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# Online Supplement to "Exact methods and a two-stage iterative heuristic for the carrier-vehicle traveling salesman problem"

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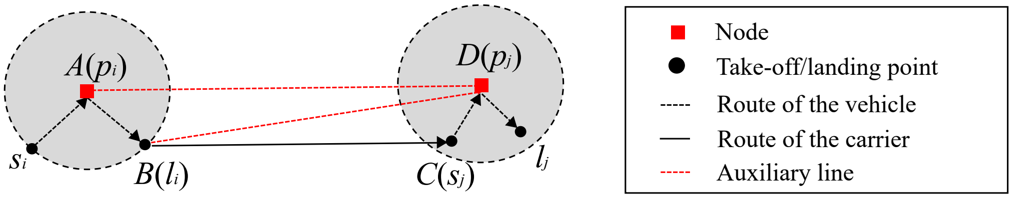
# A Proof of Propositions

## A.1 Proof of Proposition 1

Figure 2 illustrates the routing structure between two nodes in the CVTSP. In any feasible solution, the path from node $i$ to node $j$ consists of three line segments: $AB$, $BC$, and $CD$. Here, $A$ and $D$ represent nodes $i$ and $j$, respectively; $B$ is the landing point for node $i$, and $C$ is the take-off point for node $j$. The segments $AB$, $BD$, and $AD$ form a triangle. By the triangle inequality, we have $AB + BD \geq AD$. Similarly, since $BC$, $CD$, and $BD$ also form a triangle, we have $BC + CD \geq BD$. Combining the two inequalities gives $AB + BC + CD \geq AD$. Given that the lengths of $AB$, $BC$, $CD$, and $AD$ are $t_{i,2}V_v$, $T_{ij}V_c$, $t_{j,1}V_v$, and $d_{ij}$, respectively, substituting these values yields inequality (14). □

**Figure 2 Illustration of the routes between targets $i$ and $j$**



## A.2 Proof of Proposition 2

We note that the travel time from target $i$ to target $j$ comprises three components, i.e., $t_{i,2}, T_{ij}, t_{j,1}$, and the minimum distance traveled by the carrier and the vehicle should be at least $d_{ij}$. Consequently, finding the minimum travel time $\beta_{ij}$ is equivalent to find the minimum value of $t_{i,2} + T_{ij} + t_{j,1}$. Given that the vehicle's speed is faster than that of the carrier, i.e., $V_v > V_c$, it is beneficial to let the vehicle travel as long as possible. According to (17)–(18), we know the travel distance $(t_{i,2} + t_{j,1})V_v$ of the vehicle between $i$ and $j$ is at most $a(V_c + V_v)$. The minimum distance traveled by the carrier is then $d_{ij} - a(V_c + V_v)$. In case that $d_{ij} < a(V_c + V_v)$, the travel distance of the vehicle from $i$ to $j$ is $d_{ij}$, and the travel distance of the carrier is zero. The above explanations show the validity of (23). Similarly, (24) and (25) are valid. □

## A.3 Proof of Proposition 3

We first establish the validity of equation (42). The travel time between targets $i$ and $j$ consists of three components: $t_{i,2}$, $T_{ij}$, and $t_{j,1}$. Hence, determining the maximum travel time $\theta_{ij}$ reduces to finding the maximum of the expression $t_{i,2} + T_{ij} + t_{j,1}$.

The upper bounds for $t_{i,2}$ and $t_{j,1}$ are given by $\frac{a(V_c+V_v)}{2V_v}$, as established by inequalities (17) and (18). The term $T_{ij}$, representing the travel time of the carrier between targets $i$ and $j$, can be bounded using Theorem 1 from Poikonen and Golden (2020), which states that obj(CVTSP) $\leq$ obj(TSP). This implies that the collaborative routing in the CVTSP does not exceed the cost of a conventional TSP, thereby ensuring that $T_{ij} \leq \frac{d_{ij}}{V_c}$. If this inequality were not satisfied, it would imply that carrier-only routing is more efficient, contradicting the benefit of carrier-vehicle coordination.

Substituting these bounds into the expression $t_{i,2} + T_{ij} + t_{j,1}$ yields:

$$\theta_{ij} = \frac{a(V_c+V_v)}{2V_v} + \frac{d_{ij}}{V_c} + \frac{a(V_c+V_v)}{2V_v},$$

which simplifies to equation (42).

Next, we consider equation (43), which represents the maximum travel time $\theta_{0j}$ from the depot to the first visited target $j$. This is given by:

$$\theta_{0j} = T_{0j} + t_{j,1}.$$

The maximum occurs when $T_{0j} = \frac{d_{0j}}{V_c}$ and $t_{j,1} = \frac{a(V_c+V_v)}{2V_v}$, yielding:

$$\theta_{0j} = \frac{d_{0j}}{V_c} + \frac{a(V_c+V_v)}{2V_v},$$

which corresponds to equation (43).

Finally, the validity of equation (44), concerning the travel time from the last visited target back to the depot, can be established analogously by considering the corresponding travel components and applying the same bounding arguments. □

### A.4 Proof of Proposition 4

A valid analytical cut eliminates the current suboptimal solution from the MP without excluding any feasible integer solutions from the MP's feasible region. Consider a current solution with a set $S$ of connected arcs and a new solution in the subsequent iteration with a set $\overline{S}$. We examine two cases to show validity.

In the first case, where $\overline{S} = S$, the sequence in the new solution is the same as that in the current solution. The values of $\sum_{(i,j)\in\overline{S}} (1 - x_{ij})\theta_{ij}$ and $\sum_{i,j\notin\overline{S}} x_{ij}\beta_{ij}$ are both zero. The cut is valid.

In the second case, where $\overline{S} \neq S$, the sequence in the new solution differs from the current solution. Define $C_{max}(S)$ as the makespan with connected arcs in $S$, a set $S' = S \setminus \overline{S}$ of arcs that are in $S$ but not in $\overline{S}$, and a set $S'' = \overline{S} \setminus S$ of arcs that are in $\overline{S}$ but not in $S$. The new makespan satisfies

$$C_{max}(S \setminus S' \cup S'') \geq \hat{C}_{max} - \sum_{(i,j)\in S'} (1 - x_{ij})\theta_{ij} + \sum_{(i,j)\in S''} x_{ij}\beta_{ij}.$$

Given that removing an arc $(i, j)$ from $S$ reduces the makespan by at most $\theta_{ij}$, and adding an arc $(i, j)$ to $S$ can increase the makespan by at most $\beta_{ij}$, we know that the reduction term $\sum_{(i,j)\in S'}(1-x_{ij})\theta_{ij}$ is overestimated and the increasing term $\sum_{(i,j)\in S''} x_{ij}\beta_{ij}$ is underestimated. This ensures that the right-side value is less than or equal to $C_{max}(S \setminus S' \cup S'')$. Thus, the cut is valid. $\square$

# B Performance Analysis

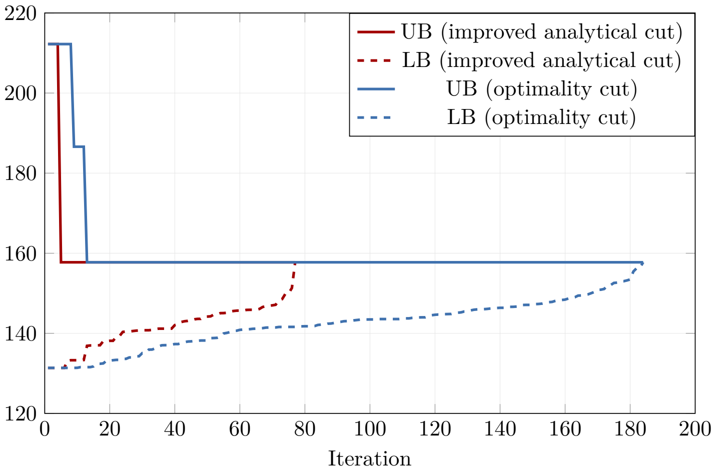
## B.1 Performance Analysis of the LBBD method

We analyze the performance of the proposed LBBD method by testing different Benders cuts using the 72 small instances in Gam72. Next, we select the best-performing Benders cuts among the proposed ones by evaluating key metrics: the obtained upper bound (UB), lower bound (LB), computation time (for both MP and SP), number of cuts added (#cuts), and number of optimal solutions (#o) found. The results are reported in Table 6.

**Table 6 Comparison results between different Benders cuts**

| | No-good cut | Optimality cut | Analytical cut | Improved analytical cut |
|---|---|---|---|---|
| UB | 221.30 | 221.30 | 221.30 | 221.30 |
| LB | 220.76 | 220.76 | 220.92 | 221.14 |
| Time (MP) | 323.78 | 248.24 | 229.82 | 110.77 |
| Time (SP) | 11.72 | 16.16 | 17.18 | 1.62 |
| #cuts | 3104 | 3106 | 3086 | 596 |
| #o | 67 | 68 | 70 | 71 |

**Figure 3 Convergence plot for LBBD with improved analytical cut and optimality cut**



We see from Table 6 that while all cuts achieve the same upper bound (221.30), the improved analytical cut performs the best. It significantly reduces computation time and requires far fewer cuts to reach convergence. It finds optimal solutions in 71 out of 72 instances, outperforming other cuts. Figure 3 further shows the convergence trend of the LBBD method with different Benders cuts. The improved analytical cut is more efficient and converges to optimality quickly.

## B.2 Performance Analysis of the TSIH

We analyze the convergence behavior of TSIH in Figures 4 and 5 for instances VLD-50 in Erd132 and Regular-50 in Poi325. The horizontal axis represents the number of iterations, and the vertical axis denotes the upper bounds obtained in each iteration. Figure 4 shows that during the iterative process, the upper bound is decreased quickly, and soon trapped into local optima. The diversification mechanism restarts the process, aiming at finding better solutions. The best solution is found in the 69th iteration after the third diversification. We can observe similar trends in Figure 5. The best solution is found in the 418th iteration after the 37th diversification. The above observations demonstrate the effectiveness of the iterative and diversification mechanisms in TSIH.

**Figure 4** **Convergence plots of instances VLD-50 in Erd132**

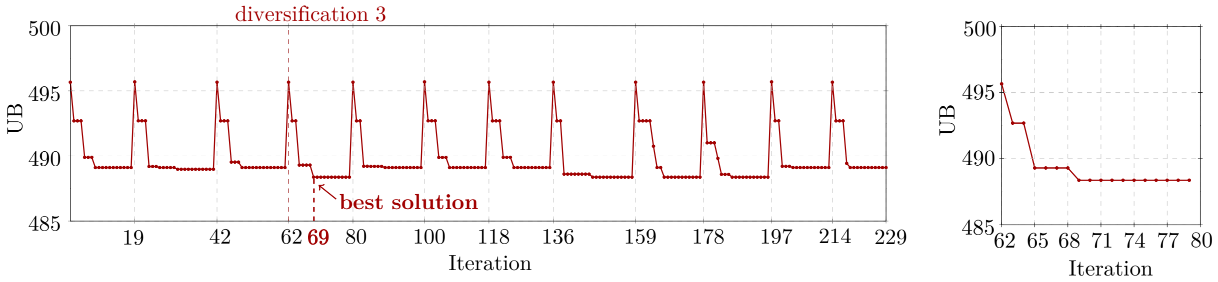
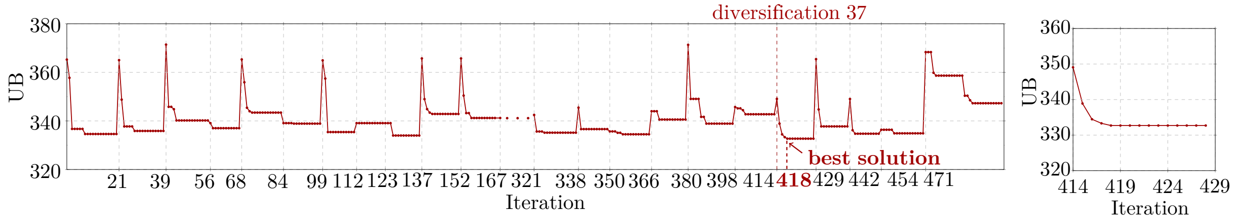


**Figure 5** **Convergence plots of instances Regular-50 in Poi325**



## C Outline of the LBBD method

**Algorithm 1** Outline of the LBBD method

```
Compute an initial upper bound $\mathcal{UB}$ with TSIH as introduced in Section 5.
Initialize $CUTS \leftarrow \emptyset$, the best upper bound $\mathcal{UB}^* \leftarrow \mathcal{UB}$ and the lower bound $\mathcal{LB} \leftarrow -\infty$.
while time limit not reached do
    Solve MP using the branch-and-cut procedure and update $\mathcal{LB}$.
    for each integer solution found during the branch-and-cut search do
        Solve SP to obtain the current upper bound $\mathcal{UB}$.
        if $\mathcal{UB} \leq \mathcal{UB}^*$ then
            $\mathcal{UB}^* \leftarrow \mathcal{UB}$.
        end if
        if $\mathcal{UB}^* = \mathcal{LB}$ then
            Terminate the algorithm.
        else
            Add the improved analytical cut (45) to the cut pool $CUTS$.
        end if
    end for
end while
Output the best upper bound $\mathcal{UB}^*$ and its corresponding solution.
```