

Online Supplement for “A Reduction-Driven Local Search for the Generalized Independent Set Problem”

1. Technical Proofs

Proof of R1. Consider whether u and v are included in a generalized independent set S . There are four cases: $u \in S$ and $v \in S$, $u \in S$ and $v \notin S$, $u \notin S$ and $v \in S$, and $u \notin S$ and $v \notin S$. When either $u \notin S$ or $v \notin S$, the edge $e = \{u, v\}$ does not exist in $G[S]$, and thus removing e has no effect on the net benefit $nb(S)$. If both $u \in S$ and $v \in S$, the edge $\{u, v\}$ has a zero penalty, which has no effect on the net benefit $nb(S)$ either. \square

Proof of R2. Assume, for the sake of contradiction, that there exists $I \in MGIS(G)$ such that $u \in I$ and $v \in I$. Assume, without loss of generality, that $\tilde{w}(u) \leq \tilde{w}(v)$. We have $nb(I) - nb(I \setminus \{u\}) = w(u) - \sum_{x \in N_r(u) \cap I} p(\{u, x\}) \leq w(u) - p(\{u, v\}) + \sum_{x \in N_r(u)} \max(0, -p(\{u, x\})) = \tilde{w}(u) - p(\{u, v\}) < 0$. In conclusion, $I \setminus \{u\}$ has a larger net benefit without containing permanent edges, which contradicts the optimality of I . Therefore, u and v cannot be simultaneously included in any $S \in MGIS(G)$, which is equivalent to the case where $\{u, v\} \in E_p$. \square

Proof of R3. We prove that u is in at least one set in $MGIS(G)$. Assume, for the sake of contradiction, that u is not included in any set in $MGIS(G)$. Let I be an arbitrary set in $MGIS(G)$. It is clear that at least one vertex in $N(u)$ must be included in I . Otherwise, $I \cup \{u\}$ implies with a larger net benefit without including any permanent edge, a contradiction, since $w(u) \geq w^+(N(u))$, $nb(I \cup \{u\} \setminus N(u)) \geq nb(I)$. So u is in at least one set in $MGIS(G)$. It holds that $\alpha(G) = \alpha(G') + w(u)$. \square

Proof of R4. We prove that u is in at least one set in $MGIS(G)$. Assume, for the sake of contradiction, that u is not included in any set in $MGIS(G)$. Let I be an arbitrary set in $MGIS(G)$. Let $S := N_p(u) \cap I$ and $S' := N_r(u) \cap I$. Now, we remove S from I and add u to I to obtain I' . Consider the difference between two net benefits $nb(I') - nb(I) \geq w(u) - \sum_{v \in S} \tilde{w}(v) - \sum_{v \in S'} p(\{u, v\}) \geq 0$. Moreover, $G[I']$ does not contain permanent edges. Therefore, $I' \in MGIS(G)$, which contradicts the assumption. So u is in at least one set in $MGIS(G)$. It holds that $\alpha(G) = \alpha(G') + w(u)$. \square

Proof of R5. We prove that u is not in any set in $MGIS(G)$. Given an arbitrary $I \in MGIS(G)$, there are two cases. When none of the neighbors of u is included in I , I does not include u because $w(u) < 0$. When any subset of neighbors of u is included in I , I does not include u either because $w(u) - \sum_{v \in S} p(\{u, v\}) < 0$ for any subset $S \subseteq N(u)$. Since I is an arbitrary set in $MGIS(G)$, u will not appear in any $I \in MGIS(G)$ and we can directly remove u from G . \square

Proof of R6. Observe that there exists a vertex set $I \in MGIS(G)$ such that $I \cap N_p[u] \neq \emptyset$, because if $I \cap N_p[u] = \emptyset$ then $nb(I \cup \{u\}) = nb(I) + w(u) \geq nb(I)$ implies $I \cup \{u\} \in MGIS(G)$.

Whenever a vertex x in $N_p(u)$ is included in I , other vertices in the clique $N_p[u] \setminus \{x\}$ must not be included in I . Then $I \cup \{u\} \setminus \{x\} \in MGIS(G)$ because $G[I \cup \{u\} \setminus \{x\}]$ does not contain permanent edges, and $nb(I \cup \{u\} \setminus \{x\}) \geq nb(I) + w(u) - w(x) + \sum_{v \in N_r(x)} p(\{v, x\}) - \sum_{v \in N_r(u)} \max(0, p(\{u, v\})) \geq nb(I) + w(u) - \sum_{v \in N_r(u)} \max(0, p(\{u, v\})) - \max(0, \tilde{w}(x)) \geq nb(I)$. In conclusion, u is in at least one set in $MGIS(G)$. \square

Proof of R7. Given an arbitrary $I \in MGIS(G)$.

When $w(u) \geq p(\{u, v\})$ and $w(u) \geq 0$. If $u \notin I$ and $v \in I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) = nb(I) + w(u) - p(\{u, v\}) \geq nb(I)$. If $u \notin I$ and $v \notin I$, then $I \cup \{u\} \in MGIS(G)$ because $G[I \cup \{u\}]$ does not contain permanent edges, and $nb(I \cup \{u\}) = nb(I) + w(u) \geq nb(I)$. In summary, u is included in at least one set in $MGIS(G)$.

When $w(u) < 0$ and $w(u) \geq p(\{u, v\})$. If $v \in I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) = nb(I) + w(u) - p(\{u, v\}) \geq nb(I)$ for $u \notin I$, and $G[I \cup \{u\}]$ does not contain permanent edges. If $v \notin I$, then $nb(I \cup \{u\}) < nb(I \setminus \{u\})$. In other words, there exists a set $S \in MGIS(G)$ such that $u \in S$ if and only if there exists a set $S' \in MGIS(G')$ such that $v \in S'$. It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, v that could appear in a set in $MGIS(G)$.

When $w(u) \geq 0$, $w(u) < p(\{u, v\})$ and $w(u) \geq \tilde{w}(v)$. If $u \notin I, v \in I$, then $nb(I \cup \{u\}) \setminus \{v\} - nb(I) = w(u) - w(v) - \sum_{x \in N_r(v) \cap I} \max(0, -p(\{v, x\})) = w(u) - \tilde{w}(v) \geq 0$, implying $I \cup \{u\} \setminus \{v\} \in MGIS(G)$. Otherwise, if $u \notin I, v \notin I$, then $nb(I \cup \{u\}) - nb(I) = w(u) \geq 0$, thus $I \cup \{u\} \in MGIS(G)$. In summary, u is included in at least one vertex set $I' \in MGIS(G)$ and v is not in I' .

When $w(u) \geq 0$, $w(u) < p(\{u, v\})$ and $w(u) < \tilde{w}(v)$. u and v cannot be both included in I . Indeed, if $u, v \in I$, then $nb(I \setminus \{u\}) = nb(I) + p(\{u, v\}) - w(u) > nb(I)$. Moreover, $G[I \setminus \{u\}]$ does not contain permanent edges. This contradicts the optimality of I . Furthermore, if $v \notin I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) - nb(I \setminus \{u\}) = w(u) \geq 0$. We conclude that there exists a vertex set in $MGIS(G)$ that includes u iff there exists a vertex set in $MGIS(G')$ that does not contain v . It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, v that could appear in a set in $MGIS(G)$.

Finally, when $w(u) < 0$ and $w(u) < p(\{u, v\})$. We have $nb(I \cup \{u\}) < nb(I \setminus \{u\})$, therefore u is not in any set in $MGIS(G)$ and we can directly remove u . \square

Proof of R8. Given an arbitrary $I \in MGIS(G)$. Because that $\{x, y\} \in E_p$, it holds that x and y cannot be both included in I .

Case 1: $w(u) \geq \max(p(\{u, x\}), p(\{u, y\}))$.

When $w(u) \geq 0$. If both x and y are not included in I and $u \notin I$, then $nb(I \cup \{u\}) = nb(I) + w(u) \geq nb(I)$. If $u \notin I$ and either $x \in I$ or $y \in I$, $nb(I \cup \{u\}) \geq nb(I) + w(u) - \max(p(\{u, x\}), p(\{u, y\})) \geq nb(I)$. In summary, there exists a vertex set in $MGIS(G)$ that includes u .

When $w(u) < 0$. If both x and y are not included in I , then $u \notin I$ because $nb(I \cup \{u\}) = nb(I \setminus \{u\}) + w(u) < nb(I \setminus \{u\})$. If either $x \in I$ or $y \in I$ and $u \notin I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) \geq nb(I) + w(u) - \max(p(\{u, x\}), p(\{u, y\})) \geq nb(I)$. We conclude that there exists a vertex set in $MGIS(G)$ that includes u iff there exists a vertex set in $MGIS(G')$ that includes either x or y . In other words, we can defer the decision of u after determining x and y . It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, x, y that could appear in a set in $MGIS(G)$.

Case 2: $p(\{u, y\}) \leq w(u) < p(\{u, x\})$. In this case, if y is in I , then $x \notin I$ and $I \cup \{u\} \in MGIS(G)$.

When $w(u) \geq 0$ and $w(u) \geq \tilde{w}(x)$. If x is in I , then y, u are not in I and $I \cup \{u\} \setminus \{x\} \in MGIS(G)$ because $nb(I \cup \{u\} \setminus \{x\}) \geq nb(I) + w(u) - \tilde{w}(x) \geq nb(I)$. If neither x nor y is included in I , then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) = nb(I) + w(u) \geq nb(I)$ for $u \notin I$. In summary, there exists a vertex set $I' \in MGIS(G)$ that includes u and not includes x .

When $w(u) \geq 0$ and $w(u) < \tilde{w}(x)$. There are five cases: $x \in I$ and $u, y \notin I$, $y \in I$ and $u, x \notin I$, $u \in I$ and $x, y \notin I$, $u, y \in I$ and $x \notin I$, and $u, x \in I$ and $y \notin I$. But u, x cannot appear in the same I , otherwise $nb(I \setminus \{u\}) = nb(I) + p(\{u, x\}) - w(u) > 0$ without containing permanent edges, which contradicts the optimality of I . If $y \in I$ and $u, x \notin I$, then $I \cup \{u\} \in MGIS(G)$. In summary, there exists a vertex set $I' \in MGIS(G)$ such that either $x \in I'$, or both $u, y \in I'$, or $u \in I'$. Since u only connects to x and y in G , we conclude that there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ that does not include x . In other words, we can defer u after the decision of x . It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, x, y that could appear in a set in $MGIS(G)$.

When $w(u) < 0$. If $y \notin I$, then $u \notin I$ because otherwise $nb(I \setminus \{u\}) > nb(I)$. Since u only connects to x and y in G , we conclude that there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ that includes y . In other words, we can defer u after the decision of y . It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, x, y that could appear in a set in $MGIS(G)$.

Case 3: $w(u) < p(\{u, x\})$ and $w(u) < p(\{u, y\})$. When $w(u) \geq 0$, the three vertices form a triangle where every edge has penalty M . The result follows from theorem 3.3 of Gu et al. (2021). In particular, (1) if $w(u) \geq 0$ and $w(u) \geq \tilde{w}(x)$, then u must be in at least one set in $MGIS(G)$. (2)

If $w(u) \geq 0$ and $\tilde{w}(y) \leq w(u) < \tilde{w}(x)$, then y must not be in at least one set in $MGIS(G)$. There exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ that does not include x . (3) If $w(u) \geq 0$ and $w(u) < \tilde{w}(y)$, then there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ that does not include both x and y . (4) When $w(u) < 0$, the result follows from R5. It is easy to verify $\alpha(G') = \alpha(G)$ in all conditions and all combinations of u, x, y that could appear in a set in $MGIS(G)$. \square

Proof of R9. Given an arbitrary $I \in MGIS(G)$. Notice that x and y can be both included in I .

Case 1: $w(u) \geq p(\{u, x\})$. In this case, if one of x, y is in I , say x , then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) - nb(I) \geq w(u) - p(\{u, x\}) \geq 0$ when $u \notin I$.

When $w(u) \geq 0$ and $w(u) \geq p(\{u, x\}) + p(\{u, y\})$. If either $x \in I$ or $y \in I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) - nb(I) \geq w(u) - \max(p(\{u, x\}) + p(\{u, y\}), p(\{u, x\})) \geq 0$ for $u \notin I$. If none of x, y is in I , then $I \cup \{u\} \in MGIS(G)$ because $w(u) \geq 0$. Therefore, u must belong to at least one set in $MGIS(G)$.

When $w(u) \geq 0$ and $w(u) < p(\{u, x\}) + p(\{u, y\})$. If both x, y are in I , then $nb(I \cup \{u\}) < nb(I)$. If either $x \notin I$ or $y \notin I$, then $nb(I \cup \{u\}) \geq nb(I)$. In other words, including u has negative profit only if both x and y are included in the same set in $MGIS(G)$. we conclude that there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that either $x \notin S'$ or $y \notin S'$. It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, x, y that could appear in a set in $MGIS(G)$.

When $w(u) < 0$, it holds that $w(u) \geq p(\{u, x\}) + p(\{u, y\})$ because $w(u) \geq p(\{u, x\})$ and $p(\{u, y\}) < 0$. If either $x \in I$ or $y \in I$, then $nb(I \cup \{u\}) \geq nb(I)$. If both $x \notin I$ and $y \notin I$, then $nb(I \cup \{u\}) < nb(I)$. In other words, including u has positive profit if and only if either x or y is included. we conclude that there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that either $x \in S'$ or $y \in S'$. It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of u, x, y that could appear in a set in $MGIS(G)$.

Case 2: $p(\{u, y\}) \leq w(u) < p(\{u, x\})$. In this case, we consider whether the three vertices u, x, y can be included in I .

When $w(u) \geq 0$ and $w(u) \geq p(\{u, x\}) + p(\{u, y\})$, the all three vertices may be included in I . All combinations of x, y, u may appear in I . Except that if x is included and y is not included in I , then $u \notin I$ because $nb(I \cup \{u\}) = nb(I \setminus \{u\}) + w(u) - p(\{u, x\}) < nb(I \setminus \{u\})$. In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$

such that either $x \notin S'$ or $y \in S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

When $w(u) \geq 0$ and $w(u) < p(\{u, x\}) + p(\{u, y\})$, the three vertices can not be included in I . Moreover, u and x cannot be both included by I because otherwise $nb(I \setminus u) > nb(I)$ without containing permanent edges, which contradicts the optimality of I . If $x \notin I$, then $I \cup \{u\} \in MGIS(G)$ because $nb(I \cup \{u\}) \geq nb(I \setminus \{u\})$. In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

When $w(u) < 0$ and $w(u) \geq p(\{u, x\}) + p(\{u, y\})$, the three vertices can be included in I . If $y \notin I$, then $nb(I \cup \{u\}) \leq nb(I \setminus \{u\})$ regardless of whether $x \in I$. If $y \in I$, then $nb(I \cup \{u\}) \geq nb(I \setminus \{u\})$ regardless of whether $x \in I$. In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $y \in S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

When $w(u) < 0$ and $w(u) < p(\{u, x\}) + p(\{u, y\})$, the three vertices can not be included in I . u is in I iff y is included and x is not included in I . In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$ and $y \in S'$. It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

Case 3: $w(u) < p(\{u, y\})$. When $w(u) \geq 0$, u is in at least one set $I \in MGIS(G)$ iff neither x nor y is included in I . In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$ and $y \notin S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

When $w(u) < 0$ and $w(u) < p(\{u, x\}) + p(\{u, y\})$, u is not in any $I \in MGIS(G)$ because otherwise $nb(I \setminus \{u\}) > nb(I)$ without containing permanent edges, a contradiction. Therefore, we can remove u directly.

When $w(u) < 0$ and $w(u) \geq p(\{u, x\}) + p(\{u, y\})$, u is in at least one set $I \in MGIS(G)$ iff both x and y are included in I . In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \in S'$ and $y \in S'$. It is easy to verify $\alpha(G') = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$. \square

Proof of R10. Observe that there exists a vertex set $I \in MGIS(G)$ that includes at least one of x, u must be included in I . Because if none of them is included in at least one set $I \in MGIS(G)$, then $nb(I \cup \{u\}) - nb(I) \geq w(u) - \sum_{v \in N_r(u)} \max(0, p(\{u, v\})) \geq 0$ and therefore $I \cup \{u\} \in MGIS(G)$.

Given an arbitrary $I \in MGIS(G)$. Furthermore, x and u cannot be both included in I since $\{u, x\} \in E_p$. In summary, there are two possible states in I : either $x \in I, u \notin I$ or $u \in I, x \notin I$. Therefore, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, u that could appear in a set in $MGIS(G)$. \square

Proof of R11. Observe that there exists a vertex set $I \in MGIS(G)$ that includes at least one of u, x, y . If none of them is included in I , then $nb(I \cup \{u\}) - nb(I) \geq w(u) - \sum_{v \in N_r(u)} \max(0, p(\{u, v\})) \geq 0$. In addition, at most one of u, x, y can be included in I because $\{u, x\} \in E_p, \{u, y\} \in E_p, \{x, y\} \in E_p$.

Given an arbitrary $I \in MGIS(G)$. When $w(u) \geq \tilde{w}(x)$. If one of x, y is included in I , say x , then $nb(I \cup \{u\} \setminus \{x\}) \geq nb(I)$. In summary, u must be included in at least one set in $MGIS(G)$.

When $\tilde{w}(y) \leq w(u) < \tilde{w}(x)$. If y is included in I , then $u \notin I$ and $nb(I \cup \{u\} \setminus \{y\}) \geq nb(I)$. Therefore, there is a vertex set $I' \in MGIS(G)$ such that $y \notin I'$ and exactly one of u, x is included in I' . We conclude that there exists a vertex set $S \in MGIS(G)$ such that $u \in S$ iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$.

When $w(u) < \tilde{w}(y)$. u is in at least one set $I \in MGIS(G)$ iff both x, y are not included in I . In other words, there exists a vertex set $S \in MGIS(G)$ that includes u iff there exists a vertex set $S' \in MGIS(G')$ such that $x \notin S'$ and $y \notin S'$. It is easy to verify $\alpha(G') + w(u) = \alpha(G)$ in all combinations of x, y, u that could appear in a set in $MGIS(G)$. \square

Proof of R12. Given an arbitrary $I \in MGIS(G)$.

When $w(u) \geq \tilde{w}(v) + w^+(N(u) \setminus N_p[v])$. If $v \in I$, then $u \notin I$. We can replace $N(u) \cap I$ with u to get a larger or equal net benefit because $nb(I \cup \{u\} \setminus N(u)) - nb(I) \geq w(u) - \tilde{w}(v) + w^+(N(u) \setminus N_p[v]) \geq w(u) - \tilde{w}(v) + w^+(N(u) \cap I) \geq 0$. Therefore, $I \cup \{u\} \setminus N(u) \in MGIS(G)$. Since $v \in N_p(u)$, there must exist at least one set in $MGIS(G)$ that does not include v .

When $w(u) \geq \tilde{w}(v) + w^+(N_p(u) \setminus N_p[v]) + \sum_{x \in N_r(u) \setminus N_p(v)} \max(0, p(\{u, x\}))$. If $v \in I$, then $u \notin I$. We can replace $N_p(u) \cap I$ with u to get a larger net benefit because $nb(I \cup \{u\} \setminus N_p(u)) - nb(I) \geq w(u) - \tilde{w}(v) + w^+(N_p(u) \setminus N_p(v)) + \sum_{x \in N_r(u) \setminus N_p(v)} \max(0, p(\{u, x\})) \geq w(u) - \tilde{w}(v) + w^+(N(u) \cap I) + \sum_{x \in N_r(u) \cap I} \max(0, p(\{u, x\})) \geq 0$. Since $v \in N_p(u)$, there must exist a set in $MGIS(G)$ that does not include v . \square

Proof of R13. Assume, for the sake of contradiction, that neither u nor v is included in any set in $MGIS(G)$. If for any $I \in MGIS(G)$, $N(v) \cap I = \emptyset$, then $I \cup \{v\} \in MGIS(G)$ because $nb(I \cup \{v\}) \geq$

$nb(I)$ without containing permanent edges, a contradiction. Note that $u \notin N(v) \cap I$. If there exists a set $I \in MGIS(G)$ such that $N(v) \cap I \neq \emptyset$, then $w(v) \geq w^+(N(v)) - \max(0, w(u)) \geq w^+(S)$. $nb(I \cup \{v\} \setminus N(v)) \geq nb(I)$ without containing permanent edges, again a contradiction. Therefore, there exists a set in $MGIS(G)$ that includes at least one of u, v . We conclude that $N_p(u) \cap N_p(v)$ must not be included in at least one set in $MGIS(G)$. \square

Proof of R14. Given an arbitrary $I \in MGIS(G)$. If u is included in I , then $N_p(v) \cap I = \emptyset$. $I \cup \{v\} \in MGIS(G)$ because $nb(I \cup \{v\}) - nb(I \setminus \{v\}) = w(v) - \sum_{x \in N_r(v)} \max(0, p(\{v, x\})) \geq 0$. The same argument holds when v is included in I . Therefore, we can fold u and v into a vertex v' . There exists $S \in MGIS(G)$ such that $u \in S$ and $v \in S$ iff there exists $S' \in MGIS(G')$ such that $v' \in S'$. \square

2. Examples of Reduction 8 and 9

Examples of $R8$ are presented in Figure 1. For example, in the first plot, u satisfies case 1 and $w(u) \geq 0$. Then by $R8.1.1$, we can remove u from the original graph G (left) to get the reduced graph G' (right) and $\alpha(G) = \alpha(G') + 10$. Similarly, examples of $R9$ are presented in Figure 2.

Figure 1 Examples of $R8$. Suppose that vertex u is adjacent to x and y , and $\{x, y\} \in E_p$. G_R represents the residual graph $G[V \setminus \{u, x, y\}]$. α and α' indicate the optimal objective value in the original graph and the graph after reduction, respectively.

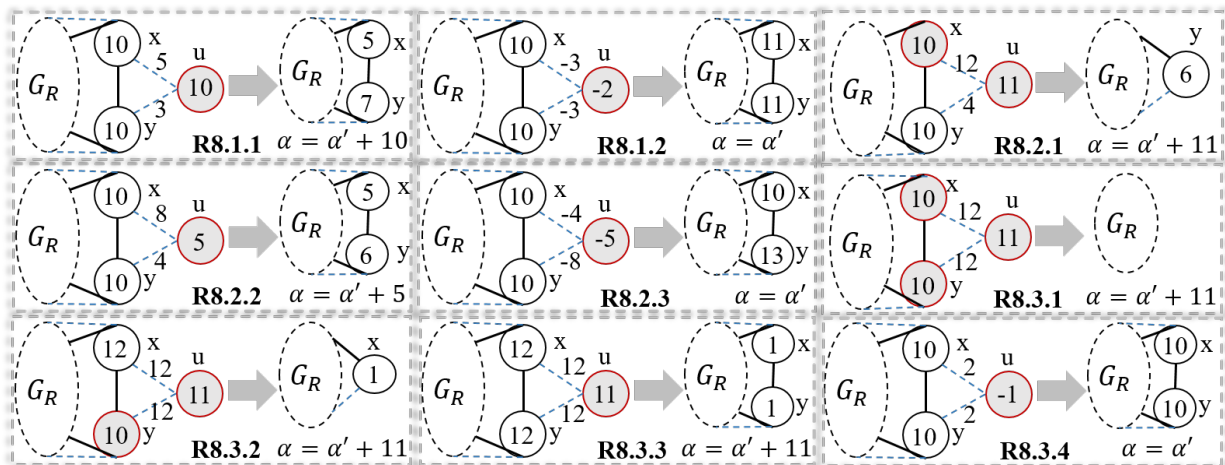
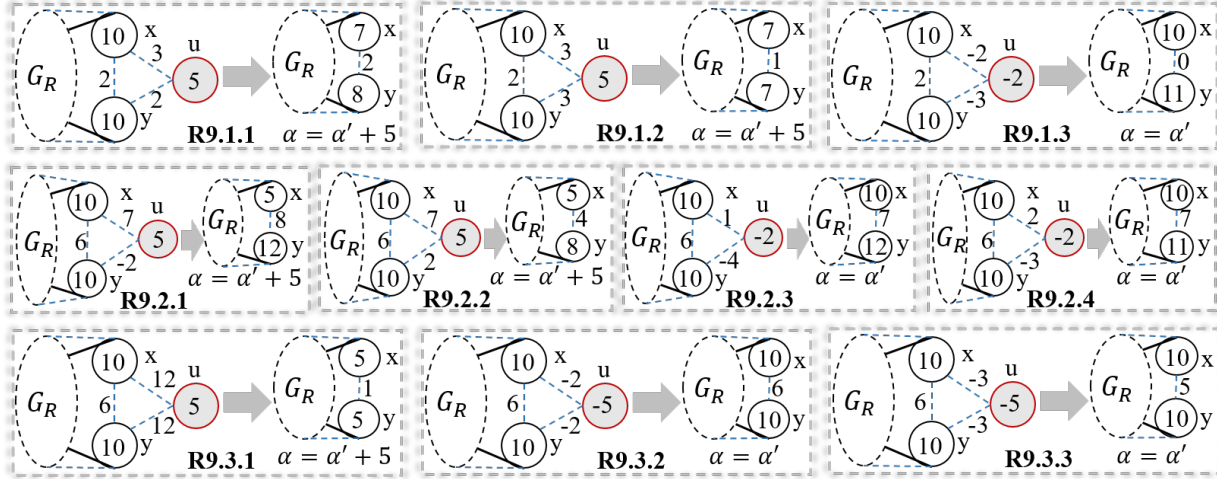


Figure 2 Examples of R9. Suppose that vertex u is adjacent to vertices x and y , and $\{x, y\} \in E_r$ or $\{x, y\} \notin E_r$.

3. Integer Programming of the GIS Problem

The GIS problem can be conveniently formulated as follows (Colombi et al. 2017):

$$\alpha(G) = \max \sum_{i \in V} w(i)x_i - \sum_{\{i,j\} \in E_r} p(\{i,j\})y_{ij} \quad (1)$$

$$\text{s.t. } x_i + x_j \leq 1 \quad \{i,j\} \in E_p \quad (2)$$

$$x_i + x_j - y_{ij} \leq 1 \quad \{i,j\} \in E_r \quad (3)$$

$$x_i \geq y_{ij}, x_j \geq y_{ij} \quad \{i,j\} \in E_r \quad (4)$$

$$x_i \in \{0, 1\} \quad i \in V \quad (5)$$

$$y_{ij} \in \{0, 1\} \quad \{i,j\} \in E_r \quad (6)$$

where binary variables x_i indicate whether the vertex i is selected, and binary variables y_{ij} indicate whether the removable edge $\{i, j\}$ is included.

4. Detailed Computational Results

The detailed comparative results on SET-2 and SET-3 are presented in Tables 1 and 2. The first four columns list instance information. The remaining columns show the best-known objective value (BKV), the number of vertices in the BKV solution ($|V_I|$), the mean running time for the heuristic algorithm to first hit the BKV ($htime$), the kernel graph size ($|V_{ker}|$) and the time of generating a kernel ($time_{ker}$). The last two rows report statistical results. Values marked with * denote the optimal objective value. Note that RLS's $htime$ includes the pre-processing time for building graph kernels. The detailed ablation results on SET-2 and SET-3 are presented in Tables 3.

Table 1 Comparative results of RLS on SET-2 instances

Instance	V	E _p	E _r	CPLEX			LA-B&B			ALS			RLS					
				BKV	V _f	htime(s)	BKV	V _f	htime(s)	BKV	V _f	htime(s)	BKV	V _f	htime(s)	V _{ker}	time _{ker} (s)	
bio-yeast	1,458	980	968	68,574*	1,111	<1	66,545	1,062	2	68,572	1,113	9	68,574*	1,111	<1	0	<1	
soc-wiki-Vote	889	727	2,187	37,358*	648	<1	35,973	615	<1	37,358*	648	8	37,358*	648	<1	0	<1	
vc-exact_131	2,980	1,191	4,169	109,641*	2,098	<1	107,812	2,045	14	109,392	2,121	14	109,641*	2,098	<1	0	<1	
web-edu	3,031	4,813	1,661	114,140*	1,752	<1	109,856	1,644	10	113,781	1,746	16	114,140*	1,753	<1	98	<1	
MANN_a81	3,321	3,118	3,362	117,327*	2,084	<1	114,616	1,999	16	116,907	2,125	14	117,327*	2,084	<1	0	<1	
tech-routers-rf	2,113	1,539	5,093	97,799*	1,668	<1	93,690	1,588	54	97,799*	1,668	11	97,799*	1,667	<1	0	<1	
vc-exact_039	6,795	2,574	8,046	259,542*	5,018	<1	254,941	4,908	253	258,137	5,164	9	259,542*	5,018	<1	0	<1	
ca-GrQc	4,158	3,179	10,243	174,782*	2,991	9	167,846	2,871	45	174,654	2,997	15	174,782*	2,994	11	282	<1	
vc-exact_038	786	10,321	3,703	12,015*	208	589	10,406	168	1,994	12,014	208	12	12,015*	207	4	697	<1	
vc-exact_078	11,349	4,289	13,450	438,671*	8,330	<1	431,371	8,128	1,074	435,412	8,671	<1	438,671*	8,330	<1	0	<1	
vc-exact_087	13,590	5,126	16,114	518,500*	10,025	2	509,734	9,809	2,319	517,500	10,083	2	518,500*	10,026	<1	0	<1	
vc-exact_107	6,402	15,646	5,594	407,878*	6,763	<1	396,456	6,488	10,646	400,223	6,846	1	407,878*	6,763	<1	0	<1	
vc-exact_151	15,783	12,144	12,519	530,138*	9,526	2	516,445	9,135	2,402	522,462	9,764	1	530,138*	9,526	<1	0	<1	
vc-exact_167	15,783	18,235	6,428	465,178*	7,797	1	449,579	7,438	10,469	457,272	7,923	2	465,178*	7,797	<1	0	<1	
bio-dimela	7,393	18,897	6,672	302,992*	4,952	<1	280,897	4,474	170	302,449	4,959	10	302,992*	4,951	<1	44	<1	
vc-exact_011	9,877	19,102	6,871	303,793*	5,369	1	286,137	4,950	4,818	301,789	5,487	7	303,793*	5,387	<1	172	<1	
web-spam	4,767	8,089	28,386	190,507*	3,285	4.61	180,446	3,130	53	190,344	3,299	14	190,507*	3,289	13	487	<1	
vc-exact_026	6,140	27,159	9,608	201,509*	3,887	1	177,902	3,372	668	201,316	3,922	17	201,509*	3,900	<1	300	<1	
vc-exact_001	6,160	19,655	20,552	215,941*	4,129	2	194,984	3,709	9,575	215,668	4,175	13	215,941*	4,128	<1	134	<1	
vc-exact_024	7,620	11,367	35,926	236,109*	4,632	10,783	226,540	4,438	244	235,440	4,735	14	236,109*	4,659	12	1,255	<1	
C1000.9	1,000	11,989	37,432	5,263	99	10,800	6,104	116	430	7,229	126	12	7,239	125	10	1,000	<1	
tech-WHOIS	7,476	42,181	14,762	325,584*	5,368	5	309,903	5,050	218	325,327	5,367	14	325,581*	5,371	13	338	<1	
p-hat700-3	700	30,398	31,242	4,025	68	10,800	3,030	52	7,436	4,473	78	<1	4,473	78	<1	700	<1	
vc-exact_008	7,537	35,678	37,155	247,091*	4,800	3	210,126	4,089	9,093	246,771	4,839	14	247,091*	4,810	<1	574	<1	
keller5	776	17,800	56,910	2,373	58	10,800	2,860	54	6,327	3,877	69	1	3,877	69	2	776	<1	
vc-exact_194	1,150	39,520	41,331	3,734	77	10,708	4,386	73	1,324	5,419	86	<1	5,419	86	<1	1,150	<1	
hamming10-4	1,024	66,369	23,231	3,256	52	8,108	3,073	45	2,069	3,660	54	<1	3,660	54	<1	1,024	<1	
brock800.2	800	26,629	84,805	1,177	49	10,800	1,996	36	10	2,473	44	<1	2,473	43	<1	800	<1	
brock800.4	800	26,831	85,126	1,583	48	10,800	1,979	39	7,215	2,501	43	<1	2,501	42	<1	800	<1	
ca-HepPh	11,204	57,721	59,898	372,096	6,094	10,800	348,738	5,571	422	370,779	6,083	6	372,079	6,098	17	2,162	<1	
p-hat700-2	700	90,788	32,134	2,870	41	10,140	2,345	37	2,078	2,971	42	<1	2,971	42	<1	700	<1	
vc-exact_196	1,534	93,273	32,809	3,916	62	10,800	4,273	57	66	5,321	76	<1	5,321	76	<1	1,534	<1	
p-hat700-1	700	135,804	47,847	730	11	10,800	844*	11	5	844*	11	<1	844*	11	<1	700	<1	
ca-AstroPh	17,903	47,135	149,837	604,288	10,468	10,800	558,736	9,752	2,544	602,320	10,521	2	604,019	10,491	14	4,985	1	
C2000.9	2,000	147,548	51,920	3,958	65	9,489	5,477	73	11	6,396	86	3	6,396	86	3	2,000	<1	
DSJC1000.5	1,000	59,694	189,980	925	39	7,645	1,515	30	694	1,861	32	1	1,861	32	<1	1,000	<1	
socfb-MIT	6,402	123,064	128,166	7,098	124	10,800	110,677	1,779	20	129,807	2,142	17	129,917	2,132	16	5,494	<1	
p-hat1500-3	1,500	66,206	210,800	4,097	132	10,800	3,877	83	704	7,501	148	14	7,501	148	16	1,500	<1	
vc-UCSB37	14,917	356,918	125,297	16,708	287	10,800	219,965	3,286	256	254,965	4,013	16	255,924	3,994	22	13,655	3	
socfb-Duke14	9,885	374,731	131,706	11,565	198	10,800	125,940	1,887	39	151,169	2,415	15	151,365	2,390	18	9,003	4	
p-hat1500-2	1,500	411,372	143,918	3,542	53	10,800	2,611	43	2,150	4,253	68	<1	4,253	68	<1	1,500	<1	
socfb-Stanford3	11,586	136,823	431,486	7,250	122	10,800	215,381	3,932	265	251,350	4,441	12	251,932	4,436	14	9,380	2	
socfb-UConn	17,206	448,038	156,829	16,996	279	10,800	231,718	3,471	322	274,637	4,329	12	275,320	4,299	18	16,025	5	
p-hat1500-1	1,500	411,744	427,583	0	0	10,800	1,184	18	5,499	1,287	19	<1	1,287	19	<1	1,500	<1	
C2000.5	2,000	489,894	509,270	0	0	10,800	1,581	24	865	1,849	26	2	1,849	26	6	2,000	1	
keller6	3,361	759,816	266,766	0	0	10,800	3,545	55	1,561	5,236	73	10	5,236	73	16	3,361	<1	
C4000.5	4,000	1,957,884	2,039,848	0	0	10,800	1,611	25	1,659	1,948	28	20	1,952	28	22	4,000	9	
# of best / # of instances				24/47	-	-	1/47	-	-	18/47	-	-	41/47	-	-	-	-	-
average running time(s)				-	-	5,819.2	-	-	2,087.4	-	-	7.9	-	-	5.9	-	-	1.4

Table 2 Comparative results of RLS on SET-3 instances

Instance	V	E _p	E _r	CPLEX			LA-B&B			ALS			RLS				
				BKV	V _f	htime(s)	BKV	V _f	htime(s)	BKV	V _f	htime(s)	BKV	V _f	htime(s)	V _{ker}	time _{ker} (s)
foodweb-wet	128	1,509	566	2,464*	44	<1	2,464*	44	8	2,464*	44	<1	2,464*	44	<1	125	<1
foodweb-dry	128	1,571	535	2,226*	42	<1	2,226*	42	8	2,226*	42	<1	2,226*	42	<1	125	<1
USAir97	332	1,589	537	10,308*	194	<1	9,507	178	407	10,308*	194	<1	10,308*	194	<1	0	<1
powergrid	4,941	1,727	4,867	193,339*	3,710	<1	190,537	3,684	108	193,095	3,754	59	193,339*	3,537	<1	0	<1
CondMat	23,133	46,769	46,670	711,517*	12,821	12	685,442	12,061	5,749	705,901	1,306	14	711,334	12,870	45	1,648	<1
Email	265,009	91,093	273,388	13,054,427*	257,435	118	OOM	-	-	OOM	-	-	13,054,427*	257,394	1	0	1
Epinion	75,879	202,653	203,087	3,052,059*	57,976	60	OOM	-	-	3,038,533	58,267	18	3,052,059*	58,008	1	253	1
Dblp	317,080	787,268	262,598	9,737,911*	171,421	790	OOM	-	-	OOM	-	-	9,734,653	171,403	64	11,851	11
cnr-2000	325,557	685,686	2,053,283	13,367,460	260,331	10,800	OOM	-	-	OOM	-	-	13,368,670	260,716	482	59,052	

Table 3 Comparative results of algorithms with or without reduction rules

Instance	V	V _{ker}	CPLEX				LA-B&B				RLS			
			without reduction		with reduction		without reduction		with reduction		without reduction		with reduction	
			BKV	htime(s)	BKV	htime(s)	BKV	htime(s)	BKV	htimes	BKV	htime(s)	BKV	htime(s)
bio-yeast	1,458	0	68,574*	<1	68,574*	<1	66,545	2	68,574*	<1	68,574*	1	68,574*	<1
soc-wiki-Vote	889	0	37,358*	<1	37,358*	<1	35,973	<1	37,358*	<1	37,358*	1	37,358*	<1
vc-exact_131	2,980	0	109,641*	<1	109,641*	<1	107,812	14	109,641*	<1	109,567	9	109,641*	<1
web-edu	3,031	98	114,140*	<1	114,140*	<1	109,856	10	114,140*	21	114,080	12	114,140*	<1
MANN_a81	3,321	0	117,327*	<1	117,327*	<1	114,616	16	117,327*	<1	117,275	14	117,327*	<1
tech-routers-rf	2,113	0	97,799*	<1	97,799*	<1	93,690	54	97,799*	<1	97,799*	1	97,799*	<1
vc-exact_039	6,795	0	259,542*	<1	259,542*	<1	254,941	253	259,542*	<1	259,475	13	259,542*	<1
ca-GrQc	4,158	282	174,782*	9	174,782*	6	167,846	45	174,525	2,998	174,768	7	174,782*	11
vc-exact_038	786	697	12,015*	589	12,015*	585	10,406	1,994	10,782	28	12,015*	13	12,015*	4
vc-exact_078	11,349	0	438,671*	<1	438,671*	1	431,371	1,074	438,671*	1	438,306	16	438,671*	<1
vc-exact_087	13,590	0	518,500*	2	518,500*	<1	509,734	2,319	518,500*	<1	518,069	15	518,500*	<1
vc-exact_107	6,402	0	407,878*	<1	407,878*	<1	396,456	10,646	407,878*	<1	406,379	14	407,878*	<1
vc-exact_151	15,783	0	530,138*	2	530,138*	<1	516,445	2,402	530,138*	<1	529,003	18	530,138*	<1
vc-exact_167	15,783	0	465,178*	1	465,178*	<1	449,579	10,469	465,178*	<1	463,446	12	465,178*	<1
bio-dmela	7,393	44	302,992*	<1	302,992*	<1	280,897	170	302,992*	<1	302,793	18	302,992*	<1
vc-exact_011	9,877	172	303,793*	1	303,793*	<1	286,137	4,818	303,589	1271	303,352	9	303,793*	<1
web-spam	4,767	487	190,507*	4.61	190,507*	<1	180,446	53	189,897	<1	190,460	18	190,507*	<1
vc-exact_026	6,140	300	201,509*	1	201,509*	<1	177,902	668	200,868	<1	201,458	13	201,503	<1
vc-exact_001	6,160	134	215,941*	2	215,941*	<1	194,984	9,575	215,941*	2	215,934	13	215,941*	<1
vc-exact_024	7,620	1,255	236,109	10,783	236,109	10,800	226,540	244	234,933	1	235,911	14	236,018	12
C1000.9	1,000	1,000	5,263	10,800	5,263	10,800	6,104	430	6,104	430	7,235	16	7,237	10
tech-WHOIS	7,476	338	325,584*	5	325,584*	<1	309,903	218	325,119	255	325,480	16	325,581	13
p-hat700-3	700	700	4,025	10,800	4,025	10,800	3,030	7,436	3,030	7,436	4,473	1	4,473	<1
vc-exact_008	7,537	574	247,091*	3	247,091*	<1	210,126	9,093	245,289	6,627	247,036	16	247,091*	<1
keller5	776	776	2,373	10,800	2,373	10,800	2,860	6,327	2,860	6,327	3,877	2	3,877	2
vc-exact_194	1,150	1,150	3,734	10,708	3,734	10,800	4,386	1,324	4,386	1,324	5,419	2	5,419	<1
hamming10-4	1,024	1,024	3,256	8,108	3,256	10,800	3,073	2,069	3,073	2,069	3,660	1	3,660	<1
brock800-2	800	800	1,177	10,800	1,177	10,800	1,996	10	1,996	10	2,473	1	2,473	<1
brock800-4	800	800	1,583	10,800	1,583	10,800	1,979	7,215	1,979	7,215	2,501	1	2,501	<1
ca-HepPh	11,204	2,165	372,096	10,800	372,229	10,800	348,738	422	368,637	2	372,079	14	372,079	17
p-hat700-2	700	700	2,870	10,140	2,870	10,140	2,345	2,078	2,345	2,078	2,971	1	2,971	<1
vc-exact_196	1,534	1,534	3,916	10,800	3,916	10,800	4,273	66	4,273	66	5,321	2	5,321	<1
p-hat700-1	700	700	730	10,800	730	10,800	844*	5	844*	5	844*	1	844*	<1
ca-AstroPh	17,903	4,985	604,288	10,800	604,342	10,800	558,736	2,544	593,899	37	603,732	15	603,945	14
C2000.9	2,000	2,000	3,958	9,489	3,958	10,800	5,477	11	5,477	11	6,396	3	6,396	3
DSJC1000.5	1,000	1,000	925	7,645	925	10,800	1,515	694	1,515	694	1,861	1	1,861	<1
socfb-MIT	6,402	5,494	7,098	10,800	118,114	10,800	110,677	20	113,623	10	129,831	16	129,872	16
p-hat1500-3	1,500	1,500	4,097	10,800	4,097	10,800	3,877	704	3,877	704	7,500	15	7,501	16
socfb-UCSB37	14,917	13,655	16,708	10,800	57,084	10,800	219,965	256	223,780	150	255,887	21	255,898	22
socfb-Duke14	9,885	9,003	11,565	10,800	127,145	10,800	125,940	39	127,269	2,231	151,321	14	151,326	18
p-hat1500-2	1,500	1,500	3,542	10,800	3,542	10,800	2,611	2,150	2,611	2,150	4,253	1	4,253	<1
socfb-Stanford3	11,586	9,380	7,250	10,800	127,708	10,800	215,381	265	223,398	7,756	251,788	19	251,833	14
socfb-UConn	17,206	16,025	16,996	10,800	51,328	10,800	231,718	322	235,104	240	275,006	16	275,034	20
p-hat1500-1	1,500	1,500	0	10,800	0	10,800	1,184	5,499	1,184	5,499	1,287	2	1,287	1
C2000.5	2,000	2,000	0	10,800	0	10,800	1,581	865	1,581	865	1,849	5	1,849	6
keller6	3,361	3,361	0	10,800	0	10,800	3,545	1,561	3,545	1,561	5,229	19	5,236	16
C4000.5	4,000	4,000	0	10,800	0	10,800	1,611	1,659	1,611	1,659	1,951	15	1,951	22
foodweb-wet	128	125	2,464*	<1	2,464*	<1	2,464*	8	2,464*	6	2,464*	<1	2,464*	<1
foodweb-dry	128	125	2,226*	<1	2,226*	<1	2,226*	8	2,226*	8	2,226*	<1	2,226*	<1
USAir97	332	0	10,308*	<1	10,308*	<1	9,507	407	10,308*	<1	10,308*	<1	10,308*	<1
powergrid	4,941	0	193,339*	<1	193,339*	<1	190,537	108	193,339*	<1	193,194	52	193,339*	<1
CondMat	23,133	1,648	711,517*	12	711,517*	<1	685,442	5,749	708,907	84	709,927	54	711,334	45
Email	265,009	0	13,054,427*	118	13,054,427*	1	OOM	–	13,054,427*	1	13,054,390	69	13,054,427*	1
Epinion	75,879	253	3,052,059*	60	3,052,059*	1	OOM	–	3,051,256	196	3,050,415	67	3,052,059*	1
Dblp	317,080	11,851	9,737,911*	790	9,737,911*	39	OOM	–	9,715,860	307	9,577,609	46	9,734,653	64
cnr-2000	325,557	59,052	13,367,460	10,800	13,641,278	10,800	OOM	–	11,347,241	10,796	13,361,167	84	13,368,670	482
WikiTalk	2,394,385	8	118,708,326*	3,566	118,708,326*	333	OOM	–	118,708,326*	333	118,672,055	56	118,708,326*	333
BerkStan	685,230	114,191	26,036,210	10,800	26,055,767	10,800	OOM	–	22,591,394	1,031	25,813,846	200	26,031,704	1,112
As-Skitter	1,696,415	31,173	66,603,360	10,800	66,830,857	10,800	OOM	–	65,907,243	10,512	59,886,304	200	66,582,254	723
soc-pokec	1,632,803	881,319	4,146,172	10,800	27,037,851	10,800	OOM	–	OOM	–	36,540,838	350	43,999,402	934
LiveJ	4,846,609	401,307	0	10,800	163,061,840	10,800	OOM	–	OOM	–	148,758,359	318	168,431,056	904
uk2002	18,483,186	4,806,660	0	100,000	567,666,923	100,000	OOM	–	OOM	–	633,365,020	1,806	687,089,901	7,678
# of best / # of instances			31/62	–	36/62	–	3/62	–	21/62	–	22/62	–	51/62	–
average running time(s)			–	7,096.5	–	6,901.0	–	2,007.5	–	1,441.1	–	60.8	–	202.6