^n) = \Theta(1)\cdot(\gamma_{x^*,x}^n - \gamma_{x^*,x}^{n+m}) + O(\beta^n)$, where the third equality holds by the uniform convergence of H_x^n, the fourth equality holds by Lemmas 1-3, Proposition 1 and the mean value theorem, and the last equality holds due to (41). Then, this implies $|\gamma_{x^*,x}^n - \gamma_{x^*,x}^{n+m}| = O(\beta^n)$, and consequently, (41) implies $|\gamma_{x,x^*}^n - \gamma_{x,x^*}^{n+m}| = O(\beta^n)$.

J Proof of Proposition 3

Since $\tilde\gamma_{x^*,x}^n$ is the root of (28), by Lemma 2, $\tilde\gamma_{x^*,x}^n$ is always negative. By Theorem 2, we also have $N_{x^*}^n/N_x^n = \Theta(1)$. Thus, by Lemmas 1-3, (30) holds. Then, (31) follows by (29) and (30). Thus, by Lemmas 1-3, (32) and (33) follow by (30) and (31), respectively. Finally, (16) and (28) imply

$$\begin{aligned}
0 &= H_{x^*}^n(\gamma_{x^*,x}^n) - H_x^n\left(\gamma_{x,x^*}^n\right) - \left[H_{x^*}(\tilde\gamma_{x^*,x}^n) - H_x\left(\tilde\gamma_{x,x^*}^n\right)\right]\\
&= H_{x^*}^n(\gamma_{x^*,x}^n) - H_{x^*}^n(\tilde\gamma_{x^*,x}^n) + H_{x^*}^n(\tilde\gamma_{x^*,x}^n) - H_{x^*}(\tilde\gamma_{x^*,x}^n)\\
&\quad - \left[H_x^n\left(\gamma_{x,x^*}^n\right) - H_x^n\left(\tilde\gamma_{x,x^*}^n\right)\right] - \left[H_x^n\left(\tilde\gamma_{x,x^*}^n\right) - H_x\left(\tilde\gamma_{x,x^*}^n\right)\right]\\
&= \Theta(1)\cdot\left(\gamma_{x^*,x}^n - \tilde\gamma_{x^*,x}^n\right) + O(\beta^n) - \Theta(1)\cdot\left(\gamma_{x,x^*}^n - \tilde\gamma_{x,x^*}^n\right) + O(\beta^n) \qquad (42)\\
&= \Theta(1)\cdot\left(\gamma_{x^*,x}^n - \tilde\gamma_{x^*,x}^n\right) + O(\beta^n), \qquad (43)
\end{aligned}$$

where (42) follows from the uniform convergence of H_x^n, and (43) holds due to (17), (29) and Theorem 2. Thus, $\left|\gamma_{x^*,x}^n - \tilde\gamma_{x^*,x}^n\right| = O(\beta^n)$. Then, (17) and (29) imply $\left|\gamma_{x,x^*}^n - \tilde\gamma_{x,x^*}^n\right| = O(\beta^n)$.

K Proof of Proposition 4

Note that $\tilde\gamma_{x^*,x}^n$ and $\tilde\gamma_{x^*,x}^m$ are the roots of (28) at stage n and m, respectively. Suppose that $\frac{N_x^m}{N_{x^*}^m} > \frac{N_x^n}{N_{x^*}^n}$. By Lemma 2 and Proposition 3, $H_x(\gamma)$ is increasing in γ and $\tilde\gamma_{x^*,x}^n < 0$ for all n, we can see that $H_{x^*}(\tilde\gamma_{x^*,x}^n) - H_x\left(-\frac{N_{x^*}^m}{N_x^m}\tilde\gamma_{x^*,x}^n\right) > H_{x^*}(\tilde\gamma_{x^*,x}^n) - H_x\left(-\frac{N_{x^*}^n}{N_x^n}\tilde\gamma_{x^*,x}^n\right) = 0$. Also, by Lemma 2, the LHS of (28) is increasing in γ. Thus, we must have $\tilde\gamma_{x^*,y}^m < \tilde\gamma_{x^*,y}^n$. With a similar argument, we can also show that $\tilde\gamma_{x^*,y}^m > \tilde\gamma_{x^*,y}^n$ if $\frac{N_x^m}{N_{x^*}^m} < \frac{N_x^n}{N_{x^*}^n}$. Therefore, we have $\left(\frac{N_x^m}{N_{x^*}^m} - \frac{N_x^n}{N_{x^*}^n}\right)\left(\tilde\gamma_{x^*,x}^n - \tilde\gamma_{x^*,x}^m\right) \ge 0$, and the equality holds if and only if $\frac{N_x^m}{N_{x^*}^m} = \frac{N_x^n}{N_{x^*}^n}$. The other inequality can be shown with the same argument by noting that $\tilde\gamma_{x,x^*}^n$ is the root of $H_{x^*}\left(-\frac{N_x^n}{N_{x^*}^n}\tilde\gamma_{x,x^*}^n\right) - H_x\left(\tilde\gamma_{x,x^*}^n\right) = 0$.

L Lemma 6

Lemma 6. *Under Assumption 1 and Algorithm 1, between two samples of the optimal alternative, the number of samples that can be allocated to any suboptimal alternatives is $O(n\beta^n)$. Symmetrically, between two samples of any suboptimal alternatives (i.e., two time stages when (22) fails), the number of samples that can be allocated to the optimal alternative is also $O(n\beta^n)$.*

Proof. Consider a stage n such that $\Delta^n > 1$ and $\Delta^{n+1} \le 1$. From Algorithm 1, x^* will be sampled at stage n and a suboptimal alternative will be sampled at stage $n+1$. Define $s_n \triangleq \inf\{l > 0 : \Delta^{n+l} > 1\}$. Note that stage

$n+s_n$ is the first time that x^* will be sampled after stage n. Then, in order to show that between two samples of x^*, the number of samples that can be allocated to any suboptimal alternatives is $O\left(n\beta^n\right)$, it is sufficient to show $s_n = O\left(n\beta^n\right)$.

For all large n, by Lemmas 1-3, Propositions 1 and 3 and the mean value theorem, we have

$$\begin{aligned}
\Delta^n &= \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\gamma^n_{x^*,x}\right)}{I^n_x\left(\gamma^n_{x,x^*}\right)} = \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\gamma^n_{x^*,x}\right) - I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right) + I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right)}{I^n_x\left(\gamma^n_{x,x^*}\right) - I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right) + I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right)} \\
&= \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right) + O\left(\beta^n\right)}{I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right) + O\left(\beta^n\right)} = \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right)}{I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right)} + O\left(\beta^n\right). \qquad (44)
\end{aligned}$$

Consider a stage $n+m$, where $0 < m \leq s_n$. Let $\lambda_n \triangleq \frac{m}{n\beta^n}$. At stage n, since $\Delta^n > 1$, by (44), we must have $\sum_{x\neq x^*} \frac{I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right)}{I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right)} > 1 - C_1\beta^n$, where C_1 is a fixed positive constant. At stage $n+m$, by (44), we also have $\sum_{x\neq x^*} \frac{I^{n+m}_{x^*}\left(\gamma^{n+m}_{x^*,x}\right)}{I^{n+m}_x\left(\gamma^{n+m}_{x,x^*}\right)} \geq \sum_{x\neq x^*} \frac{I^{n+m}_{x^*}\left(\tilde{\gamma}^{n+m}_{x^*,x}\right)}{I^{n+m}_x\left(\tilde{\gamma}^{n+m}_{x,x^*}\right)} - C_2\beta^n$, where C_2 is a fixed positive constant. Therefore, in order to have $\Delta^{n+m} > 1$, it is sufficient to have $\sum_{x\neq x^*} \frac{I^{n+m}_{x^*}\left(\tilde{\gamma}^{n+m}_{x^*,x}\right)}{I^{n+m}_x\left(\tilde{\gamma}^{n+m}_{x,x^*}\right)} - C_2\beta^n \geq \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right)}{I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right)} + C_1\beta^n$, or, equivalently,

$$\sum_{x\neq x^*} \frac{I^{n+m}_{x^*}\left(\tilde{\gamma}^{n+m}_{x^*,x}\right)}{I^{n+m}_x\left(\tilde{\gamma}^{n+m}_{x,x^*}\right)} - \sum_{x\neq x^*} \frac{I^n_{x^*}\left(\tilde{\gamma}^n_{x^*,x}\right)}{I^n_x\left(\tilde{\gamma}^n_{x,x^*}\right)} \geq C_3\beta^n, \qquad (45)$$

where $C_3 = C_1 + C_2$ is a fixed positive constant.

On the one hand, note that there must be at least one alternative $y \neq x^*$ such that $L^m_y \geq \frac{\lambda_n n\beta^n}{M}$. Then, for all large enough n,

$$\begin{aligned}
-\frac{\gamma^{n+m}_{x^*,y}}{\gamma^{n+m}_{y,x^*}} + \frac{\gamma^n_{x^*,y}}{\gamma^n_{y,x^*}} &= \frac{N^n_y + L^m_y}{N^n_{x^*}+1} - \frac{N^n_y}{N^n_{x^*}} = \frac{L^m_y}{N^n_{x^*}+1} - \left(\frac{N^n_y}{N^n_{x^*}} - \frac{N^n_y}{N^n_{x^*}+1}\right) \qquad (46)\\
&\geq \frac{\lambda_n\beta^n}{M} - \frac{N^n_y}{N^n_{x^*}}\frac{1}{N^n_{x^*}+1} \geq \frac{\lambda_n\beta^n}{M} - \frac{C_1}{n} \geq \left(\frac{\lambda_n}{M} - C_4\right)\beta^n, \qquad (47)
\end{aligned}$$

where C_4 is a fixed positive constant, (46) holds due to (17), the first inequality holds since $L^m_y \geq \frac{\lambda_n n\beta^n}{M}$, the second inequality holds since $N^n_x = \Theta(n)$ for all x, and the third inequality holds since $\lim_{n\to\infty}(\sqrt{n}\beta^n)^{-1} = 0$. In the meantime, Propositions 1 and 3 imply that

$$\begin{aligned}
-\frac{\gamma^{n+m}_{x^*,y}}{\gamma^{n+m}_{y,x^*}} + \frac{\gamma^n_{x^*,y}}{\gamma^n_{y,x^*}} &= -\frac{\tilde{\gamma}^{n+m}_{x^*,y}}{\tilde{\gamma}^{n+m}_{y,x^*}} + \frac{\tilde{\gamma}^n_{x^*,y}}{\tilde{\gamma}^n_{y,x^*}} + O\left(\beta^n\right) = \frac{1}{\tilde{\gamma}^{n+m}_{y,x^*}\tilde{\gamma}^n_{y,x^*}}\left(\tilde{\gamma}^n_{x^*,y}\tilde{\gamma}^{n+m}_{y,x^*} - \tilde{\gamma}^n_{y,x^*}\tilde{\gamma}^{n+m}_{x^*,y}\right) + O\left(\beta^n\right)\\
&= \frac{1}{\tilde{\gamma}^{n+m}_{y,x^*}\tilde{\gamma}^n_{y,x^*}}\left[\tilde{\gamma}^n_{x^*,y}\left(\tilde{\gamma}^{n+m}_{y,x^*} - \tilde{\gamma}^n_{y,x^*}\right) + \tilde{\gamma}^n_{y,x^*}\left(\tilde{\gamma}^n_{x^*,y} - \tilde{\gamma}^{n+m}_{x^*,y}\right)\right] + O\left(\beta^n\right)\\
&= \Theta(1)\cdot\left(\tilde{\gamma}^n_{y,x^*} - \tilde{\gamma}^{n+m}_{y,x^*}\right) + \Theta(1)\cdot\left(\tilde{\gamma}^n_{x^*,y} - \tilde{\gamma}^{n+m}_{x^*,y}\right) + O\left(\beta^n\right). \qquad (48)
\end{aligned}$$

Note that $\frac{N^{n+m}_y}{N^{n+m}_{x^*}} > \frac{N^n_y}{N^n_{x^*}}$ for all large enough n when $\liminf_{n\to\infty}\lambda_n > 0$. Then, by (47), (48) and Proposition 4, for all large n, we have $\max\left\{\tilde{\gamma}^n_{x^*,y} - \tilde{\gamma}^{n+m}_{x^*,y}, \tilde{\gamma}^n_{y,x^*} - \tilde{\gamma}^{n+m}_{y,x^*}\right\} \geq C_6\left(\frac{\lambda_n}{M} - C_5\right)\beta^n$ and $\min\left\{\tilde{\gamma}^n_{x^*,y} - \tilde{\gamma}^{n+m}_{x^*,y}, \tilde{\gamma}^n_{y,x^*} - \tilde{\gamma}^{n+m}_{y,x^*}\right\} > 0$, where C_5 and C_6 are fixed positive constants. Consequently, we have

$$\begin{aligned}
&I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y})/I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*}) - I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})/I^n_y(\tilde{\gamma}^n_{y,x^*})\\
=\;&\left[I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y})I^n_y(\tilde{\gamma}^n_{y,x^*}) - I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})\right] / \left[I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})I^n_y(\tilde{\gamma}^n_{y,x^*})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{I^n_y(\tilde{\gamma}^n_{y,x^*})\left[I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y}) - I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})\right] + I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})\left[I^n_y(\tilde{\gamma}^n_{y,x^*}) - I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})\right]}{I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})I^n_y(\tilde{\gamma}^n_{y,x^*})} \\
&= \Theta(1)\cdot\left[I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y}) - I^n_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y}) + I^n_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y}) - I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})\right] \\
&\quad +\Theta(1)\cdot\left[I^n_y(\tilde{\gamma}^n_{y,x^*}) - I^n_y(\tilde{\gamma}^{n+m}_{y,x^*}) + I^n_y(\tilde{\gamma}^{n+m}_{y,x^*}) - I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})\right] \qquad (49) \\
&= \Theta(1)\cdot\left[I^n_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y}) - I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})\right] + \Theta(1)\cdot\left[I^n_y(\tilde{\gamma}^n_{y,x^*}) - I^n_y(\tilde{\gamma}^{n+m}_{y,x^*})\right] + O(\beta^n) \qquad (50) \\
&= \Theta(1)\cdot(\tilde{\gamma}^n_{x^*,y} - \tilde{\gamma}^{n+m}_{x^*,y}) + \Theta(1)\cdot(\tilde{\gamma}^n_{y,x^*} - \tilde{\gamma}^{n+m}_{y,x^*}) + O(\beta^n) \qquad (51) \\
&\geq C_7\left(\frac{\lambda_n}{M} - C_8\right)\beta^n, \qquad (52)
\end{aligned}
$$

where C_7 and C_8 are fixed positive constants, (49) holds by Proposition 3, (50) holds by Proposition 3 and the uniform convergence of H^n_x and Ψ^n_x, and (51) holds by Lemmas 1-3, Proposition 3 and the mean value theorem.

On the other hand, for any suboptimal alternative $x \neq y$ and all large enough n,

$$
-\frac{\gamma^{n+m}_{x^*,x}}{\gamma^{n+m}_{x,x^*}} + \frac{\gamma^n_{x^*,x}}{\gamma^n_{x,x^*}} = \frac{N^n_x + L^m_x}{N^n_{x^*}+1} - \frac{N^n_x}{N^n_{x^*}} \geq \frac{N^n_x}{N^n_{x^*}+1} - \frac{N^n_x}{N^n_{x^*}} = -\frac{N^n_x}{N^n_{x^*}}\frac{1}{N^n_{x^*}+1} \geq -\frac{C_9}{n} \geq -C_9\beta^n, \qquad (53)
$$

where C_9 is a fixed positive constant, the first equality holds due to (17), the second inequality holds by Theorem 2, and the last inequality holds since $\lim_{n\to\infty}(\sqrt{n}\beta^n)^{-1} = 0$. At the same time, from (48), we also have that

$$
-\frac{\gamma^{n+m}_{x^*,x}}{\gamma^{n+m}_{x,x^*}} + \frac{\gamma^n_{x^*,x}}{\gamma^n_{x,x^*}} = \Theta(1)\cdot\left(\tilde{\gamma}^n_{x,x^*} - \tilde{\gamma}^{n+m}_{x,x^*}\right) + \Theta(1)\cdot\left(\tilde{\gamma}^n_{x^*,x} - \tilde{\gamma}^{n+m}_{x^*,x}\right) + O\left(\beta^n\right). \qquad (54)
$$

Then, from (53) and (54), by Proposition 4, there must exist some fixed positive constant C_{10} such that, for all large n, we have $\min\left\{\tilde{\gamma}^n_{x^*,x} - \tilde{\gamma}^{n+m}_{x^*,x}, \tilde{\gamma}^n_{x,x^*} - \tilde{\gamma}^{n+m}_{x,x^*}\right\} \geq -C_{10}\beta^n$, which implies

$$
\frac{I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,x})}{I^{n+m}_x(\tilde{\gamma}^{n+m}_{x,x^*})} - \frac{I^n_{x^*}(\tilde{\gamma}^n_{x^*,x})}{I^n_x(\tilde{\gamma}^n_{x,x^*})} = \Theta\left(\tilde{\gamma}^n_{x,x^*} - \tilde{\gamma}^{n+m}_{x,x^*}\right) + \Theta\left(\tilde{\gamma}^n_{x^*,x} - \tilde{\gamma}^{n+m}_{x^*,x}\right) + O\left(\beta^n\right) \geq -C_{11}\beta^n, \qquad (55)
$$

where C_{11} is a positive constant, and the equality holds by (51). Finally, (52) and (55) lead to $\sum_{x\neq x^*}\frac{I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,x})}{I^{n+m}_x(\tilde{\gamma}^{n+m}_{x,x^*})} - \sum_{x\neq x^*}\frac{I^n_{x^*}(\tilde{\gamma}^n_{x^*,x})}{I^n_x(\tilde{\gamma}^n_{x,x^*})} = \frac{I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,y})}{I^{n+m}_y(\tilde{\gamma}^{n+m}_{y,x^*})} - \frac{I^n_{x^*}(\tilde{\gamma}^n_{x^*,y})}{I^n_y(\tilde{\gamma}^n_{y,x^*})} + \sum_{x\neq y,x^*}\frac{I^{n+m}_{x^*}(\tilde{\gamma}^{n+m}_{x^*,x})}{I^{n+m}_x(\tilde{\gamma}^{n+m}_{x,x^*})} - \sum_{x\neq y,x^*}\frac{I^n_{x^*}(\tilde{\gamma}^n_{x^*,x})}{I^n_x(\tilde{\gamma}^n_{x,x^*})} \geq C_7(\frac{\lambda_n}{M} - C_8)\beta^n - MC_{11}\beta^n = \left[C_7(\frac{\lambda_n}{M} - C_8) - MC_{11}\right]\beta^n$. Note that M, C_7, C_8, C_{11} are all fixed positive constants. Thus, this implies that (45) will be satisfied if $\lambda_n \geq C_{12} = M\left(\frac{MC_{11}+C_3}{C_7} + C_8\right)$, with C_{12} being a positive constant. This implies that, with $m = (C_{12}+1)n\beta^n$, we will have $\Delta^{n+m} > 1$ and thus $x^n = x^*$ at stage $n+m$. Consequently, by the definition of s_n, we must have $s_n \leq (C_{12}+1)n\beta^n$. Therefore, we have $s_n = O\left(n\beta^n\right)$.

With a symmetric argument, we can show that, between two samples of any suboptimal alternatives (not necessarily same), the number of samples that can be allocated to x^* is also $O\left(n\beta^n\right)$. □

M Lemma 7

Lemma 7. *Under Assumption 1 and Algorithm 1, for any time n at which some suboptimal z is sampled, let $m_n \triangleq \inf\left\{l > 0 : x^{n+l} = z\right\}$. Then, there exist a fixed positive constant C and a time stage n_0 such that, for all $n \geq n_0$, we have $L^{(n,n+m_n)}_{x^*} \leq Cn\beta^n$.*

Proof. Define $t_n \triangleq \sup\left\{l < m_n : x^{n+l} \neq x^*\right\}$. For convenience, we abbreviate $L^{(n,n+m_n)}_{x^*}$ to $L^{m_n}_{x^*}$ and so forth. Obviously, if $t_n = 0$, then Lemma 6 implies $L^{m_n}_{x^*} = O(n\beta^n)$. Suppose $t_n > 0$. Then, to show $L^{m_n}_{x^*} = O(n\beta^n)$, it is sufficient to show $L^{t_n}_{x^*} = O(n\beta^n)$, as $L^{(n+t_n,n+m_n)}_{x^*} = O(n\beta^n)$ due to Lemma 6. To establish the proof, we introduce the following technical lemma:

Lemma 8. *For a time stage n where some suboptimal alternative $z \neq x^*$ is sampled, let $m_n \triangleq \inf\left\{l > 0 : x^{n+l} = z\right\}$ and $t_n \triangleq \sup\left\{l < m_n : x^{n+l} \neq x^*\right\}$. Let $\lambda_n \triangleq \frac{L_{x^*}^{(n,n+t_n)}}{n\beta^n}$. Then, under Algorithm 1, there exists a fixed positive constant C_1 such that, for any large enough n at which $x^n = z$, if $\lambda_n \geq C_1$, then there exists some suboptimal alternative $y \neq z$ and a time stage $n + s_n$, such that y is sampled at stage $n + s_n$ and $\frac{N_y^{n+s_n}}{N_{x^*}^{n+s_n}} > \frac{N_y^n}{N_{x^*}^n}(1 + \lambda_n C_2 \beta^n)$ holds, where $s_n \leq t_n$ and C_2 is a fixed positive constant.*

Let $\lambda_n \triangleq L_{x^*}^{t_n}/n\beta^n$. By Lemma 8, there exists a fixed positive constant C_1 such that, for all large enough n at which z is sampled, if $\lambda_n \geq C_1$, then there exists some suboptimal $y \neq z$ and a time stage $n + s_n$, such that y is sampled at stage $n + s_n$ and $\frac{N_y^{n+s_n}}{N_{x^*}^{n+s_n}} > \frac{N_y^n}{N_{x^*}^n}(1 + \lambda_n C_2 \beta^n)$, where $s_n \leq t_n$ and C_2 is a fixed positive constant. Then, this leads to

$$\frac{L_y^{s_n}}{L_{x^*}^{s_n}} > \frac{N_y^n}{N_{x^*}^n}(1 + \lambda_n C_2 \beta^n) + \frac{N_y^n}{L_{x^*}^{s_n}} \lambda_n C_2 \beta^n. \tag{56}$$

The proof will be established by showing contradiction. More specifically, we will show that, if λ_n is larger than a certain positive constant, then we will have $\Gamma_{x^*,z}^{n+s_n} < \Gamma_{x^*,y}^{n+s_n}$ and y will not be sampled at stage $n + s_n$, which contradicts the definition of s_n. For notation convenience, let $\tilde{s}_n = n + s_n$. Since $x^n = z$, by Step 3 of Algorithm 1, we must have $\Gamma_{x^*,z}^n \leq \Gamma_{x^*,y}^n$, i.e.,

$$N_z^n I_z^n\left(\gamma_{z,x^*}^n\right) + N_{x^*}^n I_{x^*}^n\left(\gamma_{x^*,z}^n\right) \leq N_y^n I_y^n\left(\gamma_{y,x^*}^n\right) + N_{x^*}^n I_{x^*}^n\left(\gamma_{x^*,y}^n\right). \tag{57}$$

Note that $L_z^{s_n} = 1$. Then, there exist fixed positive constants C_3, C_4, C_5, C_6 and C_7 such that

$$\begin{aligned}
\Gamma_{x^*,z}^{\tilde{s}_n} &= N_z^{\tilde{s}_n} I_z^{\tilde{s}_n}(\gamma_{z,x^*}^{\tilde{s}_n}) + N_{x^*}^{\tilde{s}_n} I_{x^*}^{\tilde{s}_n}(\gamma_{x^*,z}^{\tilde{s}_n}) = N_z^{\tilde{s}_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^{\tilde{s}_n}) + N_{x^*}^{\tilde{s}_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^{\tilde{s}_n}) + O(n\beta^n(1 + \lambda_n\beta^n)) && (58)\\
&\leq N_z^{\tilde{s}_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + N_{x^*}^{\tilde{s}_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_3 n\beta^n(1 + \lambda_n\beta^n) && (59)\\
&= N_z^n \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + N_{x^*}^n \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_3 n\beta^n(1 + \lambda_n\beta^n)\\
&= N_z^n I_z^n(\gamma_{z,x^*}^n) + N_{x^*}^n I_{x^*}^n(\gamma_{x^*,z}^n) + O(n\beta^n) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_3 n\beta^n(1 + \lambda_n\beta^n) && (60)\\
&\leq N_y^n I_y^n(\gamma_{y,x^*}^n) + N_{x^*}^n I_{x^*}^n(\gamma_{x^*,y}^n) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_4 n\beta^n(1 + \lambda_n\beta^n) && (61)\\
&= N_y^n \tilde{I}_y(\tilde{\gamma}_{y,x^*}^n) + N_{x^*}^n \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,y}^n) + O(n\beta^n) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_4 n\beta^n(1 + \lambda_n\beta^n) && (62)\\
&\leq N_y^n \tilde{I}_y(\tilde{\gamma}_{y,x^*}^{\tilde{s}_n}) + N_{x^*}^n \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,y}^{\tilde{s}_n}) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_5 n\beta^n(1 + \lambda_n\beta^n) && (63)\\
&= N_y^n I_y^{\tilde{s}_n}(\gamma_{y,x^*}^{\tilde{s}_n}) + N_{x^*}^n I_{x^*}^{\tilde{s}_n}(\gamma_{x^*,y}^{\tilde{s}_n}) + O(n\beta^n) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_5 n\beta^n(1 + \lambda_n\beta^n) && (64)\\
&\leq \Gamma_{x^*,y}^{\tilde{s}_n} - L_y^{s_n} I_y^{\tilde{s}_n}(\gamma_{y,x^*}^{\tilde{s}_n}) - L_{x^*}^{s_n} I_{x^*}^{\tilde{s}_n}(\gamma_{x^*,y}^{\tilde{s}_n}) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_6 n\beta^n(1 + \lambda_n\beta^n)\\
&= \Gamma_{x^*,y}^{\tilde{s}_n} - L_y^{s_n} I_y^{\tilde{s}_n}(\gamma_{y,x^*}^{\tilde{s}_n}) - L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,y}^{\tilde{s}_n}) + O(\lambda_n n(\beta^n)^2) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_6 n\beta^n(1 + \lambda_n\beta^n) && (65)\\
&= \Gamma_{x^*,y}^{\tilde{s}_n} - L_y^{s_n} I_y^{\tilde{s}_n}(\gamma_{y,x^*}^{\tilde{s}_n}) - L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,y}^{\tilde{s}_n}) + L_z^{s_n} \tilde{I}_z(\tilde{\gamma}_{z,x^*}^n) + L_{x^*}^{s_n} \tilde{I}_{x^*}(\tilde{\gamma}_{x^*,z}^n) + C_7 n\beta^n(1 + \lambda_n\beta^n),
\end{aligned}$$

where (58), (60), (62), (64) and (65) hold due to Lemmas 1-3, Proposition 3, the mean value theorem and the uniform convergence of H_x^n and Ψ_x^n, (59) and (63) hold due to (2), (10)-(11) and (28)-(29), and (61) holds due to (57). Therefore, $\Gamma_{x^*,z}^{\tilde{s}_n} < \Gamma_{x^*,y}^{\tilde{s}_n}$ is equivalent to

$$L_y^{s_n} I_y^{\tilde{s}_n}\left(\gamma_{y,x^*}^{\tilde{s}_n}\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{s}_n}\right) > L_z^{s_n} \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) + C_7 n\beta^n(1 + \lambda_n\beta^n). \tag{66}$$

We will proceed the proof in two situations, depending on whether $\tilde{\gamma}_{x^*,y}^{\tilde{s}_n} \leq \tilde{\gamma}_{x^*,z}^n$ holds. First, suppose $\tilde{\gamma}_{x^*,y}^{\tilde{s}_n} \leq \tilde{\gamma}_{x^*,z}^n$. Then, there exists a fixed positive constant C_8 such that, for all large enough n, if $\max\left\{\frac{3C_7}{C_8} + 1, C_1\right\} \leq \lambda_n \leq 2\max\left\{\frac{3C_7}{C_8} + 1, C_1\right\} + 1$, we have $L_y^{s_n} I_y^{\tilde{s}_n}\left(\gamma_{y,x^*}^{\tilde{s}_n}\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{s}_n}\right) \geq L_y^{s_n} I_y^{\tilde{s}_n}\left(\gamma_{y,x^*}^{\tilde{s}_n}\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) \geq N_y^n \lambda_n C_2 \beta^n I_y^{\tilde{s}_n}\left(\gamma_{y,x^*}^{\tilde{s}_n}\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) \geq \lambda_n C_8 n\beta^n + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) > 3C_7 n\beta^n + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) \geq C_7 n\beta^n(1 + \lambda_n\beta^n) + \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{s_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right)$, where the first and last inequality hold by Lemma 2 and Proposition 3, the second inequality holds due to (56), and the third inequality holds by Theorem 2 and Proposition 1. Consequently,

(66) holds, which implies $\Gamma^{\tilde{s}_n}_{x^*,z} < \Gamma^{\tilde{s}_n}_{x^*,y}$ and y cannot be sampled at stage $\tilde{s}_n$. Therefore, we have the desired contradiction.

Second, suppose $\tilde{\gamma}^{\tilde{s}_n}_{x^*,y} > \tilde{\gamma}^{n}_{x^*,z}$. Note that (66) is equivalent to

$$\frac{C_7 n\beta^n(1+\lambda_n\beta^n)+\tilde{I}_z\left(\tilde{\gamma}^n_{z,x^*}\right)}{L^{s_n}_{x^*}I^{\tilde{s}_n}_y\left(\gamma^{\tilde{s}_n}_{y,x^*}\right)}+\frac{\tilde{I}_{x^*}\left(\tilde{\gamma}^n_{x^*,z}\right)-\tilde{I}_{x^*}\left(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y}\right)}{I^{\tilde{s}_n}_y\left(\gamma^{\tilde{s}_n}_{y,x^*}\right)}<\frac{L^{s_n}_y}{L^{s_n}_{x^*}}. \tag{67}$$

For all large enough n, by Lemma 2, Proposition 1 and Proposition 3, we can see that

$$\frac{C_7 n\beta^n(1+\lambda_n\beta^n)+\tilde{I}_z\left(\tilde{\gamma}^n_{z,x^*}\right)}{L^{s_n}_{x^*}I^{\tilde{s}_n}_y\left(\gamma^{\tilde{s}_n}_{y,x^*}\right)}\le\frac{C_9 n\beta^n(1+\lambda_n\beta^n)}{L^{s_n}_{x^*}}, \tag{68}$$

where C_9 is a fixed positive constant. In the meantime, by Theorem 2, (56) implies

$$\frac{L^{s_n}_y}{L^{s_n}_{x^*}} > \frac{N^n_y}{N^n_{x^*}}(1+\lambda_n C_2\beta^n)+\frac{N^n_y}{L^{s_n}_{x^*}}\lambda_n C_2\beta^n \ge \frac{N^n_y}{N^n_{x^*}}+\lambda_n C_{10}\beta^n+\frac{\lambda_n C_{11}n\beta^n}{L^{s_n}_{x^*}}, \tag{69}$$

where C_{10} and C_{11} are fixed positive constants. Furthermore, similarly as in (60)-(64), we also have $N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,z}) < N^n_z\tilde{I}_z(\tilde{\gamma}^n_{z,x^*})+N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,z}) = N^n_z I^n_z(\gamma^n_{z,x^*})+N^n_{x^*}I^n_{x^*}(\gamma^n_{x^*,z})+O(n\beta^n) \le N^n_y I^n_y(\gamma^n_{y,x^*})+N^n_{x^*}I^n_{x^*}(\gamma^n_{x^*,y})+C_{12}n\beta^n = N^n_y\tilde{I}_y(\tilde{\gamma}^n_{y,x^*})+N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,y})+O(n\beta^n)+C_{12}n\beta^n \le N^n_y\tilde{I}_y(\tilde{\gamma}^{\tilde{s}_n}_{y,x^*})+N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})+C_{13}n\beta^n = N^n_y I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})+O(n\beta^n)+N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})+C_{13}n\beta^n \le N^n_y I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})+N^n_{x^*}\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})+C_{14}n\beta^n$, where C_{12}, C_{13} and C_{14} are fixed positive constants. Consequently, there exists a fixed positive constant C_{15} such that

$$\frac{N^n_y}{N^n_{x^*}}\ge\frac{N^n_{x^*}\tilde{I}_{x^*}\left[\tilde{I}_{x^*}\left(\tilde{\gamma}^n_{x^*,z}\right)-\tilde{I}_{x^*}\left(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y}\right)\right]-C_{14}n\beta^n}{N^n_{x^*}I^{\tilde{s}_n}_y\left(\gamma^{\tilde{s}_n}_{y,x^*}\right)}\ge\frac{\tilde{I}_{x^*}\left(\tilde{\gamma}^n_{x^*,z}\right)-\tilde{I}_{x^*}\left(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y}\right)}{I^{\tilde{s}_n}_y\left(\gamma^{\tilde{s}_n}_{y,x^*}\right)}-C_{15}\beta^n, \tag{70}$$

where (70) holds by Theorem 2 and Proposition 1. Then, for all large enough n, if $\max\left\{\frac{2C_9}{C_{11}}+1, \frac{C_{15}}{C_{10}}+1, C_1\right\}\le\lambda_n\le 2\max\left\{\frac{2C_9}{C_{11}}+1, \frac{C_{15}}{C_{10}}+1, C_1\right\}+1$, (68)-(70) imply that $\frac{L^{s_n}_y}{L^{s_n}_{x^*}} > \frac{N^n_y}{N^n_{x^*}}+\lambda_n C_{10}\beta^n+\frac{\lambda_n C_{11}n\beta^n}{L^{s_n}_{x^*}} \ge \frac{\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,z})-\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})}{I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})}+(\lambda_n C_{10}-C_{15})\beta^n+\frac{\lambda_n C_{11}n\beta^n}{L^{s_n}_{x^*}} \ge \frac{\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,z})-\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})}{I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})}+\frac{C_9 n\beta^n(1+\lambda_n\beta^n)}{L^{s_n}_{x^*}} \ge \frac{\tilde{I}_{x^*}(\tilde{\gamma}^n_{x^*,z})-\tilde{I}_{x^*}(\tilde{\gamma}^{\tilde{s}_n}_{x^*,y})}{I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})}+\frac{C_7 n\beta^n(1+\lambda_n\beta^n)+\tilde{I}_z(\tilde{\gamma}^n_{z,x^*})}{L^{s_n}_{x^*}I^{\tilde{s}_n}_y(\gamma^{\tilde{s}_n}_{y,x^*})}$. Thus, (67) holds, which implies $\Gamma^{\tilde{s}_n}_{x^*,z} < \Gamma^{\tilde{s}_n}_{x^*,y}$ and y cannot be sampled at stage $\tilde{s}_n$. Therefore, we have the desired contradiction. Consequently, we must have $\lambda_n = \Theta(1)$ and thus $L^{m_n}_{x^*} = O(n\beta^n)$. □

N Lemma 9

Lemma 9. *Under Assumption 1 and Algorithm 1, for any time n at which some suboptimal z is sampled, let $m_n \triangleq \inf\left\{l > 0 : x^{n+l} = z\right\}$. Then, there exist a fixed positive constant C and a time stage n_0 such that, for all $n \ge n_0$ and any $y \ne z$, we have $L^{(n,n+m_n)}_y \le Cn\beta^n$.*

Proof. First, if $y = x^*$, then the desired result holds by Lemma 7. For any $y \ne x^*$, let $t_n \triangleq \sup\left\{l < m_n : x^{n+l} = y\right\}$. Then, it is sufficient to show $L^{t_n}_y = O\left(n\beta^n\right)$, as $L^{m_n}_y = L^{t_n}_y + 1$. Since $x^{n+t_n} = y$, by Step 3 of Algorithm 1, we must have $\Gamma^{n+t_n}_{x^*,y} \le \Gamma^{n+t_n}_{x^*,z}$, i.e., $N^{n+t_n}_y I^{n+t_n}_y\left(\gamma^{n+t_n}_{y,x^*}\right)+N^{n+t_n}_{x^*}I^{n+t_n}_{x^*}\left(\gamma^{n+t_n}_{x^*,y}\right) \le N^{n+t_n}_z I^{n+t_n}_z\left(\gamma^{n+t_n}_{z,x^*}\right)+N^{n+t_n}_{x^*}I^{n+t_n}_{x^*}\left(\gamma^{n+t_n}_{x^*,z}\right)$. Note that $L^{t_n}_z = 1$ and $L^{t_n}_{x^*} = O\left(n\beta^n\right)$. For notation convenience, denote $\tilde{t}_n \triangleq n + t_n$. Then, for all large enough n,

$$\begin{aligned}
L^{t_n}_y I^{\tilde{t}_n}_y\left(\gamma^{\tilde{t}_n}_{y,x^*}\right) &= N^{\tilde{t}_n}_y I^{\tilde{t}_n}_y\left(\gamma^{\tilde{t}_n}_{y,x^*}\right)+N^{\tilde{t}_n}_{x^*}I^{\tilde{t}_n}_{x^*}\left(\gamma^{\tilde{t}_n}_{x^*,y}\right)-N^n_y I^{\tilde{t}_n}_y\left(\gamma^{\tilde{t}_n}_{y,x^*}\right)-N^{\tilde{t}_n}_{x^*}I^{\tilde{t}_n}_{x^*}\left(\gamma^{\tilde{t}_n}_{x^*,y}\right)\\
&\le N^{\tilde{t}_n}_z I^{\tilde{t}_n}_z\left(\gamma^{\tilde{t}_n}_{z,x^*}\right)+N^{\tilde{t}_n}_{x^*}I^{\tilde{t}_n}_{x^*}\left(\gamma^{\tilde{t}_n}_{x^*,z}\right)-N^n_y I^{\tilde{t}_n}_y\left(\gamma^{\tilde{t}_n}_{y,x^*}\right)-N^{\tilde{t}_n}_{x^*}I^{\tilde{t}_n}_{x^*}\left(\gamma^{\tilde{t}_n}_{x^*,y}\right)\\
&= N^{\tilde{t}_n}_z\tilde{I}_z\left(\tilde{\gamma}^{\tilde{t}_n}_{z,x^*}\right)+N^{\tilde{t}_n}_{x^*}\tilde{I}_{x^*}\left(\tilde{\gamma}^{\tilde{t}_n}_{x^*,z}\right)-N^n_y\tilde{I}_y\left(\tilde{\gamma}^{\tilde{t}_n}_{y,x^*}\right)-N^{\tilde{t}_n}_{x^*}\tilde{I}_{x^*}\left(\tilde{\gamma}^{\tilde{t}_n}_{x^*,y}\right)+O\left(n\beta^n\right)
\end{aligned} \tag{71}$$

$$\leq N_z^{\tilde{t}_n} \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + N_{x^*}^{\tilde{t}_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) - N_y^n \tilde{I}_y\left(\tilde{\gamma}_{y,x^*}^{\tilde{t}_n}\right) - N_{x^*}^{\tilde{t}_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + C_1 n\beta^n \tag{72}$$

$$= \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) - L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + N_z^n \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + N_{x^*}^n \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right)$$
$$- N_y^n \tilde{I}_y\left(\tilde{\gamma}_{y,x^*}^{\tilde{t}_n}\right) - N_{x^*}^n \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + C_1 n\beta^n$$

$$\leq \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) - L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + N_z^n \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + N_{x^*}^n \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right)$$
$$- N_y^n \tilde{I}_y\left(\tilde{\gamma}_{y,x^*}^n\right) - N_{x^*}^n \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^n\right) + C_1 n\beta^n \tag{73}$$

$$= \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) - L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + N_z^n I_z^n\left(\gamma_{z,x^*}^n\right) + N_{x^*}^n I_{x^*}^n\left(\gamma_{x^*,z}^n\right)$$
$$- N_y^n I_y^n\left(\gamma_{y,x^*}^n\right) - N_{x^*}^n I_{x^*}^n\left(\gamma_{x^*,y}^n\right) + O\left(n\beta^n\right) + C_1 n\beta^n \tag{74}$$

$$\leq \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) - L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,y}^{\tilde{t}_n}\right) + C_2 n\beta^n \tag{75}$$

$$\leq \tilde{I}_z\left(\tilde{\gamma}_{z,x^*}^n\right) + L_{x^*}^{t_n} \tilde{I}_{x^*}\left(\tilde{\gamma}_{x^*,z}^n\right) + C_2 n\beta^n = O\left(n\beta^n\right) + C_2 n\beta^n \leq C_3 n\beta^n, \tag{76}$$

where C_1, C_2 and C_3 are fixed positive constants, (71) and (74) hold due to Lemmas 1-3, Proposition 3, the mean value theorem and the uniform convergence of H_x^n and Ψ_x^n, (72) and (73) hold due to (2), (10)-(11) and (28)-(29), (75) holds since $x^n = z$, and (76) holds due to Lemma 2 and Proposition 3. Consequently, by Proposition 1, there exists a fixed positive constant C_4 such that, for all large enough n, $L_y^{t_n} \leq C_3 n\beta^n / I_y^{\tilde{t}_n}\left(\gamma_{y,x^*}^{\tilde{t}_n}\right) \leq C_4 n\beta^n$. □

O Proof of Theorem 3

For all large enough n, if $x^n = x^*$, by Step 3 of Algorithm 1, we have $\Delta^n > 1$. Let $m_n = \inf\{l > 0 : \Delta^{n+l} > 1\}$. Note that $\Delta^{n+m_n} > 1$. By Lemma 6, we know that $m_n = O\left(n\beta^n\right)$. Then, for all $x \neq x^*$ and any $0 \leq s \leq m_n$, Proposition 2 implies that $\left|\gamma_{x^*,x}^n - \gamma_{x^*,x}^{n+s}\right| = O\left(\beta^n\right)$, $\left|\gamma_{x,x^*}^n - \gamma_{x,x^*}^{n+s}\right| = O\left(\beta^n\right)$. Consequently, as in the proof of Lemma 6, for all $x \neq x^*$ and any $0 \leq s \leq m_n$, this implies $\Delta^{n+s} - \Delta^n = \sum_{x \neq x^*}\left[\frac{I_{x^*}^{n+s}\left(\gamma_{x^*,x}^{n+s}\right)}{I_x^{n+s}\left(\gamma_{x,x^*}^{n+s}\right)} - \frac{I_{x^*}^n\left(\gamma_{x^*,x}^n\right)}{I_x^n\left(\gamma_{x,x^*}^n\right)}\right] = \sum_{x \neq x^*}\left[\Theta\left(\gamma_{x,x^*}^n - \gamma_{x,x^*}^{n+s}\right) + \Theta\left(\gamma_{x^*,x}^n - \gamma_{x^*,x}^{n+s}\right) + O\left(\beta^n\right)\right] = O\left(\beta^n\right)$. Therefore, there exists some fixed positive constant C_1 such that, for all n and any $0 \leq s \leq m_n$, $\Delta^{n+s} \geq 1 - C_1\beta^n$. Consequently, $\liminf_{n\to\infty} \Delta^n = 1$. Similarly, we can show that $\limsup_{n\to\infty} \Delta^n = 1$. Therefore, we have $\lim_{n\to\infty} \Delta^n = 1$.

P Proof of Theorem 4

For any $y, z \neq x^*$, define $r_{y,z}^n \triangleq \frac{\Gamma_{x^*,y}^n}{\Gamma_{x^*,z}^n}$. For any n at which $x^n = z$, let $m_n \triangleq \inf\left\{l > 0 : x^{n+l} = z\right\}$. Obviously, $r_{y,z}^n \geq 1$ and $r_{y,z}^{n+m_n} \geq 1$. For any $0 \leq t \leq m_n$, $r_{y,z}^{n+t} - r_{y,z}^n = \frac{\Gamma_{x^*,y}^{n+t}\Gamma_{x^*,z}^n - \Gamma_{x^*,z}^{n+t}\Gamma_{x^*,y}^n}{\Gamma_{x^*,z}^{n+t}\Gamma_{x^*,z}^n} = \frac{\left(\Gamma_{x^*,y}^{n+t} - \Gamma_{x^*,y}^n\right)\Gamma_{x^*,z}^n + \left(\Gamma_{x^*,z}^n - \Gamma_{x^*,z}^{n+t}\right)\Gamma_{x^*,y}^n}{\Gamma_{x^*,z}^{n+t}\Gamma_{x^*,z}^n} \geq -\frac{\left|\Gamma_{x^*,y}^{n+t} - \Gamma_{x^*,y}^n\right|\Gamma_{x^*,z}^n + \left|\Gamma_{x^*,z}^n - \Gamma_{x^*,z}^{n+t}\right|\Gamma_{x^*,y}^n}{\Gamma_{x^*,z}^{n+t}\Gamma_{x^*,z}^n}$.

Note that, by Theorem 2 and Proposition 1, we have $\Gamma_{x^*,y}^{n+t}, \Gamma_{x^*,y}^n, \Gamma_{x^*,z}^{n+t}, \Gamma_{x^*,z}^n = \Theta(n)$. In addition, by Lemmas 7 and 9, we also have $m = O(n\beta^n)$. Consequently, we have

$$\begin{aligned}
\left|\Gamma_{x^*,y}^{n+t} - \Gamma_{x^*,y}^n\right| &= \left|N_y^{n+t} I_y^{n+t}(\gamma_{y,x^*}^{n+t}) + N_{x^*}^{n+t} I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) - N_y^n I_y^n(\gamma_{y,x^*}^n) - N_{x^*}^n I_{x^*}^n(\gamma_{x^*,y}^n)\right| \\
&\leq \left|N_y^{n+t} I_y^{n+t}(\gamma_{y,x^*}^{n+t}) - N_y^n I_y^n(\gamma_{y,x^*}^n)\right| + \left|N_{x^*}^{n+t} I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) - N_{x^*}^n I_{x^*}^n(\gamma_{x^*,y}^n)\right| \\
&\leq N_y^n \left|I_y^{n+t}(\gamma_{y,x^*}^{n+t}) - I_y^n(\gamma_{y,x^*}^n)\right| + L_y^t I_y^{n+t}(\gamma_{y,x^*}^{n+t}) + N_{x^*}^n \left|I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) - I_{x^*}^n(\gamma_{x^*,y}^n)\right| + L_{x^*}^t I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) \\
&= \Theta(n) \cdot \left|I_y^{n+t}(\gamma_{y,x^*}^{n+t}) - I_y^n(\gamma_{y,x^*}^n)\right| + \Theta(n) \cdot \left|I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) - I_{x^*}^n(\gamma_{x^*,y}^n)\right| + O(n\beta^n).
\end{aligned}$$

Moreover, $m = O(n\beta^n)$ and Proposition 2 imply that, for all $0 \leq t \leq m_n$, $\left|\gamma_{y,x^*}^{n+t} - \gamma_{y,x^*}^n\right| = O(\beta^n)$. Then, there must exist some fixed positive constant C_1 such that $\left|I_y^{n+t}(\gamma_{y,x^*}^{n+t}) - I_y^n(\gamma_{y,x^*}^n)\right| = \left|\tilde{I}_y(\gamma_{y,x^*}^{n+t}) - \tilde{I}_y(\gamma_{y,x^*}^n) + O(\beta^n)\right| \leq \left|\tilde{I}_y(\gamma_{y,x^*}^{n+t}) - \tilde{I}_y(\gamma_{y,x^*}^n)\right| + C_1\beta^n = O(1) \cdot \left|\gamma_{y,x^*}^{n+t} - \gamma_{y,x^*}^n\right| + C_1\beta^n = O(\beta^n)$, where the first equality holds due to the uniform convergence of H_x^n and Ψ_x^n, and the second equality holds due to Lemma 2 and the mean value theorem. Similarly, we also have $\left|I_{x^*}^{n+t}(\gamma_{x^*,y}^{n+t}) - I_{x^*}^{n}(\gamma_{x^*,y}^{n})\right| = O(\beta^n)$. Consequently, we have $\left|\Gamma_{x^*,y}^{n+t} - \Gamma_{x^*,y}^{n}\right| = O(n\beta^n)$. Similarly, we also have $\left|\Gamma_{x^*,z}^{n} - \Gamma_{x^*,z}^{n+t}\right| = O(n\beta^n)$. Then, we have $\frac{\left|\Gamma_{x^*,y}^{n+t}-\Gamma_{x^*,y}^{n}\right|\Gamma_{x^*,z}^{n}+\left|\Gamma_{x^*,z}^{n}-\Gamma_{x^*,z}^{n+t}\right|\Gamma_{x^*,y}^{n}}{\Gamma_{x^*,z}^{n+t}\Gamma_{x^*,z}^{n}} = \frac{O(n\beta^n)\cdot\Theta(n)+O(n\beta^n)\cdot\Theta(n)}{\Theta(n)\cdot\Theta(n)} = O(\beta^n)$. Therefore, there must exist a fixed positive constant C_2 such that, for all large n at which $x^n = z$ and all $0 \le t \le m_n$, $r_{y,z}^{n+t} \ge r_{y,z}^n - C_2\beta^n \ge 1 - C_2\beta^n$. Then, this leads to $\liminf_{n\to\infty} r_{y,z}^n = 1$. With a symmetric argument, one can show that $\limsup_{n\to\infty} r_{y,z}^n = 1$. Therefore, we have $\lim_{n\to\infty} r_{y,z}^n = 1$.

Q Proof of Lemma 8

As one can see from the definition of m_n and t_n, stage $n+m_n$ is the first time at which z is sampled after stage n, and stage $n+t_n$ is the last time at which a suboptimal alternative is sampled before stage $n+m_n$. First, similarly as in (44), for all x and all large enough n, we have

$$\begin{aligned}\frac{I_{x^*}^n(\gamma_{x^*,x}^n)}{I_x^n(\gamma_{x,x^*}^n)} &= \frac{I_{x^*}^n(\tilde\gamma_{x^*,x}^n)}{I_x^n(\tilde\gamma_{x,x^*}^n)} + O(\beta^n) = \frac{I_{x^*}^n(\tilde\gamma_{x^*,x}^n) - \tilde I_{x^*}(\tilde\gamma_{x^*,x}^n) + \tilde I_{x^*}(\tilde\gamma_{x^*,x}^n)}{I_x^n(\tilde\gamma_{x,x^*}^n) - \tilde I_x(\tilde\gamma_{x,x^*}^n) + \tilde I_x(\tilde\gamma_{x,x^*}^n)} + O(\beta^n)\\ &= \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^n) + O(\beta^n)}{\tilde I_x(\tilde\gamma_{x,x^*}^n) + O(\beta^n)} + O(\beta^n) = \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^n)}{\tilde I_x(\tilde\gamma_{x,x^*}^n)} + O(\beta^n),\end{aligned} \tag{77}$$

where the third equality holds by the uniform convergence of H_x^n and Ψ_x^n.

Since $x^n = z$ at stage n, we must have

$$\Delta^n = \sum_{x\neq x^*} \frac{I_{x^*}^n(\gamma_{x^*,x}^n)}{I_x^n(\gamma_{x,x^*}^n)} \le 1. \tag{78}$$

At the same time, since it is some suboptimal alternative that is sampled at stage $n+t_n$, from the proof of Theorem 3, we must also have

$$\Delta^{n+t_n} = \sum_{x\neq x^*} \frac{I_{x^*}^{n+t_n}(\gamma_{x^*,x}^{n+t_n})}{I_x^{n+t_n}(\gamma_{x,x^*}^{n+t_n})} \ge 1 - C_3\beta^n, \tag{79}$$

where C_3 is a fixed positive constant. Combining (78) and (79), we have $\sum_{x\neq x^*} \frac{I_{x^*}^{n+t_n}(\gamma_{x^*,x}^{n+t_n})}{I_x^{n+t_n}(\gamma_{x,x^*}^{n+t_n})} \ge \sum_{x\neq x^*} \frac{I_{x^*}^n(\gamma_{x^*,x}^n)}{I_x^n(\gamma_{x,x^*}^n)} - C_3\beta^n$. Then, (77) implies that, for some fixed positive constant C_4, $\sum_{x\neq x^*} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^{n+t_n})}{\tilde I_x(\tilde\gamma_{x,x^*}^{n+t_n})} \ge \sum_{x\neq x^*} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^n)}{\tilde I_x(\tilde\gamma_{x,x^*}^n)} - C_4\beta^n$, which leads to

$$\sum_{x\neq x^*,z} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^{n+t_n})}{\tilde I_x(\tilde\gamma_{x,x^*}^{n+t_n})} - \sum_{x\neq x^*,z} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^n)}{\tilde I_x(\tilde\gamma_{x,x^*}^n)} \ \ge\ \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,z}^n)}{\tilde I_z(\tilde\gamma_{z,x^*}^n)} - \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,z}^{n+t_n})}{\tilde I_z(\tilde\gamma_{z,x^*}^{n+t_n})} - C_4\beta^n. \tag{80}$$

Note that $\frac{N_z^{n+t_n}}{N_{x^*}^{n+t_n}} < \frac{N_z^n}{N_{x^*}^n}$ for all large n when $\liminf_{n\to\infty}\lambda_n > 0$. Repeating the arguments established in (46)-(52), we have that, for all large n, $\tilde I_{x^*}(\tilde\gamma_{x^*,z}^n)/\tilde I_z(\tilde\gamma_{z,x^*}^n) - \tilde I_{x^*}(\tilde\gamma_{x^*,z}^{n+t_n})/\tilde I_z(\tilde\gamma_{z,x^*}^{n+t_n}) = \Theta(1)\cdot(\tilde\gamma_{x^*,z}^{n+t_n} - \tilde\gamma_{x^*,z}^n) + \Theta(1)\cdot(\tilde\gamma_{z,x^*}^{n+t_n} - \tilde\gamma_{z,x^*}^n) + O(\beta^n) \ge C_5(\lambda_n - C_6)\beta^n$, where C_5 and C_6 are two positive constants. Then, (80) implies that $\sum_{x\neq x^*,z} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^{n+t_n})}{\tilde I_x(\tilde\gamma_{x,x^*}^{n+t_n})} - \sum_{x\neq x^*,z} \frac{\tilde I_{x^*}(\tilde\gamma_{x^*,x}^n)}{\tilde I_x(\tilde\gamma_{x,x^*}^n)} \ge (C_5\lambda_n - C_7)\beta^n$, where $C_7 = C_4 + C_5C_6$ is a fixed positive constant. Therefore, there must exist some suboptimal alternative $y \neq z$ such that $\tilde I_{x^*}(\tilde\gamma_{x^*,y}^{n+t_n})/\tilde I_y(\tilde\gamma_{y,x^*}^{n+t_n}) - \tilde I_{x^*}(\tilde\gamma_{x^*,y}^n)/\tilde I_y(\tilde\gamma_{y,x^*}^n) \ge \frac{C_5\lambda_n - C_7}{M-2}\beta^n$. At the same time, from the arguments in (52) and (77), we also have that $\tilde I_{x^*}(\tilde\gamma_{x^*,y}^{n+t_n})/\tilde I_y(\tilde\gamma_{y,x^*}^{n+t_n}) - \tilde I_{x^*}(\tilde\gamma_{x^*,y}^n)/\tilde I_y(\tilde\gamma_{y,x^*}^n) = \Theta(1)\cdot(\tilde\gamma_{x^*,y}^n - \tilde\gamma_{x^*,y}^{n+t_n}) + \Theta(1)\cdot(\tilde\gamma_{y,x^*}^n - \tilde\gamma_{y,x^*}^{n+t_n}) + O(\beta^n)$. Note that, by Proposition 4, we have $(\tilde\gamma_{x^*,y}^n - \tilde\gamma_{x^*,y}^{n+t_n})(\tilde\gamma_{y,x^*}^n - \tilde\gamma_{y,x^*}^{n+t_n}) \ge 0$. Thus, for $\lambda_n \ge \frac{C_7}{C_5} + 1$, we must have $\max\left\{\tilde\gamma_{x^*,y}^n - \tilde\gamma_{x^*,y}^{n+t_n}, \tilde\gamma_{y,x^*}^n - \tilde\gamma_{y,x^*}^{n+t_n}\right\} \ge \frac{C_5\lambda_n - C_8}{C_9(M-2)}\beta^n$ and $\min\left\{\tilde\gamma_{x^*,y}^n - \tilde\gamma_{x^*,y}^{n+t_n}, \tilde\gamma_{y,x^*}^n - \tilde\gamma_{y,x^*}^{n+t_n}\right\} > 0$, where C_8 and C_9 are two positive constants. Then, repeating the arguments in (48), we obtain that $N_y^{n+t_n}/N_{x^*}^{n+t_n} - N_y^n/N_{x^*}^n =$

$-\gamma_{x^*,y}^{n+t_n}/\gamma_{y,x^*}^{n+t_n} + \gamma_{x^*,y}^{n}/\gamma_{y,x^*}^{n} = \Theta(1)\cdot(\tilde{\gamma}_{y,x^*}^{n} - \tilde{\gamma}_{y,x^*}^{n+t_n}) + \Theta(1)\cdot(\tilde{\gamma}_{x^*,y}^{n} - \tilde{\gamma}_{x^*,y}^{n+t_n}) + O(\beta^n) \geq (C_{10}\lambda_n - C_{11})\beta^n$, where C_{10} and C_{11} are fixed positive constants.

Finally, let $s_n \triangleq \sup\left\{l \leq t_n : x^{n+l} = y\right\}$, i.e., stage $n+s_n$ is the last time at which y is sampled till $n+t_n$. Then, there must exist a fixed positive constant C_{12} such that, for all large enough n, if $\lambda_n \geq \max\left\{\frac{C_7}{C_5}+1, \frac{C_{11}}{C_{10}}+1\right\}$, then $\frac{N_y^{n+s_n}}{N_{x^*}^{n+s_n}} - \frac{N_y^n}{N_{x^*}^n} \geq \frac{N_y^{n+t_n}-1}{N_{x^*}^{n+t_n}} - \frac{N_y^n}{N_{x^*}^n} \geq (C_{10}\lambda_n - C_{11})\beta^n - \frac{C_{12}}{n} > \frac{C_{10}\lambda_n - C_{11}}{2}\beta^n$, where the second inequality holds by Theorem 2, and the last inequality holds by $\lim_{n\to\infty}(\sqrt{n}\beta^n)^{-1} = 0$. Consequently, by Theorem 2, for all large enough n, there exists a positive constant C_{13} such that $\frac{N_y^{n+s_n}}{N_{x^*}^{n+s_n}} > \frac{N_y^n}{N_{x^*}^n}(1 + \frac{C_{10}\lambda_n - C_{11}}{C_{13}}\beta^n)$. Obviously, we can let $C_1 = \max\left\{\frac{C_7}{C_5}+1, \frac{2C_{11}}{C_{10}}+1\right\}$ and $C_2 = \frac{C_{10}}{2C_{13}}$. Then, for all large enough n, if $\lambda_n \geq C_1$, we have $\frac{N_y^{n+s_n}}{N_{x^*}^{n+s_n}} > \frac{N_y^n}{N_{x^*}^n}(1 + \lambda_n C_2 \beta^n)$.

R Empirical Behavior of $I_{x^*,n}^n\left(\gamma_{x^*,n,x}^n\right)$, $I_x^n\left(\gamma_{x,x^*,n}^n\right)$ and $\Gamma_{x^*,n,x}^n$

We construct a simple setting to illustrate the empirical behavior of the proposed point estimators. Let $M = 2$, i.e., there are two alternatives. Let F_1 and F_2 be $\mathcal{N}(1,1)$ and $\mathcal{N}(0.6,1)$, respectively. Then, $x^* = 1$, the optimal allocation is $\alpha_1 = \alpha_2 = 0.5$, and the true optimal large deviation rates are $I_1(u_{1,2}) = I_2(u_{1,2}) = \Gamma_{1,2}(\alpha_1,\alpha_2) = 0.02$. Figure 4 illustrates the empirical performance (in a single replication) of $I_1^n\left(\gamma_{1,2}^n\right)$, $I_2^n\left(\gamma_{2,1}^n\right)$ and $\Gamma_{1,2}^n$, where the sampling budget is allocated using EA. As the total budget n increases, the empirical results demonstrate convergence of these point estimators toward their corresponding true rates.

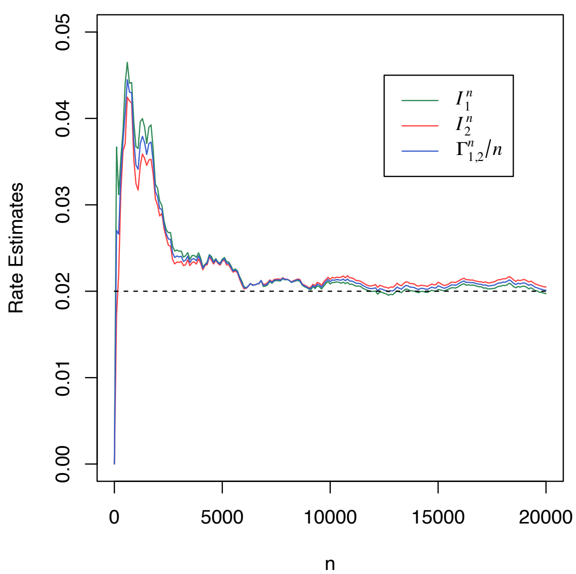


Figure 4: Point estimators of optimal large deviation rates.

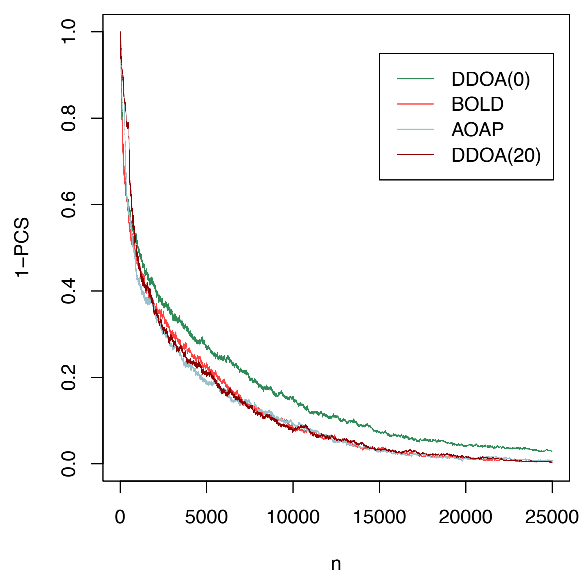


Figure 5: DDOA, BOLD and AOAP under normal sampling distributions.

S Comparison Between DDOA and AOAP Under Normal Sampling Distributions

Since the closed-form expression of the AOAP selection criterion is derived under normal sampling distributions in Peng et al. (2018), we also evaluate the performance of AOAP under the same setting as in Figure 1a. Figure 5 shows that its performance does not differ significantly from that of BOLD and DDOA(20).