

**Online supplemental materials to
“Theoretical and numerical comparison of nine single-level
reformulations for bilevel programs ”**

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Appendix A: Proofs of Theorems 1–3

Proof of Theorem 1 We first show the weak duality. Since $\nabla_z L(x, z, u, v) = 0$ for any $y \in Y(x)$ and $(z, u, v) \in \mathcal{F}_1(x)$, it follows from the pseudoconvexity of $L(x, \cdot, u, v)$ that $L(x, y, u, v) \geq L(x, z, u, v)$. Then, for $u \in \mathbb{R}_+^p$, $y \in Y(x)$, and $(z, u, v) \in \mathcal{F}_1(x)$, we have

$$f(x, y) \geq f(x, y) + u^T g(x, y) = L(x, y, u, v) \geq L(x, z, u, v) = f(x, z) + u^T g(x, z).$$

By the arbitrariness of $y \in Y(x)$ and $(z, u, v) \in \mathcal{F}_1(x)$, we have

$$\min_{y \in Y(x)} f(x, y) \geq \max_{(z, u, v) \in \mathcal{F}_1(x)} f(x, z) + u^T g(x, z).$$

Next, we show the strong duality. Since P_x satisfies the GCQ at $y_x \in S(x)$, there is (u_x, v_x) such that

$$\nabla_y L(x, y_x, u_x, v_x) = 0, \quad u_x^T g(x, y_x) = 0, \quad u_x \geq 0, \quad h(x, y_x) = 0,$$

from which we have $(y_x, u_x, v_x) \in \mathcal{F}_1(x)$. Then, we have

$$\min_{y \in Y(x)} f(x, y) \geq \max_{(z, u, v) \in \mathcal{F}_1(x)} f(x, z) + u^T g(x, z) \geq f(x, y_x) + u_x^T g(x, y_x) = f(x, y_x).$$

Since $y_x \in S(x)$, the above inequalities are all equalities. Therefore, we have

$$\min_{y \in Y(x)} f(x, y) = f(x, y_x) = f(x, y_x) + u_x^T g(x, y_x) = \max_{(z, u, v) \in \mathcal{F}_1(x)} f(x, z) + u^T g(x, z).$$

This completes the proof.

Proof of Theorem 2 We first show the weak duality. For any $y \in Y(x)$ and $(z, u, v) \in \mathcal{F}_i(x)$ ($i = 2, 4$), we have

$$u^T g(x, y) + v^T h(x, y) - (u^T g(x, z) + v^T h(x, z)) \leq 0. \tag{A.1}$$

Since $u^T g(x, \cdot) + v^T h(x, \cdot)$ is quasiconvex, (A.1) implies

$$(y - z)^T (\nabla g(x, z)u + \nabla h(x, z)v) \leq 0. \tag{A.2}$$

Since $\nabla f(x, z) + \nabla g(x, z)u + \nabla h(x, z)v = 0$, it follows that

$$(y - z)^T (\nabla f(x, z) + \nabla g(x, z)u + \nabla h(x, z)v) = 0. \tag{A.3}$$

(A.2)-(A.3) implies $(y - z)^T \nabla f(x, z) = -(y - z)^T (\nabla g(x, z)u + \nabla h(x, z)v) \geq 0$. Since $f(x, \cdot)$ is pseudoconvex, we have $f(x, y) \geq f(x, z)$. By the arbitrariness of $y \in Y(x)$ and $(z, u, v) \in \mathcal{F}_i(x)$

($i = 2, 4$), we have

$$\min_{y \in Y(x)} f(x, y) \geq \max_{(z, u, v) \in \mathcal{F}_i(x)} f(x, z) \quad (i = 2, 4).$$

Next, we show the strong duality. Since P_x satisfies the GCQ at $y_x \in S(x)$, there is (u_x, v_x) such that $\nabla_y L(x, y_x, u_x, v_x) = 0$, $u_x^T g(x, y_x) = 0$, $u_x \geq 0$, $h(x, y_x) = 0$, from which we have $(y_x, u_x, v_x) \in \mathcal{F}_i(x)$ ($i = 2, 4$). Then, we have

$$\min_{y \in Y(x)} f(x, y) \geq \max_{(z, u, v) \in \mathcal{F}_i(x)} f(x, z) \geq f(x, y_x) \quad (i = 2, 4).$$

Since $y_x \in S(x)$, the above inequalities are all equalities. Therefore, we have

$$\min_{y \in Y(x)} f(x, y) = f(x, y_x) = \max_{(z, u, v) \in \mathcal{F}_i(x)} f(x, z) \quad (i = 2, 4).$$

This completes the proof.

Proof of Theorem 3 Since the proof of this theorem is similar to Theorem 2.2 and so we omit it here.

Appendix B: Proofs of Theorems 8 and 9

For simplicity, we ignore the upper constraint $(x, y) \in \Omega$ throughout this appendix.

Proof of Theorem 8 (i) Note that the case of MDP has been shown in Theorem 9 of Li et al. (2024b), while other cases of eMDP/TWDP/TMDP/eTMDP/ETMDP/eETMDP can be shown in a similar way. For completeness, we provide a proof for the case of eMDP as an example. In fact, if $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is a KKT point of WDP, there is $(\zeta^g, \zeta^h, \zeta^u, \hat{\alpha}, \hat{\beta}) \in \mathbb{R}^{2p+q+1+m}$ such that

$$\nabla_x F(\bar{x}, \bar{y}) + \hat{\alpha}(\nabla_x f(\bar{x}, \bar{y}) - \nabla_x L(\bar{x}, \bar{z}, \bar{u}, \bar{v})) + \nabla_{zx}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\hat{\beta} + \nabla_x g(\bar{x}, \bar{y})\zeta^g + \nabla_x h(\bar{x}, \bar{y})\zeta^h = 0, \quad (\text{B.1})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \hat{\alpha}\nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y})\zeta^g + \nabla_y h(\bar{x}, \bar{y})\zeta^h = 0, \quad (\text{B.2})$$

$$\nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\hat{\beta} = 0, \quad (\text{B.3})$$

$$-\hat{\alpha}g(\bar{x}, \bar{z}) + \nabla_z g(\bar{x}, \bar{z})^T \hat{\beta} - \zeta^u = 0, \quad (\text{B.4})$$

$$-\hat{\alpha}h(\bar{x}, \bar{z}) + \nabla_z h(\bar{x}, \bar{z})^T \hat{\beta} = 0, \quad (\text{B.5})$$

$$0 \leq \hat{\alpha} \perp f(\bar{x}, \bar{y}) - L(\bar{x}, \bar{z}, \bar{u}, \bar{v}) \leq 0, \quad (\text{B.6})$$

$$0 \leq \zeta^g \perp g(\bar{x}, \bar{y}) \leq 0, \quad 0 \leq \zeta^u \perp \bar{u} \geq 0, \quad (\text{B.7})$$

$$\nabla_z L(\bar{x}, \bar{z}, \bar{u}, \bar{v}) = 0, \quad h(\bar{x}, \bar{y}) = 0. \quad (\text{B.8})$$

By setting $\eta^g = \zeta^g$, $\eta^h = \zeta^h$, $\eta^u = \zeta^u$, $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$, $\xi^g = (\hat{\alpha}, \dots, \hat{\alpha})^T \in \mathbb{R}^p$, and $\xi^h = -(\hat{\alpha}, \dots, \hat{\alpha})^T \in \mathbb{R}^q$, (B.1)-(B.8) can be rewritten equivalently as

$$\begin{aligned} & \nabla_x F(\bar{x}, \bar{y}) + \alpha(\nabla_x f(\bar{x}, \bar{y}) - \nabla_x f(\bar{x}, \bar{z})) - \sum_{i=1}^p \nabla_x g_i(\bar{x}, \bar{z})\bar{u}_i \xi_i^g + \sum_{i=1}^q \nabla_x h_i(\bar{x}, \bar{z})\bar{v}_i \xi_i^h \\ & + \nabla_{zx}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\beta + \nabla_x g(\bar{x}, \bar{y})\eta^g + \nabla_x h(\bar{x}, \bar{y})\eta^h = 0, \end{aligned} \quad (\text{B.9})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \alpha\nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y})\eta^g + \nabla_y h(\bar{x}, \bar{y})\eta^h = 0, \quad (\text{B.10})$$

$$-\alpha\nabla_z f(\bar{x}, \bar{z}) + \nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\beta - \sum_{i=1}^p \nabla_z g_i(\bar{x}, \bar{z})\bar{u}_i \xi_i^g + \sum_{i=1}^q \nabla_z h_i(\bar{x}, \bar{z})\bar{v}_i \xi_i^h = 0, \quad (\text{B.11})$$

$$-\text{diag}(g_1(\bar{x}, \bar{z}), \dots, g_p(\bar{x}, \bar{z}))\xi^g + \nabla_z g(\bar{x}, \bar{z})^T \beta - \eta^u = 0, \quad (\text{B.12})$$

$$\text{diag}(h_1(\bar{x}, \bar{z}), \dots, h_q(\bar{x}, \bar{z}))\xi^h + \nabla_z h(\bar{x}, \bar{z})^T \beta = 0, \quad (\text{B.13})$$

$$0 \leq \eta^g \perp g(\bar{x}, \bar{y}) \leq 0, \quad 0 \leq \eta^u \perp \bar{u} \geq 0, \quad (\text{B.14})$$

$$\nabla_z L(\bar{x}, \bar{z}, \bar{u}, \bar{v}) = 0, \quad h(\bar{x}, \bar{y}) = 0. \quad (\text{B.15})$$

By (B.6), we further have

$$\widehat{\alpha}(f(\bar{x}, \bar{y}) - L(\bar{x}, \bar{z}, \bar{u}, \bar{v})) = \alpha(f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{z})) - \alpha(\bar{u}^T g(\bar{x}, \bar{z}) + \bar{v}^T h(\bar{x}, \bar{z})) = 0.$$

Since $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is feasible to eMDP, we have $f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{z}) \leq 0$, $\bar{u} \circ g(\bar{x}, \bar{z}) \geq 0$, and $\bar{v} \circ h(\bar{x}, \bar{z}) = 0$, which implies

$$0 \leq \alpha \perp f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{z}) \leq 0, \quad (\text{B.16})$$

$$0 \leq \alpha \perp \bar{u}^T g(\bar{x}, \bar{z}) \geq 0 \quad (\text{B.17})$$

by the nonnegativity of α . Moreover, it is easy to verify that (B.17) is equal to

$$0 \leq \xi^g \perp \bar{u} \circ g(\bar{x}, \bar{z}) \geq 0. \quad (\text{B.18})$$

Thus, (B.9)-(B.15) and (B.16)-(B.18) mean that $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is a KKT point of eMDP.

(ii) We take the case of eMDP as an example since the other four cases TMDP/eTMDP/ETMDP/eETMDP can be shown in a similar way. In fact, if $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is a KKT point of MDP, there is $(\zeta^g, \zeta^h, \zeta^u, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) \in \mathbb{R}^{2p+q+2+m}$ such that

$$\begin{aligned} & \nabla_x F(\bar{x}, \bar{y}) + \widehat{\alpha}(\nabla_x f(\bar{x}, \bar{y}) - \nabla_x f(\bar{x}, \bar{z})) - \widehat{\gamma}(\nabla_x g(\bar{x}, \bar{z})\bar{u} + \nabla_x h(\bar{x}, \bar{z})\bar{v}) \\ & + \nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\widehat{\beta} + \nabla_x g(\bar{x}, \bar{y})\zeta^g + \nabla_x h(\bar{x}, \bar{y})\zeta^h = 0, \end{aligned} \quad (\text{B.19})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \widehat{\alpha}\nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y})\zeta^g + \nabla_y h(\bar{x}, \bar{y})\zeta^h = 0, \quad (\text{B.20})$$

$$-\widehat{\alpha}\nabla_z f(\bar{x}, \bar{z}) - \widehat{\gamma}(\nabla_z g(\bar{x}, \bar{z})\bar{u} + \nabla_z h(\bar{x}, \bar{z})\bar{v}) + \nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\widehat{\beta} = 0, \quad (\text{B.21})$$

$$-\widehat{\gamma}g(\bar{x}, \bar{z}) + \nabla_y g(\bar{x}, \bar{z})^T \widehat{\beta} - \zeta^u = 0, \quad (\text{B.22})$$

$$-\widehat{\gamma}h(\bar{x}, \bar{z}) + \nabla_z h(\bar{x}, \bar{z})^T \widehat{\beta} = 0, \quad (\text{B.23})$$

$$0 \leq \widehat{\alpha} \perp f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{z}) \leq 0, \quad (\text{B.24})$$

$$0 \leq \widehat{\gamma} \perp \bar{u}^T g(\bar{x}, \bar{z}) + \bar{v}^T h(\bar{x}, \bar{z}) \geq 0, \quad (\text{B.25})$$

$$0 \leq \zeta^g \perp g(\bar{x}, \bar{y}) \leq 0, \quad 0 \leq \zeta^u \perp \bar{u} \geq 0, \quad (\text{B.26})$$

$$\nabla_z L(\bar{x}, \bar{z}, \bar{u}, \bar{v}) = 0, \quad h(\bar{x}, \bar{y}) = 0. \quad (\text{B.27})$$

By letting $\eta^g = \zeta^g$, $\eta^h = \zeta^h$, $\eta^u = \zeta^u$, $\alpha = \widehat{\alpha}$, $\beta = \widehat{\beta}$, $\xi^g = (\widehat{\gamma}, \dots, \widehat{\gamma})^T \in \mathbb{R}^p$, and $\xi^h = -(\widehat{\gamma}, \dots, \widehat{\gamma})^T \in \mathbb{R}^q$, (B.19)-(B.27) are equivalent to

$$\begin{aligned} & \nabla_x F(\bar{x}, \bar{y}) + \alpha(\nabla_x f(\bar{x}, \bar{y}) - \nabla_x f(\bar{x}, \bar{z})) - \sum_{i=1}^p \nabla_x g_i(\bar{x}, \bar{z})\bar{u}_i \xi_i^g + \sum_{i=1}^q \nabla_x h_i(\bar{x}, \bar{z})\bar{v}_i \xi_i^h \\ & + \nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\beta + \nabla_x g(\bar{x}, \bar{y})\eta^g + \nabla_x h(\bar{x}, \bar{y})\eta^h = 0, \end{aligned} \quad (\text{B.28})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \alpha\nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y})\eta^g + \nabla_y h(\bar{x}, \bar{y})\eta^h = 0, \quad (\text{B.29})$$

$$-\alpha\nabla_z f(\bar{x}, \bar{z}) + \nabla_{zz}^2 L(\bar{x}, \bar{z}, \bar{u}, \bar{v})\beta - \sum_{i=1}^p \nabla_z g_i(\bar{x}, \bar{z})\bar{u}_i \xi_i^g + \sum_{i=1}^q \nabla_z h_i(\bar{x}, \bar{z})\bar{v}_i \xi_i^h = 0, \quad (\text{B.30})$$

$$-\text{diag}(g_1(\bar{x}, \bar{z}), \dots, g_p(\bar{x}, \bar{z}))\xi^g + \nabla_z g(\bar{x}, \bar{z})^T \beta - \eta^u = 0, \quad (\text{B.31})$$

$$\text{diag}(h_1(\bar{x}, \bar{z}), \dots, h_q(\bar{x}, \bar{z}))\xi^h + \nabla_z h(\bar{x}, \bar{z})^T \beta = 0, \quad (\text{B.32})$$

$$0 \leq \alpha \perp f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{z}) \leq 0, \quad (\text{B.33})$$

$$0 \leq \eta^g \perp g(\bar{x}, \bar{y}) \leq 0, \quad 0 \leq \eta^u \perp \bar{u} \geq 0, \quad (\text{B.34})$$

$$\nabla_z L(\bar{x}, \bar{z}, \bar{u}, \bar{v}) = 0, \quad h(\bar{x}, \bar{y}) = 0. \quad (\text{B.35})$$

By (B.25), we have $0 \leq \hat{\gamma} \perp \bar{u}^T g(\bar{x}, \bar{z}) + \bar{v}^T h(\bar{x}, \bar{z}) \geq 0$. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is feasible to eMDP, we have $h(\bar{x}, \bar{z}) = 0$, which implies

$$0 \leq \xi^g \perp \bar{u} \circ g(\bar{x}, \bar{z}) \geq 0. \quad (\text{B.36})$$

(B.28)-(B.36) mean that $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ is a KKT point of eMDP.

(iii) This result is almost the same to (ii) and so we omit its proof here.

(iv) This result is almost the same to (ii) and so we omit its proof here.

Proof of Theorem 9 For a feasible point $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ of MPCC, we define the following index sets:

$$I_{0+} = \{i : g_i(\bar{x}, \bar{y}) = 0, \bar{u}_i > 0\}, \quad I_{-0} = \{i : g_i(\bar{x}, \bar{y}) < 0, \bar{u}_i = 0\}, \quad I_{00} = \{i : g_i(\bar{x}, \bar{y}) = 0, \bar{u}_i = 0\}.$$

Recall that $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is strongly stationary to MPCC if there is $(\lambda^g, \lambda^h, \lambda^u, \lambda^L) \in \mathbb{R}^{2p+q+m}$ such that

$$\nabla_x F(\bar{x}, \bar{y}) + \nabla_{yx}^2 L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \lambda^L + \nabla_x g(\bar{x}, \bar{y}) \lambda^g + \nabla_x h(\bar{x}, \bar{y}) \lambda^h = 0, \quad (\text{B.37})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \nabla_{yy}^2 L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \lambda^L + \nabla_y g(\bar{x}, \bar{y}) \lambda^g + \nabla_y h(\bar{x}, \bar{y}) \lambda^h = 0, \quad (\text{B.38})$$

$$\nabla_y g(\bar{x}, \bar{y})^T \lambda^L - \lambda^u = 0, \quad (\text{B.39})$$

$$\nabla_y h(\bar{x}, \bar{y})^T \lambda^L = 0, \quad (\text{B.40})$$

$$\nabla_y L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) = 0, \quad g(\bar{x}, \bar{y}) \leq 0, \quad \bar{u} \geq 0, \quad h(\bar{x}, \bar{y}) = 0, \quad (\text{B.41})$$

$$\lambda_i^g = 0, \quad i \in I_{-0}, \quad (\text{B.42})$$

$$\lambda_i^u = 0, \quad i \in I_{0+}, \quad (\text{B.43})$$

$$\lambda_i^g \geq 0, \quad \lambda_i^u \geq 0, \quad i \in I_{00}. \quad (\text{B.44})$$

Note that the cases of WDP and MDP have been shown in Theorem 4.2 of Li et al. (2023) and Theorem 10 of Li et al. (2024b), respectively. Actually, the other cases of eMDP/TWDP/TMDP/eTMDP/ETMDP/eETMDP can be shown in a similar way. For completeness, we give a proof for the case of eMDP as an example. In fact, if $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is a KKT point of eMDP, there exists $(\eta^g, \eta^h, \eta^u, \alpha, \beta, \xi^g, \xi^h) \in \mathbb{R}^{3p+2q+1+m}$ such that

$$\nabla_x F(\bar{x}, \bar{y}) + \nabla_{yx}^2 L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \beta + \sum_{i=1}^p \nabla_x g_i(\bar{x}, \bar{y}) (\eta_i^g - \bar{u}_i \xi_i^g) + \sum_{i=1}^q \nabla_x h_i(\bar{x}, \bar{y}) (\eta_i^h + \bar{v}_i \xi_i^h) = 0, \quad (\text{B.45})$$

$$\nabla_y F(\bar{x}, \bar{y}) + \alpha \nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y}) \eta^g + \nabla_y h(\bar{x}, \bar{y}) \eta^h = 0, \quad (\text{B.46})$$

$$-\alpha \nabla_y f(\bar{x}, \bar{y}) + \nabla_{yy}^2 L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \beta - \sum_{i=1}^p \nabla_y g_i(\bar{x}, \bar{y}) u_i \xi_i^g + \sum_{i=1}^q \nabla_y h_i(\bar{x}, \bar{y}) v_i \xi_i^h = 0, \quad (\text{B.47})$$

$$-\text{diag}(g_1(\bar{x}, \bar{y}), \dots, g_p(\bar{x}, \bar{y})) \xi^g + \nabla_y g(\bar{x}, \bar{y})^T \beta - \eta^u = 0, \quad (\text{B.48})$$

$$\text{diag}(h_1(\bar{x}, \bar{y}), \dots, h_q(\bar{x}, \bar{y})) \xi^h + \nabla_y h(\bar{x}, \bar{y})^T \beta = 0, \quad (\text{B.49})$$

$$0 \leq \eta^g \perp g(\bar{x}, \bar{y}) \leq 0, \quad (\text{B.50})$$

$$0 \leq \eta^u \perp \bar{u} \geq 0, \quad (\text{B.51})$$

$$\nabla_z L(\bar{x}, \bar{y}, \bar{u}, \bar{v}) = 0, \quad h(\bar{x}, \bar{y}) = 0. \quad (\text{B.52})$$

Adding (B.46) and (B.47) together yields

$$\nabla_y F(\bar{x}, \bar{y}) + \nabla_{yy}^2 L(\bar{x}, \bar{y}, \bar{u}, \bar{v})\beta + \sum_{i=1}^p \nabla_y g_i(\bar{x}, \bar{y})(\eta_i^g - \bar{u}_i \xi_i^g) + \sum_{i=1}^q \nabla_y h_i(\bar{x}, \bar{y})(\eta_i^h + \bar{v}_i \xi_i^h) = 0. \quad (\text{B.53})$$

Set $\lambda^g = \eta^g - \bar{u} \circ \xi^g$, $\lambda^h = \eta^h + \bar{v} \circ \xi^h$, $\lambda^u = \eta^u + \xi^g \circ g(\bar{x}, \bar{y})$, and $\lambda^L = \beta$. We obtain (B.37)-(B.41) from (B.45) and (B.48)-(B.53) immediately. Next, we show (B.42)-(B.44).

- If $i \in I_{-0}$, which means $g_i(\bar{x}, \bar{y}) < 0$ and $\bar{u}_i = 0$, we have $\eta_i^g = 0$ by (B.50) and hence $\lambda_i^g = \eta_i^g - \bar{u}_i \xi_i^g = 0$. Namely, (B.42) holds.
- If $i \in I_{0+}$, which means $g_i(\bar{x}, \bar{y}) = 0$ and $\bar{u}_i > 0$, we have $\eta_i^u = 0$ by (B.51) and hence $\lambda_i^u = \eta_i^u + \xi_i^g g_i(\bar{x}, \bar{y}) = 0$. Namely, (B.43) holds.
- If $i \in I_{00}$, which means $g_i(\bar{x}, \bar{y}) = \bar{u}_i = 0$, we have $\lambda_i^g = \eta_i^g \geq 0$ and $\lambda_i^u = \eta_i^u \geq 0$ by (B.50) and (B.51) respectively. Namely, (B.44) holds.

This completes the proof.

Appendix C: Proof of Theorem 11

We only show the case of TWDP(t) because the cases of TMDP(t)/eTMDP(t)/ETMDP(t)/eETMDP(t) can be shown in a similar way. For simplicity, we take the upper constraint $(x, y) \in \Omega$ away from TWDP(t). In addition, we denote

$$\begin{aligned} \bar{w} &= (\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}), \quad w^k = (x^k, y^k, z^k, u^k, v^k), \quad w = (x, y, z, u, v), \\ F_0(w) &= F(x, y), \quad G_0(w) = f(x, y) - f(x, z) - u^T g(x, z), \\ G_i(w) &= g_i(x, y) \quad (1 \leq i \leq p), \quad \bar{G}_i(w) = -u_{i-p} \quad (p+1 \leq i \leq 2p), \quad H_i(w) = h_i(x, y) \quad (1 \leq i \leq q), \\ H_i(w) &= h_{i-q}(x, z) \quad (q+1 \leq i \leq 2q), \quad \bar{H}_i(w) = \nabla_{z_{i-2q}} L(x, z, u, v) \quad (2q+1 \leq i \leq 2q+m). \end{aligned}$$

Then, TWDP and TWDP(t_k) become

$$\begin{aligned} \min \quad & F_0(w) \\ \text{s.t.} \quad & G_i(w) \leq 0 \quad (i = 0, 1, \dots, 2p), \\ & H_i(w) = 0 \quad (i = 1, \dots, 2q+m) \end{aligned} \quad (\text{C.1})$$

and

$$\begin{aligned} \min \quad & F_0(w) \\ \text{s.t.} \quad & G_0(w) \leq t_k, \\ & G_i(w) \leq 0 \quad (i = 1, \dots, 2p), \quad H_i(w) = 0 \quad (i = 1, \dots, 2q+m). \end{aligned} \quad (\text{C.2})$$

Obviously, \bar{w} is a feasible point of TWDP and, without loss of generality, we may assume that the whole sequence $\{w^k\}$ converges to \bar{w} . Since w^k is a KKT point of (C.2), it follows from Lemma A.1 in Steffensen

and Ulbrich (2010) that there exists $(\lambda_0^k, \lambda^k, \mu^k) \in \mathbb{R} \times \mathbb{R}^{2p+1} \times \mathbb{R}^{q+m}$ such that

$$\begin{aligned} \nabla F_0(w^k) + \lambda_0^k \nabla G_0(w^k) + \nabla G(w^k) \lambda^k + \nabla H(w^k) \mu^k &= 0, \\ \lambda_0^k (G_0(w^k) - t_k) &= 0, \quad \lambda_0^k \geq 0, \\ G(w^k)^T \lambda^k &= 0, \quad \lambda^k \geq 0, \end{aligned} \tag{C.3}$$

and the gradients

$$\{\nabla G_i(w^k) : \lambda_i^k > 0, 0 \leq i \leq 2p\} \cup \{\nabla H_i(w^k) : \mu_i^k \neq 0\} \tag{C.4}$$

are linearly independent. To show that \bar{w} is a KKT point of (C.1), it is sufficient to show the boundedness of the sequence $\{\lambda_0^k, \lambda^k, \mu^k\}$.

Assume by contradiction that $\{\lambda_0^k, \lambda^k, \mu^k\}$ is unbounded. Taking a subsequence if necessary, we may assume

$$\lim_{k \rightarrow \infty} \frac{(\lambda_0^k, \lambda^k, \mu^k)}{\|(\lambda_0^k, \lambda^k, \mu^k)\|} = (\lambda_0^*, \lambda^*, \mu^*). \tag{C.5}$$

In particular, for every k sufficiently large, we have

$$\lambda_i^* > 0 \Rightarrow \lambda_i^k > 0, \quad \mu_i^* \neq 0 \Rightarrow \mu_i^k \neq 0. \tag{C.6}$$

Dividing by $\|(\lambda_0^k, \lambda^k, \mu^k)\|$ in (C.3) and taking a limit, we have

$$\lambda_0^* \nabla G_0(\bar{w}) + \sum_{\lambda_i^* > 0} \lambda_i^* \nabla G_i(\bar{w}) + \sum_{\mu_i^* \neq 0} \mu_i^* \nabla H_i(\bar{w}) = 0, \quad \lambda_0^* \geq 0. \tag{C.7}$$

Since $\|(\lambda_0^*, \lambda^*, \mu^*)\| = 1$ by (C.5), (C.7) implies that the gradients

$$\{\nabla G_i(\bar{w}) : \lambda_i^* > 0, 0 \leq i \leq 2p\} \cup \{\nabla H_i(\bar{w}) : \mu_i^* \neq 0\}$$

are positive-linearly dependent. Since problem (C.1) satisfies the CPLD in Qi and Wei (2000) at \bar{w} , it is not difficult to see that the gradients $\{\nabla G_i(w) : \lambda_i^* > 0, 0 \leq i \leq 2p\} \cup \{\nabla H_i(w) : \mu_i^* \neq 0\}$ are linearly dependent in a neighborhood. This, together with (C.6), contradicts the linear independence of the vectors in (C.4). Consequently, $\{\lambda_0^k, \lambda^k, \mu^k\}$ is bounded. This completes the proof.