Online Appendix of "Entropic-based robust vehicle rental revenue management with substitution and repositioning"

A. Proof of results

For simplification, before providing any further proof, we make the following clarification.

1. To facilitate the forthcoming derivation, we first define the following notations: $\hat{h}_{ijd}^{t-1,\tau-1} = \prod_{s=0}^{t-2\wedge\tau-1}(1-h_{ijd}^{t-1-s,\tau-1-s})$ and $\check{h}_{ijd}^{t-1,\tau-1} = h_{ijd}^{t-1,\tau-1}\prod_{s=1}^{t-2\wedge\tau-1}(1-h_{ijd}^{t-1-s,\tau-1-s})$. $\hat{q}_l^{t-1,\tau-1,s} = \prod_{\bar{\tau}=0}^{t-2\wedge\tau-1}\left(1-q_l^{\tau+s-1-\bar{\tau}}-a_l^{\tau+s-1-\bar{\tau}}\right)$ and $\check{q}_l^{t,\tau,s} = q_l^{\tau+s}\prod_{\bar{\tau}=1}^{t-1\wedge\tau}\left(1-q_l^{\tau+s-\bar{\tau}}-a_l^{\tau+s-\bar{\tau}}\right)$.
2. Since $\bar{S}$ is the maximum lead time of customers, the state variable $\tilde{\zeta}_{ijd}^{t,\tau}(l,s)=0$ when $\tau+s>\bar{S}$ so that Equation (4c) can be rewritten as $\tilde{y}_{ijd}^{t,0}(\tilde{\boldsymbol{\zeta}},l,s)=\sum_{\tau\in[0,\bar{S}]}\text{Bin}\left(\tilde{\zeta}_{ijd}^{t,\tau}(l,s),q_l^{\tau+s}\right)$.
3. In (2b), when substitution occurs, the probability of pickup and cancellation is not affected. Since $r_{ijl}^{t,\tau}=\bar{r}_{ijll}^{t,s,\tau}$, $\forall l\in[D], i,j\in[I], t\in[T], s\in[0,\bar{S}], \tau\in[0,S]$, the term $d\neq l$ can be removed from all the relevant equations (*i.e.*, (1b), (4a), (7c)) without loss of generality.

Proof of Proposition 1.

$$\begin{aligned}\mathbb{G}_\theta\left(\text{Bin}\left(\left\lfloor b\tilde{\xi}\right\rfloor,q\right)\right) &= \theta\log\mathbb{E}_{\mathbb{P}}\left[\exp\left(\text{Bin}\left(\left\lfloor b\tilde{\xi}\right\rfloor,q\right)/\theta\right)\right] = \theta\log\mathbb{E}_{\xi}\left[\mathbb{E}_{\mathbb{P}}\left[\exp\left(\text{Bin}\left(\left\lfloor b\tilde{\xi}\right\rfloor,q\right)/\theta\right)\middle|\tilde{\xi}\right]\right]\\ &= \theta\log\mathbb{E}_{\xi}\left[\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)/\theta\right] = \mathbb{G}_\theta\left(\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)\right)\end{aligned}$$

where $\delta(q)=\theta\log(1-q+q\exp(1/\theta))$. The first equation is derived by the conditional expectation, while the second incorporates the moment-generating function of the binomial distribution. □

Proof of Proposition 2. Based on Proposition 1, we have $\mathbb{G}_\theta\left(\text{Bin}\left(\left\lfloor b\tilde{\xi}\right\rfloor,q\right)\right)=\mathbb{G}_\theta\left(\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)\right)$, where $\delta(q)=\theta\log(1-q+q\exp(1/\theta))$ Since $q\in[0,1]$, we have $\delta(q)\in[0,1]$.

Similarly, we have $\mathbb{G}_\theta\left(\text{Bin}\left(b\tilde{\xi},q\right)\right)=\mathbb{G}_\theta\left(b\tilde{\xi}\delta(q)\right)$. Since $\mathbb{G}_\theta(\cdot)$ is a monotonically increasing function and $\delta(q)\geq 0$, we have $\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)\leq b\tilde{\xi}\delta(q)$, and thus

$$\mathbb{G}_\theta\left(\text{Bin}\left(\left\lfloor b\tilde{\xi}\right\rfloor,q\right)\right)=\mathbb{G}_\theta\left(\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)\right)\leq\mathbb{G}_\theta\left(\text{Bin}\left(b\tilde{\xi},q\right)\right)=\mathbb{G}_\theta\left(b\tilde{\xi}\delta(q)\right)$$

Because of the monotonicity of $\mathbb{G}_\theta(\cdot)$, we have

$$\mathbb{G}_\theta\left(b\tilde{\xi}\delta(q)\right)-\mathbb{G}_\theta\left(\left\lfloor b\tilde{\xi}\right\rfloor\delta(q)\right)=\mathbb{G}_\theta\left(b\tilde{\xi}\delta(q)\right)-\mathbb{G}_\theta\left(b\tilde{\xi}\delta(q)-\delta(q)\right)=\delta(q)\leq 1 \quad \square$$

Proof of Proposition 3.

$$\begin{aligned}\mathbb{G}_\theta\left(\sum_{d\in[D]}\tilde{z}_{id}^t-B_i\right) &= \sum_{j\in[I]/\{i\}}\mathbb{G}_\theta\left(-\sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\left(\tilde{y}_{ijd}^{\bar{t},0}(\tilde{\boldsymbol{\xi}})+\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\tilde{y}_{ijd}^{\bar{t},0}(\tilde{\boldsymbol{\zeta}},l,s)+\sum_{s\in[0,\bar{S}]}\bar{x}_{ijd}^{\bar{t},s}\tilde{\xi}_{ijd}^{\bar{t},s}\right)\right)\\ &+\sum_{j\in[I]/\{i\}}\mathbb{G}_\theta\left(\sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\left(\sum_{s\in[0,\bar{S}]}\bar{x}_{jid}^{\bar{t},s}\tilde{\xi}_{jid}^{\bar{t},s}+\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{\tau}\in[0,\bar{t}-1]}\text{Bin}\left(\tilde{y}_{jid}^{\bar{t}-\bar{\tau},0}(\tilde{\boldsymbol{\zeta}},l,s),\check{h}_{jil}^{\bar{t},\bar{\tau}}\right)\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{\bar{s}\in[0,\bar{t}-1]} \mathrm{Bin}\left(\tilde{y}^{\bar{t}-\bar{s},0}_{jid}(\tilde{\boldsymbol{\xi}}), \check{h}^{\bar{t},\bar{s}}_{jid}\right)\Bigg)\Bigg) + \sum_{d\in[D]} z^1_{id} + \sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\sum_{j\in[I]}\left(x^{0,\bar{t}}_{jid} - x^{0,\bar{t}}_{ijd}\right) - B_i \\
&+ \mathbb{G}_\theta\Bigg(\sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\Bigg(\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{\tau}\in[0,\bar{t}-1]} \mathrm{Bin}\left(\tilde{y}^{\bar{t}-\bar{\tau},0}_{iid}(\tilde{\boldsymbol{\zeta}},l,s), \check{h}^{\bar{t},\bar{\tau}}_{iil}\right) + \sum_{\bar{s}\in[0,\bar{t}-1]} \mathrm{Bin}\left(\tilde{y}^{\bar{t}-\bar{s},0}_{iid}(\tilde{\boldsymbol{\xi}}), \check{h}^{\bar{t},\bar{s}}_{iid}\right) \\
&\quad - \tilde{y}^{\bar{t},0}_{iid}(\tilde{\boldsymbol{\xi}}) - \sum_{l\in[D]}\sum_{s\in[0,\bar{S}]} \tilde{y}^{\bar{t},0}_{iid}(\tilde{\boldsymbol{\zeta}},l,s)\Bigg)\Bigg) + \sum_{d\in[D]}\sum_{l\in[D]}\sum_{j\in[I]}\sum_{s\in[0,\bar{S}]} \mathbb{G}_\theta\left(\sum_{\bar{t}\in[t-1]}\sum_{\bar{\tau}\in[\bar{t},S]} \mathrm{Bin}\left(y^{1,\bar{\tau}-\bar{t}+1}_{jid}(\tilde{\boldsymbol{\zeta}},l,s), \check{h}^{\bar{t},\bar{\tau}}_{jil}\right)\right) \\
&+ \sum_{d\in[D]}\sum_{j\in[I]} \mathbb{G}_\theta\left(\sum_{\bar{t}\in[t-1]}\sum_{\bar{s}\in[\bar{t},S]} \mathrm{Bin}\left(y^{1,\bar{s}-\bar{t}+1}_{jid}(\tilde{\boldsymbol{\xi}}), \check{h}^{\bar{t},\bar{s}}_{jid}\right)\right) \qquad (14)
\end{aligned}
$$

The last two terms in Equation (14) are independent on the order demand over the decision periods. Hence, we have $\mathbb{G}_\theta\left(\sum_{\bar{t}\in[t-1]}\sum_{\bar{\tau}\in[\bar{t},S]} \mathrm{Bin}\left(y^{1,\bar{\tau}-\bar{t}+1}_{jid}(\tilde{\boldsymbol{\zeta}},l,s), \check{h}^{\bar{t},\bar{\tau}}_{jil}\right)\right) = \sum_{\bar{t}\in[S]} y^{1,\bar{t}}_{jid}(\tilde{\boldsymbol{\zeta}},l,s)\check{\Upsilon}^{t,\bar{t}}_{jil}(1)$, where $\check{\Upsilon}^{t,\bar{t}}_{jid}(x) = \theta\log\left(1+\sum_{\bar{\tau}\in[(t-1)\wedge(S-\bar{t}+1)]} \check{h}^{\bar{\tau},\bar{t}+\bar{\tau}-1}_{jid}\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$, and similarly, we have $\mathbb{G}_\theta\left(\sum_{\bar{t}\in[t-1]}\sum_{\bar{s}\in[\bar{t},S]} \mathrm{Bin}\left(y^{1,\bar{s}-\bar{t}+1}_{jid}(\tilde{\boldsymbol{\xi}}), \check{h}^{\bar{t},\bar{s}}_{jid}\right)\right) = \sum_{\bar{t}\in[S]} y^{1,\bar{t}}_{jid}(\tilde{\boldsymbol{\xi}})\check{\Upsilon}^{t,\bar{t}}_{jid}(1)$. The first term in Equation (14) can be reformulated as follows:

$$
\begin{aligned}
&\mathbb{G}_\theta\Bigg(-\sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\Bigg(\tilde{y}^{\bar{t},0}_{ijd}(\tilde{\boldsymbol{\xi}}) + \sum_{l\in[D]}\sum_{s\in[0,\bar{S}]} \tilde{y}^{\bar{t},0}_{ijd}(\tilde{\boldsymbol{\zeta}},l,s) + \sum_{s\in[0,\bar{S}]} \bar{x}^{\bar{t},s}_{ijd}\tilde{\xi}^{\bar{t},s}_{ijd}\Bigg)\Bigg) \\
&= \sum_{l\in[D]}\sum_{\bar{t}\in[t-1]} \mathbb{G}_\theta\Bigg(-\sum_{s\in[0,\bar{t}-1]}\Bigg(\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\alpha^{t-\bar{t}+s+1,s+1}_{ijl}/\alpha^{t-\bar{t}+s,s}_{ijl}, q^s_l\right) \\
&\qquad + \sum_{d\in[D]}\sum_{\tau\in[0,\bar{t}-s-1]} \mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\gamma^{t-\bar{t}+s,s}_{ijld}/\alpha^{t-\bar{t}+s,s}_{ijl}, \check{q}^{t-\bar{t}+s+\tau,\tau,s}_l\right) + \bar{x}^{t-\bar{t}+s,s}_{ijl}\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\Bigg)\Bigg) \qquad (15a) \\
&+ \sum_{l\in[D]}\sum_{\bar{t}\in[t,\bar{S}+1]} \mathbb{G}_\theta\Bigg(-\sum_{s\in[\bar{t}-t+1,\bar{t}-1]}\Bigg(\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\alpha^{t-\bar{t}+s+1,s+1}_{ijl}/\alpha^{t-\bar{t}+s,s}_{ijl}, q^s_l\right) \\
&\qquad + \sum_{d\in[D]}\sum_{\tau\in[0,\bar{t}-s-1]} \mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\gamma^{t-\bar{t}+s,s}_{ijld}/\alpha^{t-\bar{t}+s,s}_{ijl}, \check{q}^{t-\bar{t}+s+\tau,\tau,s}_l\right) + \bar{x}^{t-\bar{t}+s,s}_{ijl}\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\Bigg)\Bigg) \qquad (15b) \\
&+ \sum_{l\in[D]}\sum_{\bar{t}\in[\bar{S}+2,t+\bar{S}-1]} \mathbb{G}_\theta\Bigg(-\sum_{s\in[\bar{t}-t+1,\bar{S}]}\Bigg(\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\alpha^{t-\bar{t}+s+1,s+1}_{ijl}/\alpha^{t-\bar{t}+s,s}_{ijl}, q^s_l\right) \\
&\qquad + \sum_{d\in[D]}\sum_{\tau\in[0,\bar{t}-s-1]} \mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\gamma^{t-\bar{t}+s,s}_{ijld}/\alpha^{t-\bar{t}+s,s}_{ijl}, \check{q}^{t-\bar{t}+s+\tau,\tau,s}_l\right) + \bar{x}^{t-\bar{t}+s,s}_{ijl}\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\Bigg)\Bigg) \qquad (15c) \\
&+ \sum_{d,l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]} \mathbb{G}_\theta\left(-\sum_{\tau\in[(t-1)\wedge(\bar{S}-\bar{t}+1)]} \mathrm{Bin}\left(\zeta^{1,\bar{t}}_{ijd}(l,s), \check{q}^{\tau,\bar{t}+\tau-1,s}_l\right)\right) \qquad (15d)
\end{aligned}
$$

where Equation (15d) $= \sum_{d,l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]} \zeta^{1,\bar{t}}_{ijd}(l,s)\Upsilon^{t,\bar{t},s}_l(-1)$, $\Upsilon^{t,\bar{t},s}_l(x) = \theta\log\left(1+\sum_{\tau\in[(t-1)\wedge(\bar{S}-\bar{t}+1)]} \check{q}^{\tau,\bar{t}+\tau-1,s}_l\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$. Equation (15a) can be reformulated as follows:

$$
\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]} \theta\log\mathbb{E}_{\mathbb{P}}\Bigg[\exp\Bigg(\frac{1}{\theta}\Bigg(-\sum_{s\in[0,\bar{t}-1]}\Bigg(\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{ijl}\alpha^{t-\bar{t}+s+1,s+1}_{ijl}/\alpha^{t-\bar{t}+s,s}_{ijl}, q^s_l\right)
$$

$$+\sum_{d\in[D]}\sum_{\tau\in[0,\bar{t}-s-1]}\text{Bin}\left(\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\gamma_{ijld}^{t-\bar{t}+s,s}/\alpha_{ijl}^{t-\bar{t}+s,s},\check{q}_l^{t-\bar{t}+s+\tau,\tau,s}\right)+\bar{x}_{ijl}^{t-\bar{t}+s,s}\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg]$$

$$=\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\theta\log\mathbb{E}_\xi\mathbb{E}_\mathbb{P}\Bigg[\exp\Bigg(\frac{1}{\theta}\Bigg(-\sum_{s\in[0,\bar{t}-1]}\Bigg(\text{Bin}\left(\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\alpha_{ijl}^{t-\bar{t}+s+1,s+1}/\alpha_{ijl}^{t-\bar{t}+s,s},q_l^s\right)$$

$$+\sum_{d\in[D]}\sum_{\tau\in[0,\bar{t}-s-1]}\text{Bin}\left(\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\gamma_{ijld}^{t-\bar{t}+s,s}/\alpha_{ijl}^{t-\bar{t}+s,s},\check{q}_l^{t-\bar{t}+s+\tau,\tau,s}\right)+\bar{x}_{ijl}^{t-\bar{t}+s,s}\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg|\tilde{\xi}_{ijl}^{t-1,\bar{t}-1}\Bigg]$$

$$=\begin{cases}\displaystyle\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\theta\log\mathbb{E}_\xi\Bigg[\mathbb{E}_\mathbb{P}\Bigg[\exp\Bigg(\frac{1}{\theta}\Bigg(-\sum_{s\in[0,\bar{t}-2]}\Bigg(\text{Bin}\left(\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\alpha_{ijl}^{t-\bar{t}+s+1,s+1}/\alpha_{ijl}^{t-\bar{t}+s,s},q_l^s\right)+\sum_{d\in[D]}\\ \displaystyle\qquad\sum_{\tau\in[0,\bar{t}-s-1]}\text{Bin}\left(\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\gamma_{ijld}^{t-\bar{t}+s,s}/\alpha_{ijl}^{t-\bar{t}+s,s},\check{q}_l^{t-\bar{t}+s+\tau,\tau,s}\right)+\bar{x}_{ijl}^{t-\bar{t}+s,s}\tilde{\xi}_{ijl}^{t-\bar{t}+s,s}\Bigg)\Bigg)+\tilde{\xi}_{ijl}^{t-1,\bar{t}-1}\frac{\phi_{ijl}^{t,\bar{t},\bar{t}-1}}{\theta\alpha_{ijl}^{t-1,\bar{t}-1}}\Bigg)\Bigg]\Bigg]\\ \displaystyle\phi_{ijl}^{t,\bar{t},\bar{t}-1}\geq\alpha_{ijl}^{t,\bar{t}}\rho_l^{\bar{t}-1}(-1,0)+\sum_{d\in[D]}\gamma_{ijld}^{t-1,\bar{t}-1}\check{\rho}_l^{t,\bar{t},\bar{t}-1}(-1)-x_{ijl}^{1,t-1,\bar{t}-1},\qquad\forall\ l\in[D],\ \bar{t}\in[t-1]\end{cases}$$

$$=\begin{cases}\displaystyle\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\mathbb{G}_\theta\left[\tilde{\xi}_{ijl}^{t-\bar{t},0}\frac{\phi_{ijl}^{t,\bar{t},0}}{\alpha_{ijl}^{t-\bar{t},0}}\right]\\ \displaystyle\phi_{ijl}^{t,\bar{t},\bar{t}-1}\geq\alpha_{ijl}^{t,\bar{t}}\rho_l^{\bar{t}-1}(-1,0)+\sum_{d\in[D]}\gamma_{ijld}^{t-1,\bar{t}-1}\check{\rho}_l^{t,\bar{t},\bar{t}-1}(-1)-x_{ijl}^{1,t-1,\bar{t}-1}\qquad\forall\ l\in[D],\ \bar{t}\in[t-1]\\ \displaystyle\phi_{ijl}^{t,\bar{t},s}\geq\alpha_{ijl}^{t-\bar{t}+s+1,s+1}\rho_l^s\left(-1,\frac{\phi_{ijl}^{t,\bar{t},s+1}}{\alpha_{ijl}^{t-\bar{t}+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{ijld}^{t-\bar{t}+s,s}\check{\rho}_l^{t,\bar{t},s}(-1)-x_{ijl}^{1,t-\bar{t}+s,s}\\ \qquad\forall\ l\in[D],\ t\neq 2,\ \bar{t}\in[2,t-1],\ s\in[0,\bar{t}-2]\end{cases}$$

where $\rho_l^s(x,z)=\theta\log\left(1+q_l^s\left(\exp\left(\frac{x}{\theta}\right)-1\right)+(1-q_l^s-a_l^s)\left(\exp\left(\frac{z}{\theta}\right)-1\right)\right)$ and $\check{\rho}_l^{t,\bar{t},s}(x)=\theta\log\left(1+\sum_{\tau\in[0,\bar{t}-s-1]}\check{q}_l^{t-\bar{t}+s+\tau,\tau,s}\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$. Similarly, since $\alpha_{ijl}^{1,\bar{t}-t+1}=\xi_{ijl}^{1,\bar{t}-t+1}$, Equations (15b) and (15c) have the equivalent reformulations:

$$\sum_{l\in[D]}\sum_{\bar{t}\in[t,\bar{S}+1]}\xi_{ijl}^{1,\bar{t}-t+1}\frac{\phi_{ijl}^{t,\bar{t},\bar{t}-t+1}}{\alpha_{ijl}^{1,\bar{t}-t+1}}=\sum_{l\in[D]}\sum_{\bar{t}\in[t,\bar{S}+1]}\phi_{ijl}^{t,\bar{t},\bar{t}-t+1}$$

$$\phi_{ijl}^{t,\bar{t},\bar{t}-1}\geq\alpha_{ijl}^{t,\bar{t}}\rho_l^{\bar{t}-1}(-1,0)+\sum_{d\in[D]}\gamma_{ijld}^{t-1,\bar{t}-1}\check{\rho}_l^{t,\bar{t},\bar{t}-1}(-1)-x_{ijl}^{1,t-1,\bar{t}-1}\qquad\forall\ l\in[D],\bar{t}\in[t,\bar{S}+1]$$

$$\phi_{ijl}^{t,\bar{t},s}\geq\alpha_{ijl}^{t-\bar{t}+s+1,s+1}\rho_l^s\left(-1,\frac{\phi_{ijl}^{t,\bar{t},s+1}}{\alpha_{ijl}^{t-\bar{t}+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{ijld}^{t-\bar{t}+s,s}\check{\rho}_l^{t,\bar{t},s}(-1)-x_{ijl}^{1,t-\bar{t}+s,s}$$

$$\forall\ l\in[D],\ t\neq 2,\bar{t}\in[t,\bar{S}+1],\ s\in[\bar{t}-t+1,\bar{t}-2]$$

$$\sum_{l\in[D]}\sum_{\bar{t}\in[\bar{S}+2,t+\bar{S}-1]}\xi_{ijl}^{1,\bar{t}-t+1}\frac{\phi_{ijl}^{t,\bar{t},\bar{t}-t+1}}{\alpha_{ijl}^{1,\bar{t}-t+1}}=\sum_{l\in[D]}\sum_{\bar{t}\in[\bar{S}+2,t+\bar{S}-1]}\phi_{ijl}^{t,\bar{t},\bar{t}-t+1}$$

$$\phi_{ijl}^{t,\bar{t},\bar{S}}\geq\alpha_{ijl}^{t-\bar{t}+\bar{S}+1,\bar{S}+1}\rho_l^{\bar{S}}(-1,0)+\sum_{d\in[D]}\gamma_{ijld}^{t-\bar{t}+\bar{S},\bar{S}}\check{\rho}_l^{t,\bar{t},\bar{S}}(-1)-x_{ijl}^{1,t-\bar{t}+\bar{S},\bar{S}}\qquad\forall\ l\in[D],\ \bar{t}\in[\bar{S}+2,t+\bar{S}-1]$$

$$\phi_{ijl}^{t,\bar{t},s}\geq\alpha_{ijl}^{t-\bar{t}+s+1,s+1}\rho_l^s\left(-1,\frac{\phi_{ijl}^{t,\bar{t},s+1}}{\alpha_{ijl}^{t-\bar{t}+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{ijld}^{t-\bar{t}+s,s}\check{\rho}_l^{t,\bar{t},s}(-1)-x_{ijl}^{1,t-\bar{t}+s,s}$$

$$\forall\ l\in[D],\ t\neq 2,\ \bar{t}\in[\bar{S}+2,t+\bar{S}-2],\ s\in[\bar{t}-t+1,\bar{S}-1]$$

Then for the second term in Equation (14), we can reformulate it as follows:

$$\mathbb{G}_\theta\left(\sum_{d\in[D]}\sum_{\bar{t}\in[t-1]}\left(\sum_{s\in[0,\bar{S}]}\bar{x}^{\bar{t},s}_{jid}\tilde{\xi}^{\bar{t},s}_{jid}+\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{\tau}\in[0,\bar{t}-1]}\mathrm{Bin}\left(\tilde{y}^{\bar{t}-\bar{\tau},0}_{jid}(\tilde{\boldsymbol{\zeta}},l,s),\check{h}^{\bar{t},\bar{\tau}}_{jil}\right)+\sum_{\bar{s}\in[0,\bar{t}-1]}\mathrm{Bin}\left(\tilde{y}^{\bar{t}-\bar{s},0}_{jid}(\tilde{\boldsymbol{\xi}}),\check{h}^{\bar{t},\bar{s}}_{jid}\right)\right)\right)$$

$$=\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\mathbb{G}_\theta\left(\sum_{s\in[0,\bar{t}-1]}\left(\bar{x}^{t-\bar{t}+s,s}_{jil}\tilde{\xi}^{t-\bar{t}+s,s}_{jil}+\sum_{\bar{s}\in[0,\bar{t}-s-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\alpha^{t-\bar{t}+s+1,s+1}_{jil}/\alpha^{t-\bar{t}+s,s}_{jil},q^s_l\check{h}^{t-\bar{t}+s+\bar{s},\bar{s}}_{jil}\right)\right.\right.$$
$$\left.\left.+\sum_{d\in[D]}\sum_{\bar{\tau}\in[0,\bar{t}-s-1]}\sum_{\tau\in[0,\bar{t}-s-\bar{\tau}-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\gamma^{t-\bar{t}+s,s}_{jild}/\alpha^{t-\bar{t}+s,s}_{jil},\check{q}^{t-\bar{t}+s+\bar{\tau},\bar{\tau},s}_l\check{h}^{t-\bar{t}+s+\bar{\tau}+\tau,\tau}_{jil}\right)\right)\right)\quad(16a)$$
$$+\sum_{l\in[D]}\sum_{\bar{t}\in[t,\bar{S}+1]}\mathbb{G}_\theta\left(\sum_{s\in[\bar{t}-t+1,\bar{t}-1]}\left(\bar{x}^{t-\bar{t}+s,s}_{jil}\tilde{\xi}^{t-\bar{t}+s,s}_{jil}+\sum_{\bar{s}\in[0,\bar{t}-s-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\alpha^{t-\bar{t}+s+1,s+1}_{jil}/\alpha^{t-\bar{t}+s,s}_{jil},q^s_l\check{h}^{t-\bar{t}+s+\bar{s},\bar{s}}_{jil}\right)\right.\right.$$
$$\left.\left.+\sum_{d\in[D]}\sum_{\bar{\tau}\in[0,\bar{t}-s-1]}\sum_{\tau\in[0,\bar{t}-s-\bar{\tau}-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\gamma^{t-\bar{t}+s,s}_{jild}/\alpha^{t-\bar{t}+s,s}_{jil},\check{q}^{t-\bar{t}+s+\bar{\tau},\bar{\tau},s}_l\check{h}^{t-\bar{t}+s+\bar{\tau}+\tau,\tau}_{jil}\right)\right)\right)\quad(16b)$$
$$+\sum_{l\in[D]}\sum_{\bar{t}\in[\bar{S}+2,t+\bar{S}-1]}\mathbb{G}_\theta\left(\sum_{s\in[\bar{t}-t+1,\bar{S}]}\left(\bar{x}^{t-\bar{t}+s,s}_{jil}\tilde{\xi}^{t-\bar{t}+s,s}_{jil}+\sum_{\bar{s}\in[0,\bar{t}-s-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\alpha^{t-\bar{t}+s+1,s+1}_{jil}/\alpha^{t-\bar{t}+s,s}_{jil},q^s_l\check{h}^{t-\bar{t}+s+\bar{s},\bar{s}}_{jil}\right)\right.\right.$$
$$\left.\left.+\sum_{d\in[D]}\sum_{\bar{\tau}\in[0,\bar{t}-s-1]}\sum_{\tau\in[0,\bar{t}-s-\bar{\tau}-1]}\mathrm{Bin}\left(\tilde{\xi}^{t-\bar{t}+s,s}_{jil}\gamma^{t-\bar{t}+s,s}_{jild}/\alpha^{t-\bar{t}+s,s}_{jil},\check{q}^{t-\bar{t}+s+\bar{\tau},\bar{\tau},s}_l\check{h}^{t-\bar{t}+s+\bar{\tau}+\tau,\tau}_{jil}\right)\right)\right)\quad(16c)$$
$$+\sum_{d\in[D]}\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]}\mathbb{G}_\theta\left(\sum_{\bar{\tau}\in[(\bar{S}-\bar{t}+1)\wedge(t-1)]}\sum_{\tau\in[0,t-\bar{\tau}-1]}\mathrm{Bin}\left(\tilde{\zeta}^{1,\bar{t}}_{jid}(l,s),\check{q}^{\bar{\tau},\bar{t}+\bar{\tau}-1,s}_l\check{h}^{\bar{\tau}+\tau,\tau}_{jil}\right)\right)\quad(16d)$$

Similarly, Equation (16d) $=\sum_{d,l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]}\zeta^{1,\bar{t}}_{jid}(l,s)\bar{\Upsilon}^{t,\bar{t},s}_{jil}(1)$ where $\bar{\Upsilon}^{t,\bar{t},s}_{jil}(x)=\theta\log\left(1+\sum_{\bar{\tau}\in[(\bar{S}-\bar{t}+1)\wedge(t-1)]}\sum_{\tau\in[0,t-\bar{\tau}-1]}\check{q}^{\bar{\tau},\bar{t}+\bar{\tau}-1,s}_l\check{h}^{\bar{\tau}+\tau,\tau}_{jil}\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$. Then Equation (16a), (16b), and (16c) can be reformulated as

$$\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\mathbb{G}_\theta\left(\tilde{\xi}^{t-\bar{t},0}_{jil}\frac{\psi^{t,\bar{t},0}_{jil}}{\alpha^{t-\bar{t},0}_{jil}}\right)$$
$$\psi^{t,\bar{t},\bar{t}-1}_{jil}\geq\alpha^{t,\bar{t}}_{jil}\pi^{t,\bar{t},\bar{t}-1}_{jil}(1,0)+\sum_{d\in[D]}\gamma^{t-1,\bar{t}-1}_{jild}\check{\pi}^{t,\bar{t},\bar{t}-1}_{jil}(1)+x^{1,t-1,\bar{t}-1}_{jil}\qquad\forall\ l\in[D],\ \bar{t}\in[t-1]$$
$$\psi^{t,\bar{t},s}_{jil}\geq\alpha^{t-\bar{t}+s+1,s+1}_{jil}\pi^{t,\bar{t},s}_{jil}\left(1,\frac{\psi^{t,\bar{t},s+1}_{jil}}{\alpha^{t-\bar{t}+s+1,s+1}_{jil}}\right)+\sum_{d\in[D]}\gamma^{t-\bar{t}+s,s}_{jild}\check{\pi}^{t,\bar{t},s}_{jil}(1)+x^{1,t-\bar{t}+s,s}_{jil}\forall l\in[D],t\neq 2,\bar{t}\in[2,t-1],s\in[0,\bar{t}-2]$$
$$\sum_{l\in[D]}\sum_{\bar{t}\in[t,\bar{S}+1]}\xi^{1,\bar{t}-t+1}_{jil}\frac{\psi^{t,\bar{t},\bar{t}-t+1}_{jil}}{\alpha^{1,\bar{t}-t+1}_{jil}}$$
$$\psi^{t,\bar{t},\bar{t}-1}_{jil}\geq\alpha^{t,\bar{t}}_{jil}\pi^{t,\bar{t},\bar{t}-1}_{jil}(1,0)+\sum_{d\in[D]}\gamma^{t-1,\bar{t}-1}_{jild}\check{\pi}^{t,\bar{t},\bar{t}-1}_{jil}(1)+x^{1,t-1,\bar{t}-1}_{jil}\qquad\forall\ l\in[D],\ \bar{t}\in[t,\bar{S}+1]$$
$$\psi^{t,\bar{t},s}_{jil}\geq\alpha^{t-\bar{t}+s+1,s+1}_{jil}\pi^{t,\bar{t},s}_{jil}\left(1,\frac{\psi^{t,\bar{t},s+1}_{jil}}{\alpha^{t-\bar{t}+s+1,s+1}_{jil}}\right)+\sum_{d\in[D]}\gamma^{t-\bar{t}+s,s}_{jild}\check{\pi}^{t,\bar{t},s}_{jil}(1)+x^{1,t-\bar{t}+s,s}_{jil}$$
$$\forall\ l\in[D],\ t\neq 2,\ \bar{t}\in[t,\bar{S}+1],\ s\in[\bar{t}-t+1,\bar{t}-2]$$

$$\sum_{l\in[D]}\sum_{\bar t\in[\bar S+2,t+\bar S-1]}\xi_{jil}^{1,\bar t-t+1}\frac{\psi_{jil}^{t,\bar t,\bar t-t+1}}{\alpha_{jil}^{1,\bar t-t+1}}$$

$$\psi_{jil}^{t,\bar t,\bar S}\ge\alpha_{jil}^{t-\bar t+\bar S+1,\bar S+1}\pi_{jil}^{t,\bar t,\bar S}(1,0)+\sum_{d\in[D]}\gamma_{jild}^{t-\bar t+\bar S,\bar S}\breve\pi_{jil}^{t,\bar t,\bar S}(1)+x_{jil}^{1,t-\bar t+\bar S,\bar S}\qquad\forall\ l\in[D],\ \bar t\in[\bar S+2,t+\bar S-1]$$

$$\psi_{jil}^{t,\bar t,s}\ge\alpha_{jil}^{t-\bar t+s+1,s+1}\pi_{jil}^{t,\bar t,s}\left(1,\frac{\psi_{jil}^{t,\bar t,s+1}}{\alpha_{jil}^{t-\bar t+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{jild}^{t-\bar t+s,s}\breve\pi_{jil}^{t,\bar t,s}(1)+x_{jil}^{1,t-\bar t+s,s}$$
$$\forall\ l\in[D],\ t\neq2,\ \bar t\in[\bar S+2,t+\bar S-2],\ s\in[\bar t-t+1,\bar S-1]$$

where $\pi_{jil}^{t,\bar t,s}(x,z)=\theta\log\Big(1+\sum_{\bar s\in[0,\bar t-s-1]}q_l^s\breve h_{jil}^{t-\bar t+s+\bar s,\bar s}\left(\exp\left(\frac{x}{\theta}\right)-1\right)+(1-q_l^s-a_l^s)\left(\exp\left(\frac{z}{\theta}\right)-1\right)\Big)$, and $\breve\pi_{jil}^{t,\bar t,s}(x)=\theta\log\Big(1+\sum_{\bar\tau\in[0,\bar t-s-1]}\sum_{\tau\in[0,\bar t-s-\bar\tau-1]}\breve q_l^{t-\bar t+s+\bar\tau,\bar\tau,s}\breve h_{jil}^{t-\bar t+s+\bar\tau+\tau,\tau}\left(\exp\left(\frac{x}{\theta}\right)-1\right)\Big)$.

Finally, we consider the reformulation of the fourth line in Equation (14), which also includes four terms. The first, second, and third terms are reformulated as below:

$$\sum_{l\in[D]}\sum_{\bar t\in[t-1]}\mathbb{G}_\theta\left(\tilde\xi_{iil}^{t-\bar t,0}\frac{\nu_{il}^{t,\bar t,0}}{\alpha_{iil}^{t-\bar t,0}}\right)$$

$$\nu_{il}^{t,\bar t,\bar t-1}\ge\alpha_{iil}^{t,\bar t}\eta_{il}^{t,\bar t,\bar t-1}(-1,0)+\sum_{d\in[D]}\gamma_{iild}^{t-1,\bar t-1}\breve\eta_{il}^{t,\bar t,\bar t-1}(-1)\qquad\forall\ l\in[D],\ \bar t\in[t-1]$$

$$\nu_{il}^{t,\bar t,s}\ge\alpha_{iil}^{t-\bar t+s+1,s+1}\eta_{il}^{t,\bar t,s}\left(-1,\frac{\nu_{il}^{t,\bar t,s+1}}{\alpha_{iil}^{t-\bar t+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{iild}^{t-\bar t+s,s}\breve\eta_{il}^{t,\bar t,s}(-1)\forall\ l\in[D],t\neq2,\ \bar t\in[2,t-1],\ s\in[0,\bar t-2]$$

$$\sum_{l\in[D]}\sum_{\bar t\in[t,\bar S+1]}\xi_{iil}^{1,\bar t-t+1}\frac{\nu_{il}^{t,\bar t,\bar t-t+1}}{\alpha_{iil}^{1,t-\bar t+1}}=\sum_{l\in[D]}\sum_{\bar t\in[t,\bar S+1]}\nu_{il}^{t,\bar t,\bar t-t+1}$$

$$\nu_{il}^{t,\bar t,\bar t-1}\ge\alpha_{iil}^{t,\bar t}\eta_{il}^{t,\bar t,\bar t-1}(-1,0)+\sum_{d\in[D]}\gamma_{iild}^{t-1,\bar t-1}\breve\eta_{il}^{t,\bar t,\bar t-1}(-1)\qquad\forall\ l\in[D],\ \bar t\in[t,\bar S+1]$$

$$\nu_{il}^{t,\bar t,s}\ge\alpha_{iil}^{t-\bar t+s+1,s+1}\eta_{il}^{t,\bar t,s}\left(-1,\frac{\nu_{il}^{t,\bar t,s+1}}{\alpha_{iil}^{t-\bar t+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{iild}^{t-\bar t+s,s}\breve\eta_{il}^{t,\bar t,s}(-1)$$
$$\forall\ l\in[D],\ t\neq2,\ \bar t\in[t,\bar S+1],\ s\in[\bar t-t+1,\bar t-2]$$

$$\sum_{l\in[D]}\sum_{\bar t\in[\bar S+2,t+\bar S-1]}\xi_{iil}^{1,\bar t-t+1}\frac{\nu_{il}^{t,\bar t,\bar t-t+1}}{\alpha_{iil}^{1,\bar t-t+1}}=\sum_{l\in[D]}\sum_{\bar t\in[\bar S+2,t+\bar S-1]}\nu_{il}^{t,\bar t,\bar t-t+1}$$

$$\nu_{il}^{t,\bar t,\bar S}\ge\alpha_{iil}^{t-\bar t+\bar S+1,\bar S+1}\eta_{il}^{t,\bar t,\bar S}(-1,0)+\sum_{d\in[D]}\gamma_{iild}^{t-\bar t+\bar S,\bar S}\breve\eta_{il}^{t,\bar t,\bar S}(-1)\qquad\forall\ l\in[D],\ \bar t\in[\bar S+2,t+\bar S-1]$$

$$\nu_{il}^{t,\bar t,s}\ge\alpha_{iil}^{t-\bar t+s+1,s+1}\eta_{il}^{t,\bar t,s}\left(-1,\frac{\nu_{il}^{t,\bar t,s+1}}{\alpha_{iil}^{t-\bar t+s+1,s+1}}\right)+\sum_{d\in[D]}\gamma_{iild}^{t-\bar t+s,s}\breve\eta_{il}^{t,\bar t,s}(-1)$$
$$\forall\ l\in[D],\ t\neq2,\ \bar t\in[\bar S+2,t+\bar S-2],\ s\in[\bar t-t+1,\bar S-1]$$

The fourth term is reformulated as $\sum_{d,l\in[D]}\sum_{s\in[0,\bar S]}\sum_{\bar t\in[\bar S]}\zeta_{iid}^{1,\bar t}(l,s)\hat\Upsilon_{il}^{t,\bar t,s}(-1)$. where

$$\eta_{il}^{t,\bar t,s}(x,y)=\theta\log\left(1+q_l^s\left(1-\sum_{\bar s\in[0,\bar t-s-1]}\breve h_{iil}^{t-\bar t+s+\bar s,\bar s}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right)+(1-q_l^s-a_l^s)\left(\exp\left(\frac{y}{\theta}\right)-1\right)\right)$$

$$\check{\eta}_{il}^{t,\bar{t},s}(x) = \theta \log\left(1 + \sum_{\bar{\tau}\in[0,\bar{t}-s-1]} \check{q}_l^{t-\bar{t}+s+\bar{\tau},\bar{\tau},s}\left(1 - \sum_{\tau\in[0,\bar{t}-s-\bar{\tau}-1]} \check{h}_{iil}^{t-\bar{t}+s+\bar{\tau}+\tau,\tau}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$$

$$\hat{\Upsilon}_{il}^{t,\bar{t},s}(x) = \theta \log\left(1 + \sum_{\bar{\tau}\in[(\bar{S}-\bar{t}+1)\wedge(t-1)]} \check{q}_l^{\bar{\tau},\bar{t}+\bar{\tau}-1,s}\left(1 - \sum_{\tau\in[0,t-\bar{\tau}-1]} \check{h}_{iil}^{\bar{\tau}+\tau,\tau}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$$

Combining the above results, we have the reformulation of Equation (14).

For Constraint (7b), when $t=1$, for $i\in[I]$ and $d\in[D]$, the left-hand side of equation is

$$\begin{cases} \sum_{j\in[I]}\sum_{l\in[D]}\left(\mathbb{G}_\theta\left[\tilde{\xi}_{ijl}^{1,0}\frac{\bar{\phi}_{ijld}^{1,1,0}}{\alpha_{ijl}^{1,0}}\right] + \sum_{\bar{t}\in[2,1+\bar{S}]}\bar{\phi}_{ijld}^{1,\bar{t},\bar{t}-1} + \sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]}\zeta_{ijd}^{1,\bar{t}}(l,s)\Upsilon_l^{2,\bar{t},s}(1)\right) + \sum_{j\in[I]} x_{ijd}^{0,1} - z_{id}^1 \\ \bar{\phi}_{ijdd}^{1,\bar{t},s} \geq \alpha_{ijd}^{1-\bar{t}+s+2,s+1}\rho_d^s(1,0) + \gamma_{ijdd}^{1-\bar{t}+s+1,s}\check{\rho}_d^{2,\bar{t},s}(1) + x_{ijd}^{1,1-\bar{t}+s+1,s} \\ \qquad\qquad \forall\ j\in[I],\ \bar{t}\in[1+\bar{S}],\ s=\min\{\bar{t}-1,\bar{S}\} \\ \bar{\phi}_{ijld}^{1,\bar{t},s} \geq \gamma_{ijld}^{1-\bar{t}+s+1,s}\check{\rho}_l^{2,\bar{t},s}(1) \qquad \forall\ j\in[I],\ \bar{t}\in[1+\bar{S}],\ l\in[D]/\{d\},\ s=\min\{\bar{t}-1,\bar{S}\} \end{cases} \tag{17}$$

Secondly, considering $t\geq 2$, for any $i\in[I]$ and $d\in[D]$, we have:

$$\mathbb{G}_\theta\left(\sum_{j\in[I]}\left(\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\tilde{y}_{ijd}^{t,0}(\tilde{\boldsymbol{\zeta}},l,s) + \tilde{y}_{ijd}^{t,0}(\tilde{\boldsymbol{\xi}}) + \tilde{x}_{ijd}^t\right) - \tilde{z}_{id}^t\right) = -z_{id}^1 + \text{Four terms} \tag{18}$$

The four terms can be formulated as

$$\sum_{l\in[D]}\sum_{\bar{t}\in[t]}\mathbb{G}_\theta\left[\tilde{\xi}_{ijl}^{t-\bar{t}+1,0}\frac{\bar{\phi}_{ijld}^{t,\bar{t},0}}{\alpha_{ijl}^{t-\bar{t}+1,0}}\right] + \sum_{l\in[D]}\sum_{\bar{t}\in[t+1,t+\bar{S}]}\bar{\phi}_{ijld}^{t,\bar{t},\bar{t}-t} + \sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]}\zeta_{ijd}^{1,\bar{t}}(l,s)\Upsilon_l^{t+1,\bar{t},s}(1)$$

$$\bar{\phi}_{ijdd}^{t,\bar{t},s} \geq \alpha_{ijd}^{t-\bar{t}+s+2,s+1}\rho_d^s(1,0) + \gamma_{ijdd}^{t-\bar{t}+s+1,s}\check{\rho}_d^{t+1,\bar{t},s}(1) + x_{ijd}^{1,t-\bar{t}+s+1,s},\ \forall\ j\in[I]/\{i\},\ \bar{t}\in[t+\bar{S}],\ s=\min\{\bar{t}-1,\bar{S}\}$$

$$\bar{\phi}_{ijld}^{t,\bar{t},s} \geq \gamma_{ijld}^{t-\bar{t}+s+1,s}\check{\rho}_l^{t+1,\bar{t},s}(1) \qquad \forall\ j\in[I]/\{i\},\ \bar{t}\in[t+\bar{S}],\ l\in[D]/\{d\},\ s=\min\{\bar{t}-1,\bar{S}\}$$

$$\bar{\phi}_{ijdd}^{t,\bar{t},s} \geq \alpha_{ijd}^{t-\bar{t}+s+2,s+1}\rho_d^s\left(1,\frac{\bar{\phi}_{ijdd}^{t,\bar{t},s+1}}{\alpha_{ijd}^{t-\bar{t}+s+2,s+1}}\right) + \gamma_{ijdd}^{t-\bar{t}+s+1,s}\check{\rho}_d^{t+1,\bar{t},s}(1) + x_{ijd}^{1,t-\bar{t}+s+1,s}$$
$$\forall\ j\in[I]/\{i\}\ s\in[0,\bar{S}-1],\ \bar{t}\in[s+2,s+t]$$

$$\bar{\phi}_{ijld}^{t,\bar{t},s} \geq \alpha_{ijl}^{t-\bar{t}+s+2,s+1}\rho_l^s\left(0,\frac{\bar{\phi}_{ijld}^{t,\bar{t},s+1}}{\alpha_{ijl}^{t-\bar{t}+s+2,s+1}}\right) + \gamma_{ijld}^{t-\bar{t}+s+1,s}\check{\rho}_l^{t+1,\bar{t},s}(1)\forall j\in[I]/\{i\}, s\in[0,\bar{S}-1], l\in[D]/\{d\}, \bar{t}\in[s+2,s+t]$$

$$\sum_{l\in[D]}\sum_{\bar{t}\in[t-1]}\mathbb{G}_\theta\left[\tilde{\xi}_{jil}^{t-\bar{t},0}\frac{\bar{\psi}_{jild}^{t,\bar{t},0}}{\alpha_{jil}^{t-\bar{t},0}}\right] + \sum_{l\in[D]}\sum_{\bar{t}\in[t,t+\bar{S}-1]}\bar{\psi}_{jild}^{t,\bar{t},\bar{t}-t+1} + \sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]}\zeta_{jid}^{1,\bar{t}}(l,s)\bar{\Upsilon}_{jil}^{t,\bar{t},s}(-1)$$

$$\bar{\psi}_{jidd}^{t,\bar{t},s} \geq \alpha_{jid}^{t-\bar{t}+s+1,s+1}\pi_{jid}^{t,\bar{t},s}(-1,0) + \gamma_{jidd}^{t-\bar{t}+s,s}\check{\pi}_{jid}^{t,\bar{t},s}(-1) - x_{jid}^{1,t-\bar{t}+s,s}\forall\ j\in[I]/\{i\}, \bar{t}\in[t+\bar{S}-1], s=\min\{\bar{t}-1,\bar{S}\}$$

$$\bar{\psi}_{jild}^{t,\bar{t},s} \geq \gamma_{jild}^{t-\bar{t}+s,s}\check{\pi}_{jil}^{t,\bar{t},s}(-1) \qquad \forall\ j\in[I]/\{i\}\ l\in[D]/\{d\},\ \bar{t}\in[t+\bar{S}-1],\ s=\min\{\bar{t}-1,\bar{S}\}$$

$$\bar{\psi}_{jidd}^{t,\bar{t},s} \geq \alpha_{jid}^{t-\bar{t}+s+1,s+1}\pi_{jid}^{t,\bar{t},s}\left(-1,\frac{\bar{\psi}_{jidd}^{t,\bar{t},s+1}}{\alpha_{jid}^{t-\bar{t}+s+1,s+1}}\right) + \gamma_{jidd}^{t-\bar{t}+s,s}\check{\pi}_{jid}^{t,\bar{t},s}(-1) - x_{jid}^{1,t-\bar{t}+s,s}$$
$$\forall\ j\in[I]/\{i\},\ s\in[0,\bar{S}-1],\ t\neq 2,\ \bar{t}\in[s+2,s+t-1]$$

$$\bar{\psi}_{jild}^{t,\bar{t},s} \geq \alpha_{jil}^{t-\bar{t}+s+1,s+1}\pi_{jil}^{t,\bar{t},s}\left(0,\frac{\bar{\psi}_{jild}^{t,\bar{t},s+1}}{\alpha_{jil}^{t-\bar{t}+s+1,s+1}}\right) + \gamma_{jild}^{t-\bar{t}+s,s}\check{\pi}_{jil}^{t,\bar{t},s}(-1)$$

$$\forall\ j \in [I]/\{i\},\ s \in [0, \bar{S}-1],\ t \neq 2,\ l \in [D]/\{d\},\ \bar{t} \in [s+2, s+t-1]$$

$$\sum_{l\in[D]}\sum_{\bar{t}\in[t]} \mathbb{G}_\theta\left[\tilde{\xi}_{iil}^{t-\bar{t}+1,0}\frac{\bar{\nu}_{ild}^{t,\bar{t},0}}{\alpha_{iil}^{t-\bar{t}+1,0}}\right] + \sum_{l\in[D]}\sum_{\bar{t}\in[t+1,t+\bar{S}]} \bar{\nu}_{ild}^{t,\bar{t},\bar{t}-t} + \sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[\bar{S}]} \zeta_{iid}^{1,\bar{t}}(l,s)\breve{\Upsilon}_{il}^{t,\bar{t},s}(1)$$

$$\bar{\nu}_{idd}^{t,\bar{t},s} \geq \alpha_{iid}^{t-\bar{t}+s+2,s+1}\rho_d^s(1,0) + \gamma_{iidd}^{t-\bar{t}+s+1,s}\breve{\rho}_d^{t+1,\bar{t},s}(1) \qquad \forall\ \bar{t} \in [t+\bar{S}],\ s = \min\{\bar{t}-1, \bar{S}\}$$

$$\bar{\nu}_{ild}^{t,\bar{t},s} \geq \gamma_{iild}^{t-\bar{t}+s+1,s}\breve{\rho}_l^{t+1,\bar{t},s}(1) \qquad \forall\ l \in [D]/\{d\},\ \bar{t} \in [t+\bar{S}],\ s = \min\{\bar{t}-1, \bar{S}\}$$

$$\bar{\nu}_{idd}^{t,\bar{t},s} \geq \alpha_{iid}^{t-\bar{t}+s+2,s+1}\bar{\eta}_{id}^{t,\bar{t},s}\left(1, \frac{\bar{\nu}_{idd}^{t,\bar{t},s+1}}{\alpha_{iid}^{t-\bar{t}+s+2,s+1}}\right) + \gamma_{iidd}^{t-\bar{t}+s+1,s}\breve{\eta}_{id}^{t,\bar{t},s}(1),\ \forall\ s \in [0, \bar{S}-1],\ \bar{t} \in [s+2, s+t]$$

$$\bar{\nu}_{ild}^{t,\bar{t},s} \geq \alpha_{iil}^{t-\bar{t}+s+2,s+1}\bar{\eta}_{il}^{t,\bar{t},s}\left(0, \frac{\bar{\nu}_{ild}^{t,\bar{t},s+1}}{\alpha_{iil}^{t-\bar{t}+s+2,s+1}}\right) + \gamma_{iild}^{t-\bar{t}+s+1,s}\breve{\eta}_{il}^{t,\bar{t},s}(1),\ \forall\ s \in [0, \bar{S}-1]\ l \in [D]/\{d\},\ \bar{t} \in [s+2, s+t]$$

and $\sum_{l\in[D]}\sum_{j\in[I]}\sum_{s\in[0,\bar{S}]}\sum_{\bar{t}\in[S]} y_{jid}^{1,\bar{t}}(\tilde{\boldsymbol{\zeta}}, l, s)\breve{\Upsilon}_{jil}^{t,\bar{t}}(-1) + \sum_{j\in[I]}\sum_{\bar{t}\in[S]} y_{jid}^{1,\bar{t}}(\tilde{\boldsymbol{\xi}})\breve{\Upsilon}_{jid}^{t,\bar{t}}(-1)$, where

$$\bar{\eta}_{il}^{t,\bar{t},s}(x,y) = \theta\log\left(1 + q_l^s\left(1 - \sum_{\bar{s}\in[0,\bar{t}-s-2]}\check{h}_{iil}^{t+1-\bar{t}+s+\bar{s},\bar{s}}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right) + (1-q_l^s-a_l^s)\left(\exp\left(\frac{y}{\theta}\right)-1\right)\right)$$

$$\breve{\eta}_{il}^{t,\bar{t},s}(x) = \theta\log\left(1 + \sum_{\bar{\tau}\in[0,\bar{t}-s-1]}\breve{q}_l^{t+1-\bar{t}+s+\bar{\tau},\bar{\tau},s}\left(1 - \sum_{\tau\in[0,\bar{t}-s-\bar{\tau}-2]}\check{h}_{iil}^{t+1-\bar{t}+s+\bar{\tau}+\tau,\tau}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$$

$$\breve{\Upsilon}_{il}^{t,\bar{t},s}(x) = \theta\log\left(1 + \sum_{\bar{\tau}\in[(\bar{S}-\bar{t}+1)\wedge(t)]}\breve{q}_l^{\bar{\tau},\bar{t}+\bar{\tau}-1,s}\left(1 - \sum_{\tau\in[0,t-\bar{\tau}-1]}\check{h}_{iil}^{\bar{\tau}+\tau,\tau}\right)\left(\exp\left(\frac{x}{\theta}\right)-1\right)\right)$$

$\square$

Proof of Proposition 4 Constraint $\mathbb{G}_\theta(\bar{R}-\tilde{R}) \leq 0$ is equivalent to $\mathbb{G}_\theta(-\tilde{R}) \leq \bar{R}$. The reformulation of $\mathbb{G}_\theta(-\tilde{R})$ follows the proof of Proposition 3 and is thus omitted here, and is shown below:

$$\sum_{i,j\in[I]}\sum_{l\in[D]}\left(\sum_{t\in[T]}\mathbb{G}_\theta\left[\tilde{\xi}_{ijl}^{T-t+1,0}\frac{\hat{\psi}_{ijl}^{t,0}}{\alpha_{ijl}^{T-t+1,0}}\right] + \sum_{t\in[T+1,T+\bar{S}]}\hat{\psi}_{ijl}^{t,t-T} + \sum_{d\in[D]}\sum_{s\in[0,\bar{S}]}\sum_{t\in[\bar{S}]}\zeta_{ijd}^{1,t}(l,s)\dot{\Upsilon}_{ijld}^{T+1,t,s}\right)$$
$$+ \sum_{i,j\in[I]}\sum_{t\in[T]}\sum_{d\in[D]} c_{ijd}x_{ijd}^{0,t} + \sum_{d,l\in[D]}\sum_{i,j\in[I]}\sum_{s\in[0,\bar{S}]}\sum_{t\in[S]} y_{ijd}^{1,t}(\tilde{\boldsymbol{\zeta}}, l, s)\ddot{\Upsilon}_{ijld}^{T+1,t,s} + \sum_{d\in[D]}\sum_{i,j\in[I]}\sum_{t\in[S]} y_{ijd}^{1,t}(\tilde{\boldsymbol{\xi}})\dddot{\Upsilon}_{ijd}^{T+1,t}$$

$$\hat{\psi}_{ijl}^{t,s} \geq \alpha_{ijl}^{T-t+s+2,s+1}\bar{\pi}_{ijl}^{T+1,t,s}(0) + \sum_{d\in[D]}\gamma_{ijld}^{T-t+s+1,s}\breve{\pi}_{ijld}^{T+1,t,s} + c_{ijl}x_{ijl}^{1,T-t+s+1,s}$$
$$\forall\ i,j \in [I],\ l \in [D],\ t \in [T+\bar{S}],\ s = \min\{t-1, \bar{S}\}$$

$$\hat{\psi}_{ijl}^{t,s} \geq \alpha_{ijl}^{T-t+s+2,s+1}\bar{\pi}_{ijl}^{T+1,t,s}\left(\frac{\hat{\psi}_{ijl}^{t,s+1}}{\alpha_{ijl}^{T-t+s+2,s+1}}\right) + \sum_{d\in[D]}\gamma_{ijld}^{T-t+s+1,s}\breve{\pi}_{ijld}^{T+1,t,s} + c_{ijl}x_{ijl}^{1,T-t+s+1,s}$$
$$\forall\ i,j \in [I],\ l \in [D],\ s \in [0, \bar{S}-1],\ t \in [s+2, s+T]$$

where

$$\bar{\pi}_{ijl}^{T+1,t,s}(z) = \theta\log\left(1 + \sum_{\bar{s}\in[0,t-s-1]} q_l^s\check{h}_{ijl}^{T+1-t+s+\bar{s},\bar{s}}\left(\exp\left(\frac{-r_{ijl}^{T-t+s+\bar{s}+1,\bar{s}}}{\theta}\right)-1\right) + (1-q_l^s-a_l^s)\left(\exp\left(\frac{z}{\theta}\right)-1\right)\right)$$

$$\breve{\pi}_{ijld}^{T+1,t,s} = \theta \log\left(1 + \sum_{\bar{\tau}\in[0,t-s-1]} \sum_{\tau\in[0,t-s-\bar{\tau}-1]} \breve{q}_l^{T+1-t+s+\bar{\tau},\bar{\tau},s} \breve{h}_{ijl}^{T+1-t+s+\bar{\tau}+\tau,\tau}\left(\exp\left(\frac{-\bar{r}_{ijld}^{T-t+s+\bar{\tau}+\tau+1,s,\tau}}{\theta}\right) - 1\right)\right)$$

$$\dot{\Upsilon}_{ijld}^{T+1,t,s} = \theta \log\left(1 + \sum_{\bar{\tau}\in[(\bar{S}-t+1)\wedge(T)]} \sum_{\tau\in[0,T-\bar{\tau}]} \breve{q}_l^{\bar{\tau},t+\bar{\tau}-1,s} \breve{h}_{ijl}^{\bar{\tau}+\tau,\tau}\left(\exp\left(\frac{-\bar{r}_{ijld}^{\bar{\tau}+\tau,s,\tau}}{\theta}\right) - 1\right)\right)$$

$$\ddot{\Upsilon}_{ijld}^{T+1,t,s} = \theta \log\left(1 + \sum_{\bar{\tau}\in[(T)\wedge(S-t+1)]} \breve{h}_{ijl}^{\bar{\tau},t+\bar{\tau}-1}\left(\exp\left(\frac{-\bar{r}_{ijld}^{\bar{\tau},s,t+\bar{\tau}-1}}{\theta}\right) - 1\right)\right)$$

$$\dddot{\Upsilon}_{ijd}^{T+1,t} = \theta \log\left(1 + \sum_{\bar{s}\in[(T)\wedge(S-t+1)]} \breve{h}_{ijd}^{\bar{s},t+\bar{s}-1}\left(\exp\left(\frac{-r_{ijd}^{\bar{s},t+\bar{s}-1}}{\theta}\right) - 1\right)\right).$$

□

Proof of Corollary 1 It is straightforward based on proposition of the entropic measure. □

B. Extension model with cancellation fee

Our model can be extended to incorporate the cancellation fee, whereby customers are required to pay a certain fee upon canceling their order. This cancellation fee is also counted as part of the rental company's revenue. Let $\hat{r}_{ijd}^{t,\tau}$ denote the cancellation fee for an order for vehicle type d on arc (i,j) that has been placed for τ period at time t, that is canceled without substitution. Similarly, let $\check{r}_{ijld}^{t,s,\tau}$ be the cancellation fee for an order for vehicle type l on arc (i,j) that is substituted with vehicle type d after waiting s periods since placement, that continues to wait for τ periods at time t and is subsequently canceled. Accordingly, the cancellation fee can be formulated as follows:

$$\tilde{R}' = \sum_{i,j\in[I]} \sum_{t\in[T]} \sum_{d\in[D]} \left[\sum_{\bar{\tau}\in[0,\bar{S}]} \left(\hat{r}_{ijd}^{t,\bar{\tau}} \text{Bin}\left(\tilde{\xi}_{ijd}^{t,\bar{\tau}}\left(1 - \sum_{l\in[D]} \beta_{ijdl}^{t,\bar{\tau}}\right), a_d^{\bar{\tau}}\right) + \sum_{s\in[0,\bar{S}]} \sum_{l\in[D]} \check{r}_{ijld}^{t,s,\bar{\tau}} \text{Bin}\left(\tilde{\zeta}_{ijd}^{t,\bar{\tau}}(l,s), a_l^{\bar{\tau}+s}\right)\right)\right] \tag{20}$$

Constraint (20) includes two terms that account for the cancellation fees corresponding to the customers in the waiting lists $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{\zeta}}$, respectively. The following proposition shows that Constraint (20) can be reformulated as convex constraints. We omit its proof for brevity.

Proposition 7. *Constraint $\mathbb{G}_\theta\left(\bar{R} - \tilde{R} - \tilde{R}'\right) \le 0$ can be reformulated as the following set of exponential conic constraints under the non-integral approximation.*

$$\sum_{i,j\in[I]} \sum_{l\in[D]} \left(\sum_{t\in[T]} \mathbb{G}_\theta\left[\tilde{\xi}_{ijl}^{T-t+1,0} \frac{\check{\psi}_{ijl}^{t,0}}{\alpha_{ijl}^{T-t+1,0}}\right] + \sum_{t\in[T+1,T+\bar{S}]} \check{\psi}_{ijl}^{t,t-T}\right) + \sum_{d,l\in[D]} \sum_{i,j\in[I]} \sum_{s\in[0,\bar{S}]} \sum_{t\in[\bar{S}]} \zeta_{ijd}^{1,t}(l,s)\, \dddot{\Upsilon}_{ijld}^{T+1,t,s}$$
$$+ \sum_{i,j\in[I]} \sum_{t\in[T]} \sum_{d\in[D]} c_{ijd} x_{ijd}^{0,t} + \sum_{d,l\in[D]} \sum_{i,j\in[I]} \sum_{s\in[0,\bar{S}]} \sum_{t\in[S]} y_{ijd}^{1,t}(\tilde{\boldsymbol{\zeta}}, l, s) \ddot{\Upsilon}_{ijld}^{T+1,t,s} + \sum_{d\in[D]} \sum_{i,j\in[I]} \sum_{t\in[S]} y_{ijd}^{1,t}(\tilde{\boldsymbol{\xi}})\, \dddot{\Upsilon}_{ijd}^{T+1,t} \le \bar{R}$$

$$\check{\psi}_{ijl}^{t,s} \ge \alpha_{ijl}^{T-t+s+2,s+1} \dot{\pi}_{ijl}^{T+1,t,s}(0) + \sum_{d\in[D]} \gamma_{ijld}^{T-t+s+1,s} \ddot{\pi}_{ijld}^{T+1,t,s} + c_{ijl} x_{ijl}^{1,T-t+s+1,s}$$
$$\forall\ i,j\in[I],\ l\in[D],\ t\in[T+\bar{S}],\ s=\min\{t-1,\bar{S}\}$$

$$\check{\psi}_{ijl}^{t,s} \geq \alpha_{ijl}^{T-t+s+2,s+1} \dot{\pi}_{ijl}^{T+1,t,s}\left(\frac{\hat{\psi}_{ijl}^{t,s+1}}{\alpha_{ijl}^{T-t+s+2,s+1}}\right) + \sum_{d\in[D]} \gamma_{ijld}^{T-t+s+1,s} \ddot{\pi}_{ijld}^{T+1,t,s} + c_{ijl} x_{ijl}^{1,T-t+s+1,s}$$
$$\forall\ i,j\in[I],\ l\in[D],\ s\in[0,\bar{S}-1],\ t\in[s+2,s+T]$$

where $\dot{\Upsilon}_{ijld}^{T+1,t,s}$, $\ddot{\Upsilon}_{ijld}^{T+1,s,t}$ *and* $\dddot{\Upsilon}_{ijd}^{T+1,t}$ *are defined as in the proof of Proposition 4. Denote* $\check{a}_l^{t,\tau,s} = a_l^{\tau+s}\prod_{\bar{\tau}=1}^{t-1\wedge\tau}\left(1-q_l^{\tau+s-\bar{\tau}}-a_l^{\tau+s-\bar{\tau}}\right)$*, and the other parameter is defined as follows:*

$$\dot{\pi}_{ijl}^{T+1,t,s}(z) = \theta\log\Bigg[1+\sum_{\bar{s}\in[0,t-s-1]} q_l^s \check{h}_{ijl}^{T+1-t+s+\bar{s},\bar{s}}\left(\exp\left(\frac{-r_{ijl}^{T-t+s+\bar{s}+1,\bar{s}}}{\theta}\right)-1\right)$$
$$+a_l^s\left(\exp\left(\frac{-\hat{r}_{ijl}^{T-t+s+1,s}}{\theta}\right)-1\right)+(1-q_l^s-a_l^s)\left(\exp\left(\frac{z}{\theta}\right)-1\right)\Bigg]$$
$$\ddot{\pi}_{ijld}^{T+1,t,s} = \theta\log\Bigg[1+\sum_{\bar{\tau}\in[0,t-s-1]}\sum_{\tau\in[0,t-s-\bar{\tau}-1]} \check{q}_l^{T+1-t+s+\bar{\tau},\bar{\tau},s} \check{h}_{ijl}^{T+1-t+s+\bar{\tau}+\tau,\tau}\left(\exp\left(\frac{-\bar{r}_{ijld}^{T-t+s+\bar{\tau}+\tau+1,s,\tau}}{\theta}\right)-1\right)$$
$$+\sum_{\bar{\tau}\in[0,t-s-1]} \check{a}_l^{T+1-t+s+\bar{\tau},\bar{\tau},s}\left(\exp\left(\frac{-\check{r}_{ijld}^{T+1-t+s+\bar{\tau},s,\bar{\tau}}}{\theta}\right)-1\right)\Bigg]$$
$$\dddot{\Upsilon}_{ijld}^{T+1,t,s} = \theta\log\Bigg[1+\sum_{\bar{\tau}\in[(\bar{S}-t+1)\wedge(T)]}\sum_{\tau\in[0,T-\bar{\tau}]} \check{q}_l^{\bar{\tau},t+\bar{\tau}-1,s} \check{h}_{ijl}^{\bar{\tau}+\tau,\tau}\left(\exp\left(\frac{-\bar{r}_{ijld}^{\bar{\tau}+\tau,s,\tau}}{\theta}\right)-1\right)$$
$$+\sum_{\bar{\tau}\in[(T)\wedge(S-t+1)]} \check{a}_{ijl}^{\bar{\tau},t+\bar{\tau}-1,s}\left(\exp\left(\frac{-\check{r}_{ijld}^{\bar{\tau},s,t+\bar{\tau}-1}}{\theta}\right)-1\right)\Bigg]$$

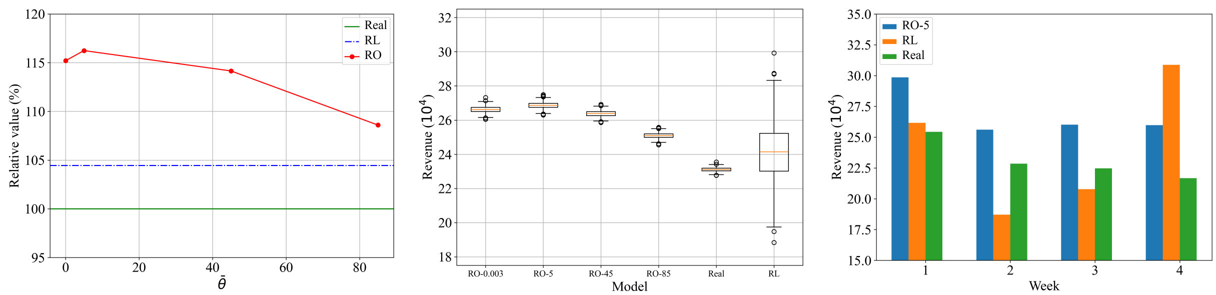
C. SAA model

The dynamics of the SAA model for every sample $e\in E$ is formulated as follows:

$$\xi_{ijd}^{t,0,e} = \hat{\lambda}_{ijd}^{t,e}, \tag{22a}$$
$$\xi_{ijd}^{t,s,e} = \xi_{ijd}^{t-1,s-1,e}\left(1-\sum_{l\in[D]}\beta_{ijdl}^{t-1,s-1}\right)(1-q_d^{s-1}-a_d^{s-1}) \qquad \forall s\in[\bar{S}], \tag{22b}$$
$$\zeta_{ijd}^{t,0,e}(l,s) = \beta_{ijld}^{t,s}\xi_{ijl}^{t,s,e}, \qquad \forall l\in[D], s\in[0,\bar{S}], \tag{22c}$$
$$y_{ijd}^{t,e}(\boldsymbol{\xi}) = \sum_{s=0}^{t-1}\xi_{ijd}^{t-s,0,e}q_d^s + \sum_{s=t}^{\bar{S}}\xi_{ijd}^{1,s-t+1}q_d^s, \tag{22d}$$
$$y_{ijd}^{t,e}(\boldsymbol{\zeta},l,s) = \sum_{s=0}^{t-1}\zeta_{ijd}^{t-s,0,e}(l,s)q_l^s + \sum_{s=t}^{\bar{S}}\zeta_{ijd}^{1,s-t+1}(l,s)q_l^s, \qquad \forall l\in[D], s\in[0,\bar{S}], \tag{22e}$$
$$x_{ijd}^{t,e} = \sum_{s\in[0,\bar{S}]}\frac{x_{ijd}^{1,t,s}}{\alpha_{ijd}^{t,s}}\xi_{ijd}^{t,s,e} + x_{ijd}^{0,t}. \tag{22f}$$

Compared with our model, SAA model (22) cannot capture the dynamics of the vehicle rental system via distributional laws, and thus employs sample realizations. For every sample $e\in E$ and every $i\in[I], d\in[D]$, the inventory $z_{id}^{t,e}$ and associated SAA model can be formulated as below:

$$\max_{\boldsymbol{\alpha},\ \boldsymbol{\gamma},\ \boldsymbol{x}\geq\boldsymbol{0}}\ R = \sum_{i,j\in[I]}\sum_{t\in[T]}\sum_{d\in[D]}\sum_{e\in[E]}\frac{1}{E}\left(\sum_{\tau=1}^{t-1} y_{ijd}^{t-\tau,e}(\boldsymbol{\xi})h_{ijd}^{\tau}r_{ijd}^{t,\tau} + \sum_{\tau=t}^{S} y_{ijd}^{1,\tau-t+1}(\boldsymbol{\xi})h_{ijd}^{\tau}r_{ijd}^{t,\tau}\right.$$



(a) Average of revenue (b) Quantiles of revenue (c) Weekly revenue

Figure 10 **Out-of-sample revenue comparison of the RO-$\bar{\theta}$, Real decision and RL algorithm**

$$+\sum_{s\in[0,\bar{S}]}\sum_{l\in[D]}\left(\sum_{\tau=1}^{t-1}y_{ijd}^{t-\tau,e}(\boldsymbol{\zeta},l,s)h_{ijl}^{\tau}\bar{r}_{ijld}^{t,s,\tau}+\sum_{\tau=t}^{S}y_{ijd}^{1,\tau-t+1}(\boldsymbol{\zeta},l,s)h_{ijl}^{\tau}\bar{r}_{ijld}^{t,s,\tau}\right)\Bigg)-\sum_{i,j\in[I]}\sum_{d\in[D]}\sum_{t\in[T]}\sum_{e\in[E]}\frac{1}{E}c_{ijd}x_{ijd}^{t,e}$$

$$\text{s.t. } z_{id}^{t,e}=z_{id}^{t-1,e}-\sum_{j\in[I]}\left(y_{ijd}^{t-1,e}(\boldsymbol{\xi})+\sum_{s\in[0,\bar{S}]}\sum_{l\in[D]}y_{ijd}^{t-1,e}(\boldsymbol{\zeta},l,s)\right)-\sum_{j\in[I]}x_{ijd}^{t-1,e}+\sum_{j\in[I]}x_{jid}^{t-1,e}$$

$$+\sum_{j\in[I]}\left(\sum_{\tau=1}^{t-1}y_{jid}^{t-\tau,e}(\boldsymbol{\xi})h_{jid}^{\tau}+\sum_{\tau=t}^{S}y_{jid}^{1,\tau-t+1}(\boldsymbol{\xi})h_{jid}^{\tau}+\sum_{s\in[0,\bar{S}]}\sum_{l\in[D]}\left(\sum_{\tau=1}^{t-1}y_{jid}^{t-\tau,e}(\boldsymbol{\zeta},l,s)h_{jil}^{\tau}+\sum_{\tau=t}^{S}y_{jid}^{1,\tau-t+1}(\boldsymbol{\zeta},l,s)h_{jil}^{\tau}\right)\right.$$

$$\sum_{d\in[D]}z_{id}^{t,e}\le B_i,\quad \forall i\in[I],t\in[2,T],e\in[E]$$

$$\sum_{j\in[I]}\left(\sum_{l\in[D]}\sum_{s\in[0,\bar{S}]}y_{ijd}^{t,e}(\boldsymbol{\zeta},l,s)+y_{ijd}^{t,e}(\boldsymbol{\xi})\right)+\sum_{j\in[I]}x_{ijd}^{t,e}\le z_{id}^{t,e},\quad \forall i\in[I],d\in[D],t\in[T],e\in[E]$$

$$\sum_{d\in[D]}\gamma_{ijld}^{t,s}\le\alpha_{ijl}^{t,s},\quad \forall i,j\in[I],l\in[D],t\in[T],s\in[0,\bar{S}]$$

D. Revenue comparison with benchmarks in practice

To assess practical effectiveness, we benchmark our model against real company decisions (Real) and a reinforcement learning (RL) baseline. For the real policy, we aggregate operational data to compute substitution proportions and repositioning volumes across stores and vehicle types. For RL, we implement a Proximal Policy Optimization (PPO) agent with a Multi-Layer policy (MLP) with learning rate 3×10^{-4}, batch size 32, 10 epochs per update, $\gamma=1.0$, and rollout length 512, trained for 50,000 steps to ensure convergence. Figure 10(a) shows that RO-$\bar{\theta}$ models consistently outperform both baselines under 1000 experiments using real data, especially improving revenues by 5–15% over real decisions. Figures 10(b) and 10(c) further display that the RL policy yields greater variability and unstable weekly performance. By contrast, RO-$\bar{\theta}$ models achieve higher medians with lower variance, and we see the RO-5 model dominates across four weeks.