

Online Supplement for “Stationary Coupled-Flow Approximation for Time-Varying Retrial Queues”

The online supplement contains the proofs of the main results (Section A) and the numerical experiments to test the accuracy of the SCF approximation (Section B).

Appendix A: Proofs of the Main Results

A.1. Proofs of Lemma 1, Theorem 1, and Corollary 1

Proof of Lemma 1. Let us rewrite $x(1 - qA(x))$ as

$$x(1 - qA(x)) = x(1 - A(x)) + (1 - q)xA(x).$$

The function $x \mapsto x(1 - A(x))$ is the rate of served customers in an $M/M/s + G$ queue, and the function $x \mapsto xA(x)$ is the rate of customers who abandon the queue in an $M/M/s + G$ queue. These two quantities are increasing in the arrival rate as proven in [Bhattacharya and Ephremides \(1991\)](#). Consequently, $x \mapsto x(1 - qA(x))$ is increasing in x . The limits of $x \mapsto x(1 - A(x))$ when $x \rightarrow 0$ and $x \rightarrow \infty$ can be deduced from those of $x \mapsto A(x)$ which tends to 0 as x tends to 0, and tends to 1 as x tends to infinity. $\square$

Proof of Theorem 1. We first prove that λ_i is increasing and bounded and next determine the equation that determines the limit of λ_i .

Increasing property. We prove the result by induction on i . We have

$$\lambda_2 - \lambda_1 = \lambda_1 A(\lambda_1) \beta r_1 + \lambda_2 A(\lambda_2) \beta r_0 - \lambda_1 A(\lambda_1) \beta r_0.$$

Consequently,

$$\lambda_2(1 - \beta r_0 A(\lambda_2)) - \lambda_1(1 - \beta r_0 A(\lambda_1)) = \lambda_1 A(\lambda_1) \beta r_1 > 0.$$

From Lemma 1, the function $x \mapsto x(1 - \beta r_0 A(x))$ is increasing in x . Therefore, we deduce that $\lambda_2 > \lambda_1$. Assume now that $\lambda_1 < \lambda_2 < \dots < \lambda_i$. With the convention that $\lambda_0 = 0$, we deduce that

$$\lambda_{i+1} - \lambda_i = \sum_{k=1}^{i+1} \lambda_k A(\lambda_k) \beta r_{i+1-k} - \sum_{k=1}^i \lambda_k A(\lambda_k) \beta r_{i-k} = \sum_{k=0}^i (\lambda_{k+1} A(\lambda_{k+1}) - \lambda_k A(\lambda_k)) \beta r_{i-k}.$$

Therefore,

$$\lambda_{i+1}(1 - \beta r_0 A(\lambda_{i+1})) - \lambda_i(1 - \beta r_0 A(\lambda_i)) = \sum_{k=0}^{i-1} (\lambda_{k+1} A(\lambda_{k+1}) - \lambda_k A(\lambda_k)) \beta r_{i-k} > 0.$$

The term on the right-hand side of this equation is positive due to the assumption $\lambda_1 < \lambda_2 < \dots < \lambda_i$ and the fact that $x \mapsto xA(x)$ is increasing in x . Consequently, $\lambda_{i+1}(1 - \beta r_0 A(\lambda_{i+1})) - \lambda_i(1 - \beta r_0 A(\lambda_i)) > 0$. From Lemma 1, this proves that $\lambda_{i+1} - \lambda_i > 0$. This proves the induction step. Thus, λ_i is increasing in i .

The relation $\alpha_i \geq \lambda_i$. We now prove by induction on i that $\alpha_i \geq \lambda_i$ for $i \geq 1$.

We have $\alpha_1 = \frac{\lambda}{1 - \beta r_0}$ and λ_1 is the unique solution to $\lambda_1(1 - A(\lambda_1)\beta r_0) = \lambda$. Therefore,

$$\alpha_1 = \lambda_1 \frac{1 - A(\lambda_1)\beta r_0}{1 - \beta r_0}.$$

Given that $A(\lambda_1) \leq 1$, we deduce that $\alpha_1 \geq \lambda_1$.

Assume now that $\alpha_k \geq \lambda_k$, for $k = 1, 2, \dots, i-1$. We have

$$\alpha_i(1 - \beta r_0) - \lambda_i(1 - \beta r_0 A(\lambda_i)) = \sum_{k=1}^{i-1} (\alpha_k - \lambda_k A(\lambda_k)) \beta r_{i-k}.$$

Given that $\alpha_k \geq \lambda_k$, for $k = 1, 2, \dots, i-1$ and $A(\lambda_k) \leq 1$, the right hand side of this equation is positive. Therefore, $\alpha_i(1 - \beta r_0) - \lambda_i(1 - \beta r_0 A(\lambda_i)) \geq 0$. Using the same argument for the case $i = 1$, we deduce that $\alpha_i \geq \lambda_i$, which proves the induction step.

The sequence $(\alpha_i)_{i \geq 1}$ has a finite limit with $\beta < 1$. We rewrite the definition of α_i as

$$\alpha_i(1 - \beta r_0) = \lambda + \sum_{k=1}^{i-1} \alpha_k \beta r_{i-k}, \quad (1)$$

for $i \geq 1$. Therefore,

$$(\alpha_{i+1} - \alpha_i)(1 - \beta r_0) = \alpha_1 \beta r_i + \sum_{k=1}^{i-1} (\alpha_{k+1} - \alpha_k) \beta r_{i-k}.$$

This relation allows us to prove by induction on i that α_i is increasing in i .

Consider now the sequence $\bar{\alpha}_m = \frac{1}{m} \sum_{k=1}^m \alpha_k$, which represents the average value of the m first terms of the sequence $(\alpha_i)_{i \geq 1}$. Since α_i is increasing in i , the sequence $(\bar{\alpha}_m)_{m \geq 1}$ is also increasing in m .

We now sum up the terms defining α_i in (1) for i from 1 to m . This leads to

$$(1 - \beta r_0) \sum_{i=1}^m \alpha_i = \lambda m + \beta \sum_{i=1}^m \sum_{k=1}^{i-1} \alpha_k r_{i-k}. \quad (2)$$

Observe now that

$$\sum_{i=1}^m \sum_{k=1}^{i-1} \alpha_k r_{i-k} = \sum_{k=1}^{m-1} \alpha_k \sum_{j=1}^{m-k} r_j.$$

The discretized retrial time is a discrete random variable defined by $P(R = j) = r_j$ for $j \geq 0$. Thus, from (2), we deduce that

$$\sum_{k=1}^m \alpha_k (1 - \beta P(R \leq m - k)) = \lambda m$$

Given that $P(R \leq m - k) \leq 1$, we deduce that

$$(1 - \beta) \sum_{k=1}^m \alpha_k < \sum_{k=1}^m \alpha_k (1 - \beta P(R \leq m - k)) = \lambda m.$$

Since $\beta < 1$, this proves that $\bar{\alpha}_m = \frac{\sum_{k=1}^m \alpha_k}{m}$ is bounded.

We now prove that α_k is also bounded. To see this, let us assume that α_k is unbounded. Since $\bar{\alpha}_m$ is bounded, there exists a finite constant $\hat{\alpha} > 0$ such that for $m \geq 1$ we have $\bar{\alpha}_m \leq \hat{\alpha}$. Moreover, since α_k is unbounded and increasing, for all $\omega > 0$, there exists j such that $\alpha_k \geq \omega$ for $k \geq j$. Consider ω such that $\omega > \hat{\alpha}$ and the corresponding value j such that $\alpha_k \geq \omega$ for $k \geq j$. For $m \geq j$, we then have

$$\hat{\alpha} \geq \bar{\alpha}_m = \frac{\sum_{k=1}^{j-1} \alpha_k + \sum_{k=j}^m \alpha_k}{m} \geq \frac{\sum_{k=1}^{j-1} \alpha_k + \omega(m - j + 1)}{m}.$$

By letting m tend to infinity in this inequality, we obtain $\omega \leq \hat{\alpha}$, which is in contradiction with the assumption $\omega > \hat{\alpha}$. Consequently, $(\alpha_k)_{k \geq 1}$ is bounded.

This proves that $(\alpha_k)_{k \geq 1}$ has a finite limit as it increases and bounds. Consequently, $(\lambda_k)_{k \geq 1}$ is also bounded and has a finite limit in the case $\beta < 1$.

The relation $u_i \geq \lambda_i$. We now prove by induction on i that $u_i \geq \lambda_i$ for $i \geq 1$. For $i = 1$, we have

$$u_1 - \lambda_1 = r_0 \frac{u_1^2}{c\mu + u_1} - r_0 \lambda_1 A(\lambda_1).$$

Consequently,

$$u_1 \left(1 - r_0 \frac{u_1}{c\mu + u_1} \right) = \lambda_1 (1 - r_0 \lambda_1 A(\lambda_1)).$$

We have $A(\lambda_1) < B(\lambda_1)$. Moreover, the blocking probability $B(x)$ satisfies the inequality $B(x) < \frac{x}{x+c\mu}$ as observed by

$$\frac{x}{c\mu + x} - B(x) = \frac{x/\mu}{c} \frac{\sum_{k=0}^{c-2} \left(\frac{x}{\mu}\right)^k / k!}{\left(1 + \frac{x}{c\mu}\right) \sum_{k=0}^c \left(\frac{x}{\mu}\right)^k / k!} > 0.$$

Consequently, $B(\lambda_1) < \frac{\lambda_1}{\lambda_1 + c\mu}$. We then deduce that

$$u_1 \left(1 - r_0 \frac{u_1}{c\mu + u_1} \right) > \lambda_1 \left(1 - r_0 \frac{\lambda_1}{c\mu + \lambda_1} \right).$$

This inequality is equivalent to

$$(u_1 - \lambda_1) \left(1 - r_0 \frac{c\mu(u_1 + \lambda_1) + \lambda_1 u_1}{(c\mu + u_1)(c\mu + \lambda_1)} \right) > 0.$$

Given that $(c\mu + u_1)(c\mu + \lambda_1) - c\mu(u_1 + \lambda_1) - \lambda_1 u_1 = (c\mu)^2 > 0$, we deduce that $r_0 \frac{c\mu(u_1 + \lambda_1) + \lambda_1 u_1}{(c\mu + u_1)(c\mu + \lambda_1)} < 1$, so $u_1 - \lambda_1 > 0$.

Assume now that $u_k \geq \lambda_k$, for $k = 1, 2, \dots, i-1$. We have

$$u_i \left(1 - r_0 \frac{u_i}{c\mu + u_i} \right) - \lambda_i (1 - \beta A(\lambda_i)) = \sum_{k=1}^{i-1} \left(\frac{u_k^2}{c\mu + u_k} - \lambda_k A(\lambda_k) \right) r_{i-k}.$$

We have

$$\frac{u_k^2}{c\mu + u_k} - \lambda_k A(\lambda_k) > \frac{u_k^2}{c\mu + u_k} - \frac{\lambda_k^2}{c\mu + \lambda_k} > 0,$$

due to $A(\lambda_k) < B(\lambda_k) < \frac{\lambda_k}{\lambda_k + c\mu}$, $u_k > \lambda_k$ and $x \mapsto \frac{x^2}{c\mu + x}$ is increasing in x . Consequently, $u_i \left(1 - r_0 \frac{u_i}{c\mu + u_i} \right) - \lambda_i (1 - \beta A(\lambda_i)) > 0$. Using the same argument for the case $i = 1$, we deduce that $u_i \geq \lambda_i$, which proves the induction step.

The sequence $(u_i)_{i \geq 1}$ has a finite limit with $\lambda < c\mu$. First, we prove that u_i is increasing in i . This can be proven by induction on i , using

$$\begin{aligned} u_{i+1} - u_i &= r_i \frac{u_1^2}{c\mu + u_1} + \sum_{k=1}^i r_{i-k} \left(\frac{u_{k+1}^2}{c\mu + u_{k+1}} - \frac{u_k^2}{c\mu + u_k} \right) \\ &= r_i \frac{u_1^2}{c\mu + u_1} + \sum_{k=1}^i r_{i-k} (u_{k+1} - u_k) \frac{c\mu(u_{k+1} + u_k) + u_k u_{k+1}}{(c\mu + u_{k+1})(c\mu + u_k)}. \end{aligned}$$

This relation is equivalent to

$$(u_{i+1} - u_i) \left(1 - r_0 \frac{c\mu(u_{i+1} + u_i) + u_i u_{i+1}}{(c\mu + u_{i+1})(c\mu + u_i)} \right) = r_i \frac{u_i^2}{c\mu + u_i} + \sum_{k=1}^{i-1} r_{i-k} (u_{k+1} - u_k) \frac{c\mu(u_{k+1} + u_k) + u_k u_{k+1}}{(c\mu + u_{k+1})(c\mu + u_k)}.$$

Given that $r_0 \frac{c\mu(u_{i+1} + u_i) + u_i u_{i+1}}{(c\mu + u_{i+1})(c\mu + u_i)} < 1$, this relation shows that the increasing property of u_i propagates in i . Thus, u_i is increasing in i for $i \geq 1$.

We now prove that $u_i < \frac{\lambda c\mu}{c\mu - \lambda}$ for $i \geq 1$. We have

$$u_i \left(1 - r_0 \frac{u_i}{c\mu + u_i} \right) - \sum_{k=1}^{i-1} u_k r_{i-k} \frac{u_k}{c\mu + u_k} = \lambda.$$

Given that u_i is increasing in i and $x \mapsto \frac{x^2}{x + c\mu}$ is increasing in x , we deduce that

$$u_i \left(1 - r_0 \frac{u_i}{c\mu + u_i} \right) - \sum_{k=1}^{i-1} u_i r_{i-k} \frac{u_i}{c\mu + u_i} = u_i \left(1 - (r_0 + r_1 + \dots + r_{i-1}) \frac{u_i}{c\mu + u_i} \right) < \lambda.$$

Given that $r_0 + r_1 + \dots + r_{i-1} < 1$, we deduce that $\frac{u_i c\mu}{c\mu + u_i} < \lambda$, which proves the result. Consequently $(u_i)_{i \geq 1}$ is increasing and bounded, thus it has a finite limit. Consequently, $(\lambda_k)_{k \geq 1}$ is also bounded and has a finite limit in the case $\lambda < c\mu$ and $\beta = 1$.

The limit. There remains to prove that the finite limit of $(\lambda_k)_{k \geq 1}$, termed λ_∞ , is solution of Equation (7) from the main paper, given as

$$\lambda_\infty (1 - \beta A(\lambda_\infty)) = \lambda. \quad (3)$$

To this end, we need to show that

$$\lim_{i \rightarrow \infty} \left(\sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k} \right) = \lambda_\infty A(\lambda_\infty).$$

First note that if λ_k tends to λ_∞ , then $\lambda_k A(\lambda_k)$ tends to $\lambda_\infty A(\lambda_\infty)$.

Consider $\epsilon > 0$. There exists $j_1 \geq 1$ such that for $k \geq j_1$, we have $|\lambda_k A(\lambda_k) - \lambda_\infty A(\lambda_\infty)| < \epsilon$. Moreover, the probability $\sum_{k=1}^{\infty} r_k = P(R \geq k)$ tends to zero as k tends to infinity (otherwise we could not have $\sum_{k=1}^{\infty} r_k = 1$). Thus, there exists $j_2 \geq 0$

such that for $k \geq j_2$ we have $\sum_{k=j_2}^{\infty} r_k < \epsilon$.

Define now $j \equiv j_2 + \max(j_1, j_2) + 1$. For $i \geq j$, using $\lambda_\infty A(\lambda_\infty) \geq \lambda_k A(\lambda_k)$, we have

$$\begin{aligned} & \left| \sum_{k=1}^i \lambda_\infty A(\lambda_\infty) r_{i-k} - \sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k} \right| \\ &= \sum_{k=1}^{j-1} r_{i-k} (\lambda_\infty A(\lambda_\infty) - \lambda_k A(\lambda_k)) + \sum_{k=j}^i r_{i-k} (\lambda_\infty A(\lambda_\infty) - \lambda_k A(\lambda_k)) \\ &\leq \epsilon (\lambda_\infty A(\lambda_\infty) - \lambda_1 A(\lambda_1)) + \sum_{k=j}^i r_{i-k} \epsilon. \end{aligned}$$

Given that $\sum_{k=j}^i r_{i-k} \leq 1$, we deduce that

$$\left| \sum_{k=1}^i \lambda_\infty A(\lambda_\infty) r_{i-k} - \sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k} \right| \leq \epsilon (1 + \lambda_\infty A(\lambda_\infty) - \lambda_1 A(\lambda_1)).$$

Consequently, for i sufficiently high, the difference $\left| \sum_{k=1}^i \lambda_{\infty} A(\lambda_{\infty}) r_{i-k} - \sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k} \right|$ can be made as small as desired. This shows that $\sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k}$ converges to $\lambda_{\infty} A(\lambda_{\infty}) \sum_{k=1}^i r_{i-k}$. Since $\sum_{k=1}^i r_{i-k}$ tends to 1 as i tends to infinity, we deduce that $\sum_{k=1}^i \lambda_k A(\lambda_k) r_{i-k}$ converges to $\lambda_{\infty} A(\lambda_{\infty})$ as i tends to infinity. This proves that the limit of $(\lambda_k)_{k \geq 1}$ is solution of (3). $\square$

Proof of Corollary 1. Assuming that the arrival rate of fresh customers is upper-bounded. We can build a constant-arrival rate sequence that converges to a finite limit from Theorem 1. This shows that the local stability condition for the queueing model approximated by SCF is also a long-run stability condition, as the long-run total arrival rate then remains upper-bounded. $\square$

A.2. Proof of Theorem 2

Proof. We prove the result by showing that λ_{i+kn} is increasing in k and next using the same sequences α_k and u_k as those considered in the proof of Theorem 1 to prove that λ_{i+kn} is bounded. The equation of the limits directly follows from Equation (8) from the main paper, given as

$$\lambda_{i+kn} = \lambda_i^f + \sum_{j=1}^{i+kn} \beta r_{i+kn-j} \lambda_j A(\lambda_j). \quad (4)$$

The following equality shows that the total arrival rate per interval increases:

$$\begin{aligned} \lambda_{i+(k+1)n} - \lambda_{i+kn} &= \sum_{j=1}^{i+(k+1)n} \beta r_{i+(k+1)n-j} \lambda_j A(\lambda_j) - \sum_{j=1}^{i+kn} \beta r_{i+kn-j} \lambda_j A(\lambda_j) \\ &= \sum_{j=1}^n \beta r_{i+(k+1)n-j} A(\lambda_j) + \sum_{j=1}^{i+kn} \beta r_{i+kn-j} (\lambda_{j+n} A(\lambda_{j+n}) - \lambda_j A(\lambda_j)) \end{aligned}$$

Next, we redefine α_k and u_k as

$$\begin{aligned} \alpha_i &= \max(\lambda_1^f, \lambda_2^f, \dots, \lambda_n^f) + \sum_{k=1}^i \alpha_k \beta r_{i-k}, \text{ for } \beta < 1 \text{ and} \\ u_i &= \max(\lambda_1^f, \lambda_2^f, \dots, \lambda_n^f) + \sum_{k=1}^i u_k r_{i-k} \frac{u_k}{c\mu + u_k}, \text{ for } \beta = 1 \text{ and } \max(\lambda_1^f, \lambda_2^f, \dots, \lambda_n^f) < c\mu, \end{aligned}$$

to prove that the total arrival rate per interval is bounded. This proves that on each interval, the total arrival rate converges to a finite limit. $\square$

A.3. Proof of Proposition 1

Proof. In the infinite time horizon case, another representation of Equation (5) from the main paper given as

$$\lambda_i = \lambda_i^f + \sum_{k=1}^i \lambda_k A(\lambda_k) \beta r_{i-k} \text{ for } i \geq 1 \quad (5)$$

involves the z -transforms of the sequences λ_i , λ_i^f and r_i for $i \geq 1$. This representation is valid when the sequence of arrival rates defined by (5) is bounded. The z -transform of Equation (5) can be expressed as

$$\tilde{\lambda}(z) = \tilde{\lambda}^f(z) + \beta \tilde{\lambda}_A(z) \tilde{r}(z), \quad (6)$$

where $\tilde{\lambda}(z) \equiv \sum_{i=1}^{\infty} \lambda_i z^{i-1}$, $\tilde{\lambda}_A(z) \equiv \sum_{i=1}^{\infty} \lambda_i A(\lambda_i) z^{i-1}$, $\tilde{\lambda}^f(z) \equiv \sum_{i=1}^{\infty} \lambda_i^f z^{i-1}$, and $\tilde{r}(z) \equiv \sum_{i=0}^{\infty} r_i z^i$, for $0 \leq z < 1$. The value of the z -transform representation is that it transforms a convolution product into a simple product.

Equation (6) can be expressed as

$$\tilde{\lambda}(z) = \frac{\tilde{\lambda}^f(z)}{1 - \beta \tilde{r}(z)} - \beta s \mu \frac{\tilde{r}(z)}{(1 - z)(1 - \beta \tilde{r}(z))}, \quad (7)$$

for $0 \leq z < 1$. In case of a constant arrival rate, exponential patience time, and an heavy workload situation ($A(\lambda(t)) \approx 1 - \frac{c\mu}{\lambda(t)}$), we have $\tilde{\lambda}^f(z) = \frac{\lambda}{1-z}$ and $\tilde{r}(z) = 1 - \frac{1-e^{-\ell\psi}}{\ell\psi} + \frac{z(1-e^{-\ell\psi})^2}{\ell\psi(1-ze^{-\ell\psi})}$, thus

$$\tilde{\lambda}(z) = \frac{\lambda - c\mu\beta}{(1-\beta)(1-z)} - \frac{\beta(1-e^{-\ell\psi})(\lambda - c\mu)}{(1-\beta)(\beta(1-e^{-\ell\psi}) + \ell\psi(1-\beta))} \left(\frac{1}{1 - z \frac{e^{-\ell\psi}\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})}{\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})}} \right). \quad (8)$$

Getting back to the total arrival rate, we deduce that

$$\lambda_i = \frac{\lambda - c\mu\beta}{(1-\beta)} - \frac{\beta(1-e^{-\ell\psi})(\lambda - c\mu)}{(1-\beta)(\beta(1-e^{-\ell\psi}) + \ell\psi(1-\beta))} \left(\frac{e^{-\ell\psi}\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})}{\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})} \right)^{i-1} \text{ for } i \geq 1.$$

This expression shows that while the limit of λ_i is not a function of ψ , the speed of convergence of λ_i to its stationary value is a function of ψ . Specifically, the term $\frac{e^{-\ell\psi}\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})}{\ell\psi(1-\beta) + \beta(1-e^{-\ell\psi})}$ which controls the exponential growth of the arrival rate is decreasing in ψ . Consequently, the convergence to the stationary state takes longer when the mean retrial time is long.

Note that Equation (1) from the main paper given as

$$\lambda(t) = \lambda^f(t) + \beta \int_0^t \lambda(x) A(\lambda(x)) r(t-x) dx \quad (9)$$

can be solved using a Laplace Transform (LT) given by

$$\bar{\lambda}(s) = \bar{\lambda}^f(s) + \bar{\lambda} \bar{A}(\lambda)(s) \beta \bar{r}(s), \quad (10)$$

where the LT of a given function $f(t)$ is defined by $\bar{f}(s) = \int_{t=0}^{\infty} f(t) e^{-st} dt$.

When letting the interval length tend to zero, Equation (10) allows us to prove that

$$\lambda(t) = \frac{\lambda - c\mu\beta}{1-\beta} - \frac{\beta(\lambda - c\mu)}{1-\beta} e^{-\gamma(1-\beta)t},$$

which shows that the duration of the transient phase follows an exponential distribution. $\square$

A.4. Proofs of Lemma 2, Theorem 3, and Corollary 2

Proof of Lemma 2. Consider the function $g_{a,b}(z) \equiv \frac{M(a+1,b,z)}{M(a,b,z)}$ for $z \geq 0$, $a \geq 0$ and $b \geq a+1$. From the third statement of 10.2, page 392 of Andrews (1998), we have

$$M(a+1, b, z) = M(a, b, z) + \frac{z}{a} M'(a, b, z),$$

where $M'(a, b, z) = \frac{\partial M(a,b,z)}{\partial z}$. This allows us to express $g(z)$ as

$$g_{a,b}(z) = 1 + \frac{z}{a} h_{a,b}(z), \text{ where } h_{a,b}(z) = \frac{M'(a, b, z)}{M(a, b, z)}.$$

We now differentiate $h(z)$ in z . This leads to

$$h'_{a,b}(z) = \frac{M''(a, b, z)M(a, b, z) - (M'(a, b, z))^2}{(M(a, b, z))^2}.$$

Recall that $M(a, b, z)$ follows Kummer's second order differential equation as expressed at statement 19, page 393 of [Andrews \(1998\)](#):

$$zM''(a, b, z) + (b - z)M'(a, b, z) - aM(a, b, z) = 0.$$

Consequently, we find that

$$h'_{a,b}(z) = \frac{a}{z} + \left(1 - \frac{b}{z}\right) h_{a,b}(z) - h_{a,b}(z)^2. \quad (11)$$

Consider the quadratic equation in X : $\frac{a}{z} + \left(1 - \frac{b}{z}\right) X - X^2 = 0$. It has two solutions given by

$$f_{a,b}^+(z) = \frac{z - b + \sqrt{(z - b)^2 + 4az}}{2z} \text{ and } f_{a,b}^-(z) = \frac{z - b - \sqrt{(z - b)^2 + 4az}}{2z}.$$

Given that $f_{a,b}^-(z) < 0$ and $h_{a,b}(z) > 0$, we have $h_{a,b}(z) > f_{a,b}^-(z)$. Consequently, we deduce from (11) that

$$h'_{a,b}(z) > 0 \text{ if and only if } h_{a,b}(z) < f_{a,b}^+(z). \quad (12)$$

Consider $h_{a,b}(z)$ in the neighborhood of $z = 0$. From Kummer's second order differential equation, we deduce that

$$bM'(a, b, 0) = aM(a, b, 0),$$

thus $h_{a,b}(0) = \frac{a}{b} = f_{a,b}^+(0)$. Moreover, in the neighborhood of $z = 0$, we have

$$M(a, b, z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2} + o(z^2), \text{ and } M'(a, b, z) = \frac{a}{b} + \frac{a(a+1)}{b(b+1)}z + o(z),$$

so,

$$h_{a,b}(z) = \frac{M'(a, b, z)}{M(a, b, z)} = \frac{a}{b} \left(1 + \frac{b-a}{b(b+1)}z\right) + o(z)$$

Given that $b > a$, we then deduce that $h'_{a,b}(z) > 0$ in the neighborhood of $z = 0$. Consequently, from (12), in the neighborhood of $z = 0^+$, we have $h_{a,b}(z) < f_{a,b}^+(z)$.

Assume that for some $z > 0$ we have $h_{a,b}(z) \geq f_{a,b}^+(z)$. Let us then define by z^* the lowest strictly positive real such that $h_{a,b}(z^*) = f_{a,b}^+(z^*)$. At $z = z^*$, we have $h'_{a,b}(z^*) = 0$ (from Equation (11)), thus $z = z^*$ is an extremum of $h_{a,b}(z)$. Moreover, since $h_{a,b}(z) < f_{a,b}^+(z)$ for $z < z^*$, $h_{a,b}(z)$ is increasing in z for $z < z^*$. This proves that $z = z^*$ is a local maximum for $h_{a,b}(z)$, consequently, $h_{a,b}(z)$ must decrease just above z^* . So, $z = z^*$ is a maximum. Since $f_{a,b}^+(z)$ is increasing in z due to $c > a$, $h_{a,b}(z)$ stays strictly below $f_{a,b}^+(z)$ just above z^* while being decreasing in z . This is in contradiction with (12). Consequently, for all $z > 0$, we have $h_{a,b}(z) < f_{a,b}^+(z)$. $\square$

Proof of Theorem 3.

Let us now express $f_{a,b}^+(z)$ in terms of the problem's parameters: $a = \frac{\lambda}{\psi}$, $b = \frac{\mu + (1-\beta)(\lambda + \psi)}{\psi(1-\beta)}$, and $z = \frac{\lambda\beta}{\psi(1-\beta)}$. We have

$$f_{a,b}^+(z) = \frac{\frac{\lambda\beta}{\psi(1-\beta)} - \frac{\mu + (1-\beta)(\lambda + \psi)}{\psi(1-\beta)} + \sqrt{\left(\frac{\lambda\beta}{\psi(1-\beta)} - \frac{\mu + (1-\beta)(\lambda + \psi)}{\psi(1-\beta)}\right)^2 + 4\frac{\lambda}{\psi}\frac{\lambda\beta}{\psi(1-\beta)}}}{2\frac{\lambda\beta}{\psi(1-\beta)}}.$$

Multiplying this expression by $\psi(1-\beta)$ leads to

$$f_{a,b}^+(z) = \frac{\lambda\beta - (\mu + (1-\beta)(\lambda + \psi)) + \sqrt{(\lambda\beta - (\mu + (1-\beta)(\lambda + \psi)))^2 + 4\lambda^2\beta(1-\beta)}}{2\lambda\beta}.$$

We deduce from $f_{a,b}^+(z) > h_{a,b}(z)$ that

$$g_{a,b}(z) < \frac{\lambda - \mu - \psi(1 - \beta) + \sqrt{(\lambda\beta - (\mu + (1 - \beta)(\lambda + \psi)))^2 + 4\lambda^2\beta(1 - \beta)}}{2\lambda(1 - \beta)}. \quad (13)$$

Consider the function $k(\psi) = -\psi(1 - \beta) + \sqrt{(\lambda\beta - (\mu + (1 - \beta)(\lambda + \psi)))^2 + 4\lambda^2\beta(1 - \beta)}$. We have

$$k'(\psi) = -(1 - \beta) + \frac{(1 - \beta)(\lambda\beta - (\mu + (1 - \beta)(\lambda + \psi)))}{\sqrt{(\lambda\beta - (\mu + (1 - \beta)(\lambda + \psi)))^2 + 4\lambda^2\beta(1 - \beta)}} < 0.$$

Thus, $k(\psi)$ is decreasing in ψ . So, the maximal value for $k(\psi)$ is attained for $\psi = 0$. We observe that when ψ tends to zero, the upper bound found in (13) tends to $\frac{\lambda_\infty}{\lambda}$. This proves that $\lambda_\infty \geq \lambda_F$.

We deduce $U_F \leq U_\infty$ from $\lambda_F \leq \lambda_\infty$, given that the function $x \mapsto \frac{x}{x + \mu}$ is increasing in x .

By substituting the expression of λ_∞ into $L_F^O - L_\infty^O$, we deduce that $L_F^O - L_\infty^O \geq 0$ is equivalent with

$$\lambda_F \leq \frac{\mu(\lambda + \mu - \sqrt{(\mu + \lambda)^2 - 4\lambda\mu\beta})}{2\beta\mu - \lambda - \mu + \sqrt{(\mu + \lambda)^2 - 4\lambda\mu\beta}}.$$

By multiplying the numerator and denominator of this expression by $2\beta\mu - \lambda - \mu - \sqrt{(\mu + \lambda)^2 - 4\lambda\mu\beta}$, we deduce that $L_F^O - L_\infty^O \geq 0$ is equivalent with $\lambda_F \leq \lambda_\infty$. $\square$

Proof of Corollary 2. Observing that $h'_{a,b}(z)$ tends to zero in Equation (11) when ψ goes to zero, we deduce that $h_{a,b}(z)$ tends to $f_{a,b}^+(z)$. Next, given that $f_{a,b}^+(z)$ tends to λ_∞ as ψ tends to zero, the SCF approximation provides the exact performance measures when ψ tends to zero. $\square$

A.5. Proof of Proposition 2

Proof. We consider the asymptotic equivalent of Equation (10) in heavy workload situations ($A(\lambda(t)) \approx 1 - \frac{c\mu}{\lambda(t)}$) with a sinusoidal arrival rate of the form $\lambda(1 + \sin(\omega t))$. We find that

$$\begin{aligned} \lambda(t) = & \frac{\lambda - c\mu\beta}{(1 - \beta)} - \frac{\beta [\psi^2(\lambda - c\mu)(1 - \beta)^2 - \omega\lambda\psi(1 - \beta) + \omega^2(\lambda - c\mu)]}{(1 - \beta)(\omega^2 + \psi^2(1 - \beta)^2)} e^{-\psi(1 - \beta)t} \\ & + \frac{\lambda\sqrt{(\omega^2 + \psi^2(1 - \beta))^2 + (\omega\beta\psi)^2}}{\omega^2 + \psi^2(1 - \beta)^2} \sin(\omega t + d), \end{aligned}$$

where $d = \arctan\left(\frac{-\omega\beta\psi}{\omega^2 + \psi^2(1 - \beta)^2}\right)$. This shows that the total arrival rate is a function of 3 terms. The first represents the stationary limit with a constant arrival rate λ as found in the previous section. The second one in $e^{-\psi(1 - \beta)t}$ vanishes as t tends to infinity, so it represents the transient effect. The third one shows how the retrial time modifies the time variations of the arrival rate of fresh customers. In this term, the amplitude of the variation is strictly increasing in ψ , which shows that only $\mathcal{H}_0$ is apparent. The retrial time induces a dephasing of the cycles due to the parameter d . When the retrial rate tends to zero (as in the SCF approximation), the dephasing effect vanishes, and the amplitude of time variations of $\lambda(t)$ becomes identical to those of the original arrival rate of fresh customers. $\square$

Appendix B: Accuracy of the SCF Approximation in the Stationary Limit

In this section, we assess the accuracy of the SCF approximation using scatter plots that compare simulation estimates with their SCF counterparts across the parameter instances of Table 1. Each point corresponds to one instance, with the simulated performance measure on the x -axis and the SCF estimate on the y -axis. Therefore, the closer the points are to the 45-degree line, the more accurate the approximation. These plots provide a more detailed illustration of the results reported in Table 1. They show that there are no outliers for which the accuracy of the SCF approximation deteriorates substantially. In the following figures, we present several parameter settings that are representative of our study.

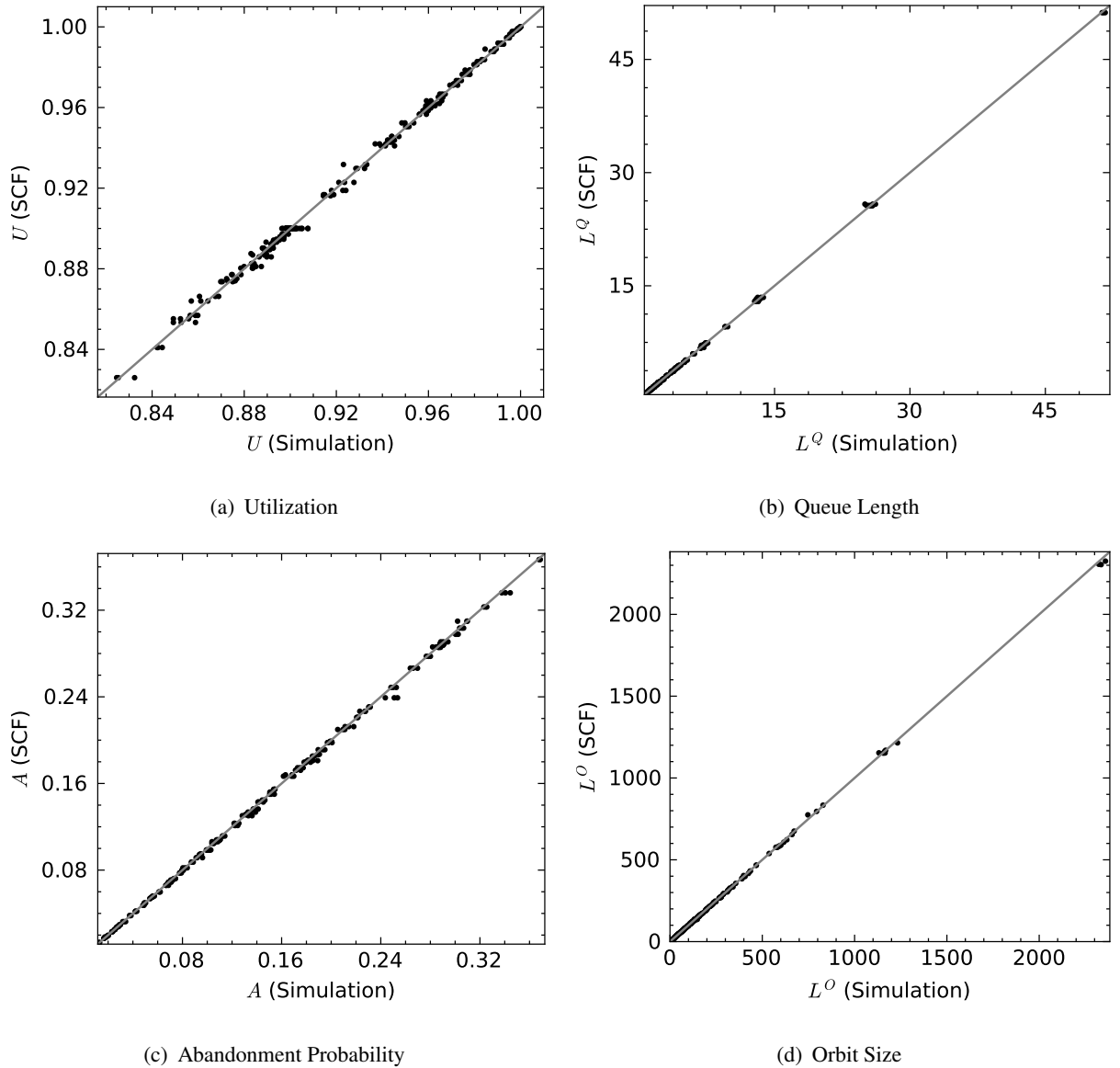
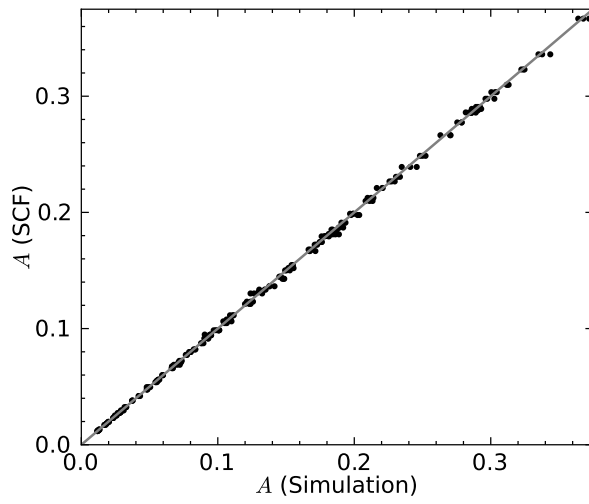
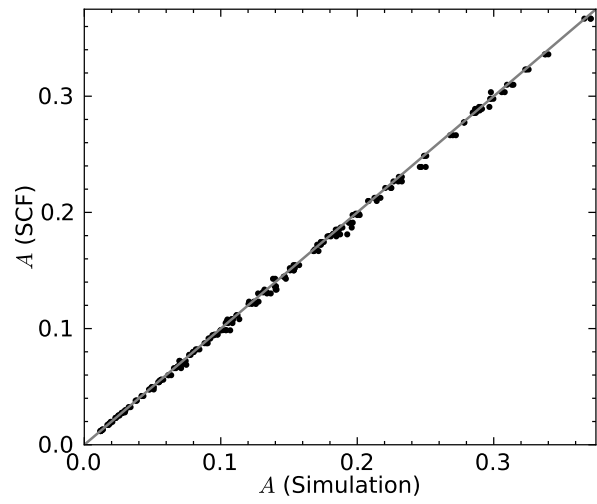


Figure 1 Comparison of Simulated and Approximated Performance Measures under a Constant Arrival Rate and Exponential Retrial Time

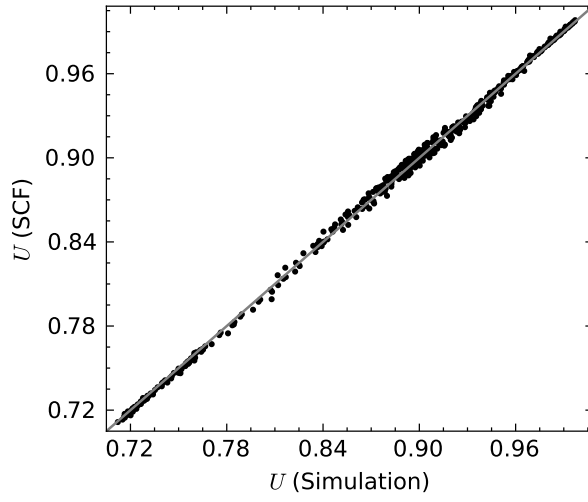


(a) Erlang Distribution ($cv_r = 0.2$)

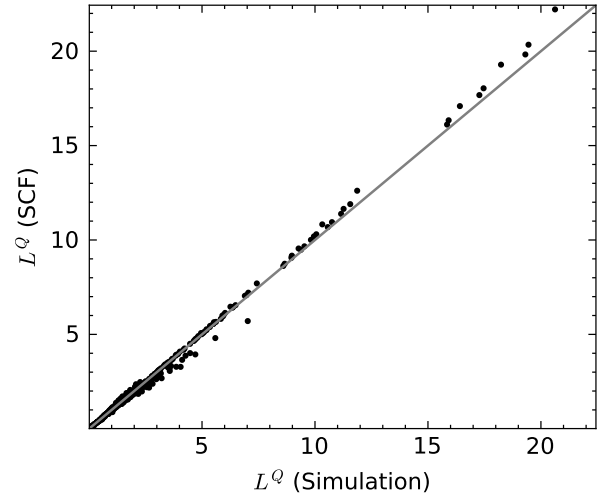


(b) 2-phase Hyperexponential Distribution ($cv_r = 5$)

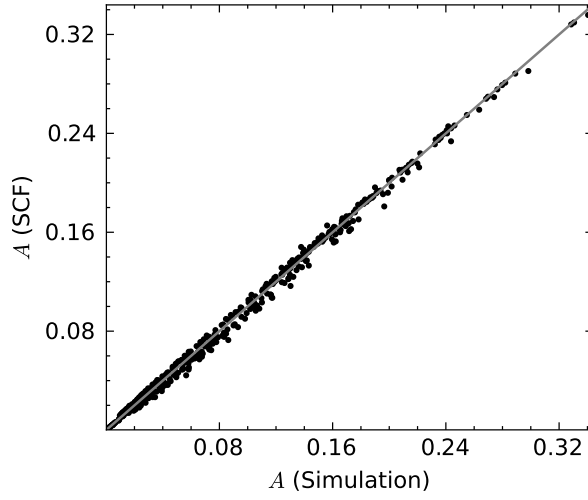
Figure 2 Comparison of Simulated and Approximated Abandonment Probability under a Constant Arrival Rate and Generally Distributed Retrial Times



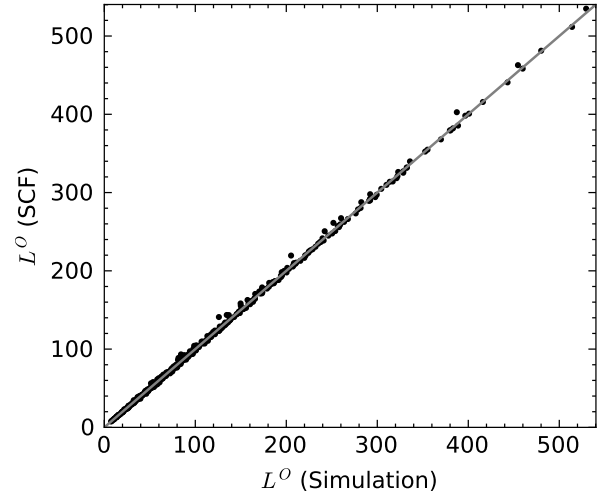
(a) Utilization



(b) Queue Length

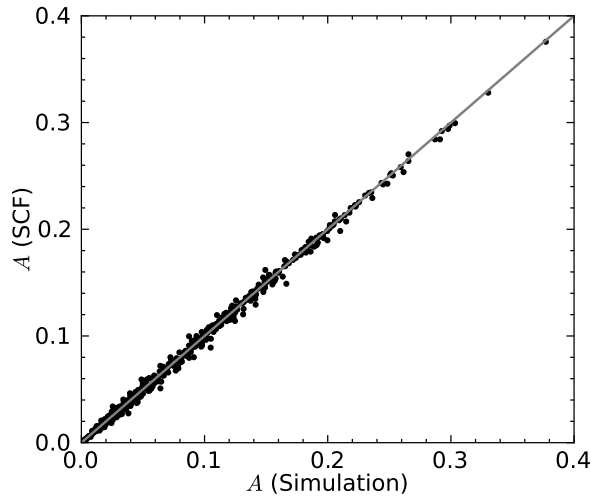


(c) Abandonment Probability

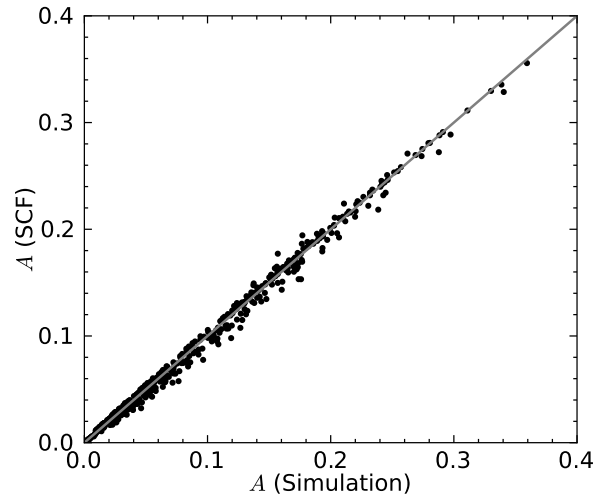


(d) Orbit Size

Figure 3 Comparison of Simulated and Approximated Performance Measures under a Cyclic Arrival Rate and Exponential Retrial Time at Four Representative Intervals of the Steady-state Cycle: trough ($j = 1$), peak ($j = 61$), increasing phase ($j = 31$), and decreasing phase ($j = 91$)



(a) Erlang Distribution ($cv_r = 0.2$)



(b) 2-phase Hyperexponential Distribution ($cv_r = 5$)

Figure 4 Comparison of Simulated and Approximated Abandonment Probability under Generally Distributed Retrial Times and a Cyclic Arrival Rate at Four Representative Intervals of the Steady-state Cycle: trough ($j = 1$), peak ($j = 61$), increasing phase ($j = 31$), and decreasing phase ($j = 91$)

Appendix C: Interaction Between Cycle Length and Mean Retrial Time under Generally Distributed Retrial Times

In this section, we investigate the interaction between cycle length and mean retrial time under general retrial-time distributions. Specifically, we compare abandonment probabilities obtained from the SCF approximation and from simulation for mean retrial times $\psi^{-1} \in \{60, 120, 180, 240\}$, using Erlang and two-phase hyperexponential retrial-time distributions with coefficients of variation $cv_r \in \{0.2, 0.5, 2, 5\}$.

Figures 5–6 show that, for retrial-time distributions with low variability, retrials may either amplify or smooth congestion depending on the alignment between ψ^{-1} and the cycle length. Specifically, congestion is amplified when the mean retrial time matches the full cycle length, whereas it is smoothed when the mean retrial time is close to half the cycle length. By contrast, Figures 7–8 indicate that, under higher retrial-time variability, this effect is no longer visible, because it is dominated by the more standard effect whereby shorter mean retrial times lead to higher abandonment probabilities.

Overall, these findings are consistent with Conjecture 2 and further support the conclusion that the cycle-length interaction effect arises primarily when retrial-time variability is low.

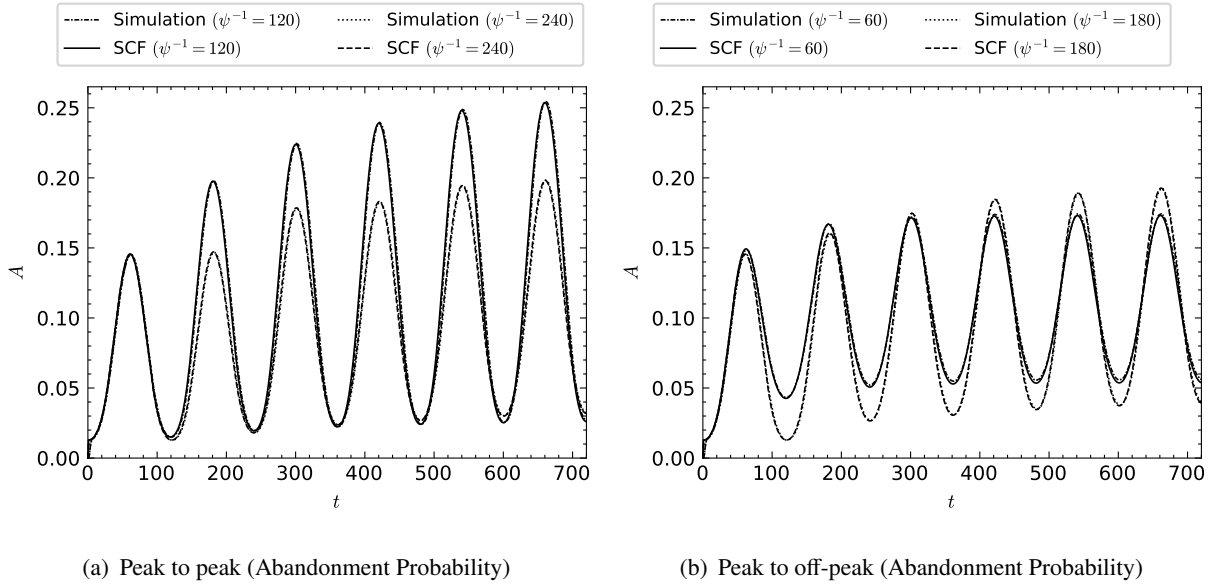
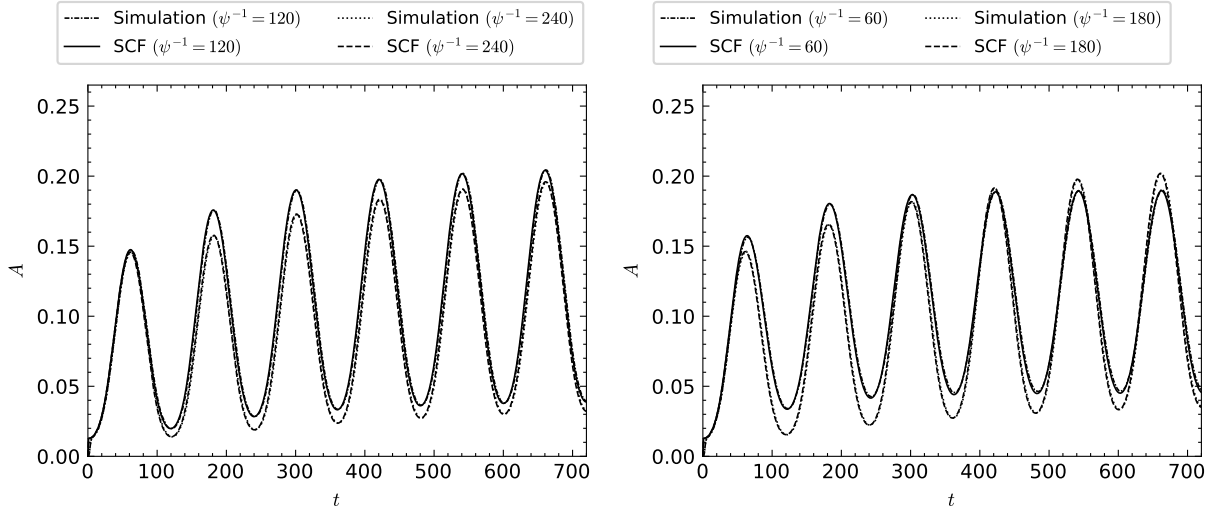


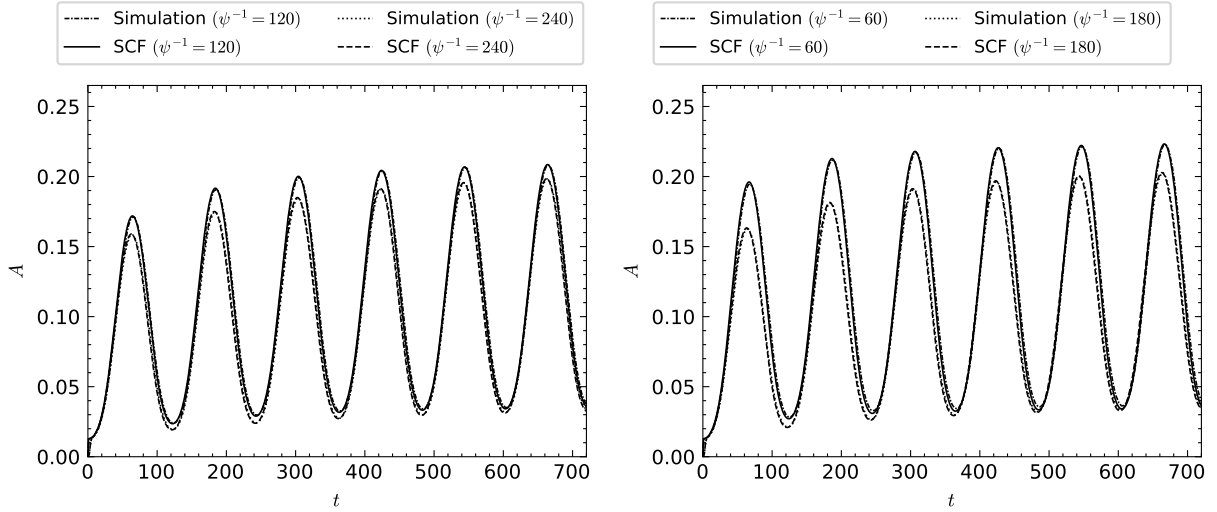
Figure 5 Evaluation of the SCF approximation under sinusoidal arrival pattern and Erlang retrials ($cv_r = 0.2$, $\mu = \nu = 1$, $c = 16$, $\beta = 1$, $\lambda(t) = 14.4 + 3.2 \cdot \sin\left(\frac{\pi}{60} \cdot t - \frac{\pi}{2}\right)$).



(a) Peak to peak (Abandonment Probability)

(b) Peak to off-peak (Abandonment Probability)

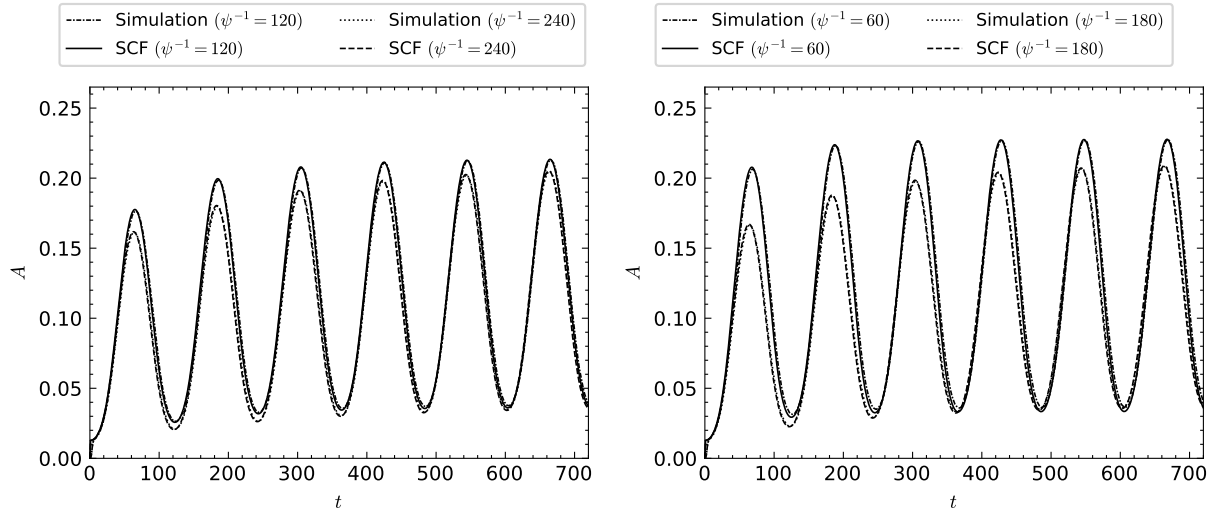
Figure 6 Evaluation of the SCF approximation under sinusoidal arrival pattern and Erlang retrials ($cv_r = 0.5$, $\mu = \nu = 1$, $c = 16$, $\beta = 1$, $\lambda(t) = 14.4 + 3.2 \cdot \sin\left(\frac{\pi}{60} \cdot t - \frac{\pi}{2}\right)$).



(a) Peak to peak (Abandonment Probability)

(b) Peak to off-peak (Abandonment Probability)

Figure 7 Evaluation of the SCF approximation under sinusoidal arrival pattern and 2-phase hyperexponential retrials ($cv_r = 2$, $\mu = \nu = 1$, $c = 16$, $\beta = 1$, $\lambda(t) = 14.4 + 3.2 \cdot \sin\left(\frac{\pi}{60} \cdot t - \frac{\pi}{2}\right)$).



(a) Peak to peak (Abandonment Probability)

(b) Peak to off-peak (Abandonment Probability)

Figure 8 Evaluation of the SCF approximation under sinusoidal arrival pattern and 2-phase hyperexponential retrials ($cv_r = 5$, $\mu = \nu = 1$, $c = 16$, $\beta = 1$, $\lambda(t) = 14.4 + 3.2 \cdot \sin\left(\frac{\pi}{60} \cdot t - \frac{\pi}{2}\right)$).

References

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