Online Appendices for : "A Solution Approach to Distributionally Robust Chance-Constrained Assignment Problems"

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Appendix A: Lifted Cover Inequality

We now provide coefficient calculations for a lifted cover inequality that is valid for $conv(\mathcal{F}_{j\omega})$.

THEOREM 1. *The lifted cover inequality*

$$\sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\mathcal{I}\backslash\mathcal{C}} \alpha_i y_{ij} + \sum_{i\in\mathcal{D}} \beta_i y_{ij} + \gamma(z_{j\omega}-1) \le |\mathcal{C}\backslash\mathcal{D}| + \sum_{i\in\mathcal{D}} \beta_i - 1 \tag{1}$$

is valid for $conv(\mathcal{F}_{j\omega})$ if

$$\gamma = \underset{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}\cap\mathcal{Y}_j}{\text{maximize}} \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\mathcal{I}\backslash\mathcal{C}} \alpha_i y_{ij} + \sum_{i\in\mathcal{D}} \beta_i y_{ij} - |\mathcal{C}\backslash\mathcal{D}| - \sum_{i\in\mathcal{D}} \beta_i + 1. \tag{2}$$

Furthermore, if $|\mathcal{C}| \le \rho_j + 1$, (1) is facet-defining for $conv(\mathcal{F}_{j\omega})$.

Proof. When $z_{j\omega}=1$, (1) is valid for $conv(\mathcal{F}_{j\omega})$ because of Lemma 2. When $z_{j\omega}=0$, due to the definition of γ, (1) is also valid for $conv(\mathcal{F}_{j\omega})$. Thus, (1) is valid for $conv(\mathcal{F}_{j\omega})$.

Consider the following $|\mathcal{I}|+1$ feasible points of $conv(\mathcal{F}_{j\omega})$: when $z_{j\omega}=1$, there exists $|\mathcal{I}|$ feasible points of $conv(\mathcal{F}_{j\omega})$ that are affinely independent and satisfy (1) at equality based on the Lemma 2; when $z_{j\omega}=0$, let $\boldsymbol{y}_j$ be the optimal solution of (2). These $|\mathcal{I}|+1$ feasible points satisfy (1) at equality and are affinely independent. Thus, (1) is facet-defining for $conv(\mathcal{F}_{j\omega})$. □

By restricting the feasible region of $\boldsymbol{y}_j$ in (2) using the chance constraints (1d), we obtain a stronger valid inequality for (CAP) in Theorem 2.

THEOREM 2. *For $k\in\Omega\backslash\{\omega\}$, let*

$$\delta_k = \underset{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}\cap\mathcal{Y}_j}{\text{maximize}} \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\mathcal{I}\backslash\mathcal{C}} \alpha_i y_{ij} + \sum_{i\in\mathcal{D}} \beta_i y_{ij} - |\mathcal{C}\backslash\mathcal{D}| - \sum_{i\in\mathcal{D}} \beta_i + 1 \tag{3a}$$

$$\text{subject to } \sum_{i\in\mathcal{I}} \xi_i^k y_{ij} \le m_j^k(k). \tag{3b}$$

Sort δ_k such that $\delta_{\bar{k}_1} \le \ldots \le \delta_{\bar{k}_{|\Omega|-1}}$. Let $q^1 := \min\left\{ l \middle| \sum_{j=1}^{l} p_{\bar{k}_j} > \varepsilon \right\}$, then the inequality (1) is valid for (CAP), where $\gamma = \delta_{\bar{k}_{q^1}}$.

Proof. Let

$$\gamma = \underset{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}\cap\mathcal{Y}_j}{\text{maximize}} \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\mathcal{I}\backslash\mathcal{C}} \alpha_i y_{ij} + \sum_{i\in\mathcal{D}} \beta_i y_{ij} - |\mathcal{C}\backslash\mathcal{D}| - \sum_{i\in\mathcal{D}} \beta_i + 1 \tag{4a}$$

$$\text{subject to } \sum_{k\in\Omega\backslash\{\omega\}} p_k \mathbb{1}\left(\sum_{i\in\mathcal{I}} \xi_i^k y_{ij} \le t_j\right) \ge 1-\varepsilon. \tag{4b}$$

$\boldsymbol{y}_j$ satisfies the chance constraint (1d) and $z_\omega = 0$ for computing γ, the inequality (1) is valid for (CAP).

Let $\hat{\boldsymbol{y}}_j$ be an optimal solution of (4). Then, there exists at least one $k' \in \{\bar{k}_1,\ldots,\bar{k}_{q^1}\}$ such that $\sum_{i\in\mathcal{I}} \xi_i^{k'} \hat{y}_{ij} \le t_j$. Otherwise, if $\sum_{i\in\mathcal{I}} \xi_i^k \hat{y}_{ij} > t_j$ for all $k\in\{\bar{k}_1,\ldots,\bar{k}_{q^1}\}$, then $\sum_{k\in\{\bar{k}_1,\ldots,\bar{k}_{q^1}\}} p_k \mathbb{1}\left(\sum_{i\in\mathcal{I}} \xi_i^k \hat{y}_{ij} > t\right) > \varepsilon$, which indicates that (4b) is violated by $\hat{\boldsymbol{y}}_j$. Therefore, $\hat{\boldsymbol{y}}_j$ is a feasible solution of (3) for $k = k'$. We have $\delta_{\bar{k}_{q^1}} \geq \delta_{k'} \geq \gamma$, and (1) is a valid inequality for (CAP) when $\gamma = \delta_{\bar{k}_{q^1}}$. □

We further restrict the feasible region of $\boldsymbol{y}_j$ in (2) by using (2b) to obtain a stronger valid inequality for (DR-CAP) in the following theorem.

Theorem 3. *For $k \in \Omega\backslash\{\omega\}$, let δ_k be defined as in Theorem 2, and sort δ_k such that $\delta_{\bar{k}_1} \leq \ldots \leq \delta_{\bar{k}_{|\Omega|-1}}$. Let $\bar{q}^1 := \min\{l \,|\, \sup_{\boldsymbol{p}\in\mathscr{P}} \sum_{j=1}^{l} p_{\bar{k}_j} > \varepsilon\}$. Then, the inequality (1) is valid for (DR-CAP) when $\gamma = \delta_{\bar{k}_{\bar{q}^1}}$. Moreover, if $\{\hat{p}_\omega\}_{\omega\in\Omega} \in \mathscr{P}$, let $\hat{q}^1 := \min\{l \,|\, \sum_{j=1}^{l} \hat{p}_{\bar{k}_j} > \varepsilon\}$. Then, $\hat{q}^1 \geq \bar{q}^1$ and the inequality (1) is valid for (DR-CAP) when $\gamma = \delta_{\bar{k}_{\hat{q}^1}}$.*

Proof. Let

$$\gamma = \underset{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}\cap\mathcal{Y}_j}{\text{maximize}} \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\mathcal{I}\backslash\mathcal{C}} \alpha_i y_{ij} + \sum_{i\in\mathcal{D}} \beta_i y_{ij} - |\mathcal{C}\backslash\mathcal{D}| - \sum_{i\in\mathcal{D}} \beta_i + 1 \tag{5a}$$

$$\text{subject to} \inf_{\boldsymbol{p}\in\mathscr{P}} \sum_{k\in\Omega\backslash\{\omega\}} p_k \mathbb{1}\left(\sum_{i\in\mathcal{I}} \xi_i^k y_{ij} \leq t_j\right) \geq 1-\varepsilon. \tag{5b}$$

$\boldsymbol{y}_j$ satisfies the chance constraint (2b) and $z_\omega = 0$ for computing γ, (1) is valid for (DR-CAP).

Let $\hat{\boldsymbol{y}}_j$ be an optimal solution of (5). Then, $\sum_{i\in\mathcal{I}} \xi_i^{k'} \hat{y}_{ij} \leq t_j$ for at least one $k' \in \{\bar{k}_1, \ldots, \bar{k}_{\bar{q}}\}$. Otherwise, if $\sum_{i\in\mathcal{I}} \xi_i^k \hat{y}_{ij} > t_j$ for all $k \in \{\bar{k}_1, \ldots, \bar{k}_{\bar{q}}\}$, we have $\sup_{\boldsymbol{p}\in\mathscr{P}} \sum_{k\in\{\bar{k}_1,\ldots,\bar{k}_{\bar{q}}\}} p_k \mathbb{1}\left(\sum_{i\in\mathcal{I}} \xi_i^k \hat{y}_{ij} > t\right) > \varepsilon$, which indicates that (5b) is violated by $\hat{\boldsymbol{y}}_j$. Therefore, $\hat{\boldsymbol{y}}_j$ is a feasible solution of (3) for $k = k'$. We have $\delta_{\bar{k}_{\bar{q}}} \geq \delta_{k'} \geq \gamma$, then (1) is a valid inequality for (DR-CAP) when $\gamma = \delta_{\bar{k}_{\bar{q}}}$. Since $\sup_{\boldsymbol{p}\in\mathscr{P}} \sum_{j=1}^{\hat{q}^1} p_{\bar{k}_j} \geq \sum_{j=1}^{\hat{q}^1} \hat{p}_{\bar{k}_j} > \varepsilon$, we have $\hat{q}^1 \geq \bar{q}^1$, which implies $\delta_{\bar{k}_{\hat{q}^1}} \geq \delta_{\bar{k}_{\bar{q}^1}} \geq \gamma$, and (1) is a valid inequality for (DR-CAP) when $\gamma = \delta_{\bar{k}_{\hat{q}^1}}$. □

Appendix B: Proof of Propositions and Theorems

B.1. Proof of Proposition 1

Let $\boldsymbol{y}_j^*$ be an optimal solution of (4). Then, there exists at least one $k' \in \{k_1, \ldots, k_q\}$ such that $\sum_{i\in\mathcal{I}} \xi_i^{k'} y_{ij}^* \leq t_j$. Otherwise, we have $\sum_{i\in\mathcal{I}} \xi_i^k y_{ij}^* > t_j$, for $k \in \{k_1, \ldots, k_q\}$. Since $\sum_{j=1}^{q} p_{k_j} > \varepsilon$, the inequality $\mathbb{P}\left\{\sum_{i\in\mathcal{I}} \xi_i y_{ij}^* \leq t_j\right\} \geq 1-\varepsilon$ is violated. This is a contradiction. Therefore, $\boldsymbol{y}_j^*$ is a feasible solution of (5) with $k = k'$. Then $m_j^\omega(k_{q+1}) \geq \bar{M}_j^\omega$, $m_j^\omega(k_{q+1})$ is an upper bound for $\bar{M}_j^\omega$. □

B.2. Proof of Theorem 1

For $j \in \mathcal{J}$ and $\omega \in \Omega$, let $\hat{M}_j^\omega := \underset{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}}{\text{maximize}}\{\sum_{i\in\mathcal{I}} \xi_i^\omega y_{ij} \,\big|\, \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{P}\{\sum_{i\in\mathcal{I}} \xi_i y_{ij} \leq t_j\} \geq 1-\varepsilon,\ \boldsymbol{y}_j \in \mathcal{Y}_j\}$. We show that $m_j^\omega(k_{\bar{q}})$ is an upper bound for $\hat{M}_j^\omega$. Let y_{ij}^* be an optimal solution of the above maximization problem, there exists at least one $k' \in \bar{\Omega} := \{1, \cdots, \bar{q}\}$ such that $\sum_{i\in\mathcal{I}} \xi_i^{k'} y_{ij}^* \leq t_j$. Otherwise, $\sum_{i\in\mathcal{I}} \xi_i^k y_{ij}^* > t_j$ for $k \in \bar{\Omega}$, we have $\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{P}\left\{\sum_{i\in\mathcal{I}} \xi_i y_{ij}^* \leq t_j\right\} = \inf_{\boldsymbol{p}\in\mathscr{P}} \sum_{\omega\in\Omega\backslash\bar{\Omega}} p_\omega \mathbb{1}\left(\sum_{i\in\mathcal{I}} \xi_i^\omega y_{ij}^* \leq t_j\right) \leq \inf_{\boldsymbol{p}\in\mathscr{P}} \sum_{\omega\in\Omega\backslash\bar{\Omega}} p_\omega = \inf_{\boldsymbol{p}\in\mathscr{P}} \left(1 - \sum_{\omega\in\bar{\Omega}} p_\omega\right) = 1 - \sup_{\boldsymbol{p}\in\mathscr{P}} \sum_{\omega\in\bar{\Omega}} p_\omega < 1-\varepsilon$, which is a contradiction. Thus, $m_j^\omega(k_{\bar{q}}) \geq \hat{M}_j^\omega$. Therefore, (DR-CAP) can be rewritten as (8). □

B.3. Proof of Proposition 2

Let $(\boldsymbol{y},\boldsymbol{z})$ be a feasible solution of the relaxation problem of the binary bilinear reformulation of (DR-CAP). We have $\sum_{i\in\mathcal{I}}\xi_i^\omega y_{ij}(z_j^\omega-1)-m_j^\omega(k_{\bar{q}})(z_j^\omega-1)=(z_j^\omega-1)(\sum_{i\in\mathcal{I}}\xi_i^\omega y_{ij}-m_j^\omega(k_{\bar{q}}))$. If $m_j^\omega(k_{\bar{q}})\geq \bar{m}'_{j\omega}$, $\sum_{i\in\mathcal{I}}\xi_j^\omega y_{ij}m_j^\omega(k_{\bar{q}})\leq 0$, which implies that $(z_{j\omega}-1)\left(\sum_{i\in\mathcal{I}}\xi_j^\omega y_{ij}m_j^\omega(k_{\bar{q}})\right)\geq 0$. Consequently, $\sum_{i\in\mathcal{I}}\xi_i^\omega y_{ij}+m_j^\omega(k_{\bar{q}})(z_j^\omega-1)\leq\sum_{i\in\mathcal{I}}\xi_i^\omega y_{ij}z_j^\omega\leq m_j^\omega(\omega)z_j^\omega$ holds. Therefore, $(\boldsymbol{y},\boldsymbol{z})$ is a feasible solution of the relaxation problem of (8). The proof can be similarly extend to (CAP). □

B.4. Proof of Proposition 3

The set $\mathcal{H}=\bigcap_{j\in\mathcal{J}}\{(\boldsymbol{y},\boldsymbol{z})|(\boldsymbol{y}_j,\boldsymbol{z}_j)\in\mathcal{G}_j\}$ implies that $\mathcal{H}\subseteq\mathcal{G}_j$. Thus, if an inequality is valid for $conv(\mathcal{G}_j)$, then it is also valid for $conv(\mathcal{H})$. If an inequality is facet-defining for $conv(\mathcal{G}_j)$, then there exits $|\mathcal{I}|+N$ affinely independent points that satisfy this inequality at equality. Because this inequality does not have coefficients with respect to a pair of $(\boldsymbol{y}_{j_1},\boldsymbol{z}_{j_1})$ for $j_1\in\mathcal{J}$ and $j_1\neq j$, we can extend the $|\mathcal{I}|+N$ affinely independent points to a set of $|\mathcal{I}|\times|\mathcal{J}|+|\mathcal{J}|\times N$ affinely independent points by appropriately setting the values of $(\boldsymbol{y}_{j_1},\boldsymbol{z}_{j_1})$ for each $j_1\in\mathcal{J}$ and $j_1\neq j$. □

B.5. Proof of Proposition 4

The inequality (11) is valid for (10) based on the definition of $\mathcal{C}$.

Consider the following $|\mathcal{C}\backslash\mathcal{D}|$ feasible points of (10): for $k\in\mathcal{C}\backslash\mathcal{D}$, set $y_{ij}=1,\forall i\in\mathcal{C}\backslash\{\mathcal{D}\cup k\}$, $y_{ij}=0,\forall i\in k\bigcup(\mathcal{I}\backslash\mathcal{C})$, and $y_{ij}=1,\forall i\in\mathcal{D}$; These $|\mathcal{C}\backslash\mathcal{D}|$ points are affinely independent and satisfy (11) at equality. When $|\mathcal{C}|\leq\rho_j+1$, these $|\mathcal{C}\backslash\mathcal{D}|$ points are feasible. □

B.6. Proof of Lemma 1

Suppose that there exists $\hat{\boldsymbol{y}}_j$ that serves as a member of the set $\{\boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}|\sum_{i\in\mathcal{I}}\xi_i^\omega y_{ij}\leq m_j^\omega(\omega),\boldsymbol{y}_j\in\mathcal{Y}_j,y_{ij}=1,\forall i\in\mathcal{D}\}$ such that $\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}\leq|\mathcal{C}\backslash\mathcal{D}|-1$ and $\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\mathcal{I}\backslash\mathcal{C}}\alpha_i\hat{y}_{ij}>|\mathcal{C}\backslash\mathcal{D}|-1$. Let $r:=\max\{k|\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\boldsymbol{\pi}(k)}\alpha_i\hat{y}_{ij}\leq|\mathcal{C}\backslash\mathcal{D}|-1\}$. We have

$$\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\boldsymbol{\pi}(r+1)}\alpha_i\hat{y}_{ij}=\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\boldsymbol{\pi}(r)}\alpha_i\hat{y}_{ij}+(|\mathcal{C}\backslash\mathcal{D}|-1-obj_{\pi_{r+1}})\hat{y}_{\pi_{r+1},j}\leq|\mathcal{C}\backslash\mathcal{D}|-1,$$

which is a contradiction. Thus, (12) is valid for (14).

Consider the following $|\mathcal{I}\backslash\mathcal{D}|$ feasible points of (14): for $k\in\mathcal{C}\backslash\mathcal{D}$, set $y_{ij}=1,\forall i\in\mathcal{C}\backslash\{\mathcal{D}\cup k\}$, and $y_{ij}=0,\forall i\in k\bigcup(\mathcal{I}\backslash\mathcal{C})$; for $k=1,\cdots,|\mathcal{I}\backslash\mathcal{C}|$, set $y_{\pi_k j}=1$, $y_{ij}=0,\forall i\in\{\pi_{k+1},\cdots,\pi_{|\mathcal{I}\backslash\mathcal{C}|}\}$, and $\{y_{ij}\}_{i\in(\mathcal{C}\backslash\mathcal{D})\bigcup\{\pi_1,\ldots,\pi_{k-1}\}}$ are the optimal solutions of (13). All these points have $y_{ij}=1,\forall i\in\mathcal{D}$. When $|\mathcal{C}|\leq\rho_j+1$, the above $|\mathcal{I}\backslash\mathcal{D}|$ points are feasible, satisfy (12) at equality and are affinely independent. □

B.7. Proof of Lemma 2

Suppose that we have $\hat{\boldsymbol{y}}_j\in\mathcal{Q}_{j\omega}$ that violates (15). $\boldsymbol{\kappa}$ can be partitioned into $\mathcal{D}^0:=\{i\in\boldsymbol{\kappa}|\hat{y}_{ij}=0\}$ and $\mathcal{D}^1:=\{i\in\boldsymbol{\kappa}|\hat{y}_{ij}=1\}$. We assume that the last element in the set $\mathcal{D}^0$ is κ_h where $h\leq|\mathcal{D}|$. Then, we have $\sum_{i\in\mathcal{C}\backslash\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\mathcal{I}\backslash\mathcal{C}}\alpha_i\hat{y}_{ij}>|\mathcal{C}\backslash\mathcal{D}|+\sum_{i\in\mathcal{D}^0}\beta_i-1$. Note that $|\mathcal{C}\backslash\mathcal{D}|+\sum_{i\in\mathcal{D}^0}\beta_i-1=|\mathcal{C}\backslash\mathcal{D}|+obj_{\kappa_h}-\sum_{i\in\boldsymbol{\kappa}(h-1)}\beta_i-|\mathcal{C}\backslash\mathcal{D}|+1+\sum_{i\in\mathcal{D}^0\backslash\kappa_h}\beta_i-1=obj_{\kappa_h}-\sum_{i\in\boldsymbol{\kappa}(h-1)}\beta_i+\sum_{i\in\mathcal{D}^0\backslash\kappa_h}\beta_i$. Based on the definition of obj_{κ_h}, we have that $\hat{\boldsymbol{y}}_j$ is a feasible solution of (16) with $l=h$. Then, $obj_{\kappa_h} - \sum_{i\in\boldsymbol{\kappa}(h-1)} \beta_i + \sum_{i\in\mathcal{D}^0\setminus\kappa_h} \beta_i \geq \sum_{i\in\mathcal{C}\setminus\mathcal{D}} \hat{y}_{ij} + \sum_{i\in\mathcal{I}\setminus\mathcal{C}} \alpha_i\hat{y}_{ij}$. This is a contradiction. Thus, (15) is valid for $conv(\mathcal{Q}_{j\omega})$.

Consider the following $|\mathcal{I}|$ feasible points of $conv(\mathcal{Q}_{j\omega})$: when $y_{ij}=1, \forall i\in\mathcal{D}$, then there exists $|\mathcal{I}\setminus\mathcal{C}|$ feasible points that are independent and satisfy the inequality (15) at equality based on Lemma 1; for $l\in\{1,\cdots,|\mathcal{D}|\}$, set $\boldsymbol{y}_j$ is the optimal solution of (16). When $|\mathcal{C}|\leq\rho_j+1$, these $|\mathcal{I}|$ points are feasible, satisfy the inequality (15) at equality and are affinely independent. □

B.8. Proof of Theorem 2

We first prove that for (CAP) if the coefficients are described in Theorem 2, then, (21) is valid for $conv(\mathcal{G}_j)$. For $k\in\{1,\ldots,|\bar{\Omega}|\}$, let $(\hat{\boldsymbol{y}}_j,\hat{\boldsymbol{z}}_j)\in\mathcal{G}_j$. If $\hat{z}_{j\omega}=1$ for $\omega\in\Omega_k$, then (21) is valid for $conv(\mathcal{G}_j)$. Otherwise, let $\boldsymbol{\tau}$ be partitioned into $\Omega_k^0=\{\omega\in\boldsymbol{\tau}|\hat{z}_{j\omega}=0\}$ and $\Omega_k^1=\{\omega\in\boldsymbol{\tau}|\hat{z}_{j\omega}=1\}$. We assume that the last element of Ω_k^0 is τ_h where $h\leq|\Omega_k|$. (21) becomes $\sum_{i\in\mathcal{C}\setminus\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\mathcal{I}\setminus\mathcal{C}}\bar{\alpha}_i\hat{y}_{ij}+\sum_{i\in\mathcal{D}}\bar{\beta}_i\hat{y}_{ij}\leq|\mathcal{C}\setminus\mathcal{D}|+\sum_{i\in\mathcal{D}}\bar{\beta}_i-1+\sum_{\omega\in\Omega_k^0}\bar{\gamma}_\omega$. Note that $|\mathcal{C}\setminus\mathcal{D}|+\sum_{i\in\mathcal{D}}\bar{\beta}_i-1+\sum_{\omega\in\Omega_k^0}\bar{\gamma}_\omega=obj_{\tau_h}-\sum_{\omega\in\boldsymbol{\tau}(h-1)}\bar{\gamma}_\omega+\sum_{\omega\in\{\Omega_k^0\setminus\tau_h\}}\bar{\gamma}_\omega$. Since $(\hat{\boldsymbol{y}}_j,\hat{\boldsymbol{z}}_j)$ satisfies (25) with $k=h$, we have $obj_{\tau_h}-\sum_{\omega\in\boldsymbol{\tau}(h-1)}\bar{\gamma}_\omega+\sum_{\omega\in\{\Omega_k^0\setminus\tau_h\}}\bar{\gamma}_\omega\geq\sum_{i\in\mathcal{C}\setminus\mathcal{D}}\hat{y}_{ij}+\sum_{i\in\mathcal{I}\setminus\mathcal{C}}\bar{\alpha}_i\hat{y}_{ij}+\sum_{i\in\mathcal{D}}\bar{\beta}_i\hat{y}_{ij}$. Thus, (21) is valid for $conv(\mathcal{G}_j)$ when $\bar{\gamma}_{\tau_l}=obj_{\tau_l}-|\mathcal{C}\setminus\mathcal{D}|+1-\sum_{i\in\mathcal{D}}\bar{\beta}_i-\sum_{\omega\in\boldsymbol{\tau}(l-1)}\bar{\gamma}_\omega$ for $l=1,\cdots,|\Omega_k|$. Note that $\bar{\alpha}_{\bar{\pi}_l}=|\mathcal{C}\setminus\mathcal{D}|-1-\min_{\omega\in\Omega_k}obj_{\bar{\pi}_l}(\omega)$, based on the definition of $obj_{\bar{\pi}_l}(\omega)$, it is easy to see that $obj_{\bar{\pi}_1}(\omega)$ is integer, and consequently $\bar{\alpha}_{\bar{\pi}_1}$ is integer. If $\bar{\alpha}_{\bar{\pi}_1}$ is integer, then $obj_{\bar{\pi}_2}(\omega)$ is integer, which implies $\bar{\alpha}_{\bar{\pi}_2}$ is integer. Using these arguments we know that $\bar{\boldsymbol{\alpha}}$ is integer. Similarly, $\bar{\boldsymbol{\beta}}$ is also integer. Since the coefficients in (21) are integers, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are binary, obj_{τ_l} is integer. $obj^r_{\tau_l}$ is an upper bound on obj_{τ_l} and obj_{τ_l} is integer, thus $\lfloor obj^r_{\tau_l}\rfloor$ is also an upper bound on obj_{τ_l}. Therefore, (21) is valid for $conv(\mathcal{G}_j)$ when $\bar{\gamma}_{\tau_l}=\lfloor obj^r_{\tau_l}\rfloor-|\mathcal{C}\setminus\mathcal{D}|+1-\sum_{i\in\mathcal{D}}\bar{\beta}_i-\sum_{\omega\in\boldsymbol{\tau}(l-1)}\bar{\gamma}_\omega$ for $l=1,\cdots,|\Omega_k|$. The proof is similarly extended to $\mathcal{G}'_j$. □

B.9. Poof of Theorem 3

The algorithm processes a finite number of nodes as it is based on branching on a finite number of binary variables. When there exists an oracle that solve (SP$_j$) to optimality, we can obtain an optimal solution of (SP$_j$) and verify the feasibility of $(\boldsymbol{y}^k,\boldsymbol{z}^k)$ from (MP) to (DR-CAP). In addition, since a finite number of integer solutions are obtained from (MP), (SP$_j$) is solved finite times and the set of feasibility cuts generated in line 12 is finite. Thus, Algorithm 1 terminates in finitely many iterations. Next, we show that the cuts (28) and (29) can remove the current infeasible solution and never cut off any feasible solutions of (DR-CAP). It can be verified that (28) and (29) can remove the current infeasible solution. Also, $\sum_{\omega\in\Omega}p^k_\omega z_{j\omega}\geq\inf_{\boldsymbol{p}\in\mathscr{P}}\sum_{\omega\in\Omega}p_\omega z_{j\omega}\geq1-\varepsilon$. Thus, (28) never cuts off any feasible solutions of (DR-CAP). We assume that $\tilde{\boldsymbol{y}}$ is a new future solution from (MP) and the corresponding set $\tilde{\mathcal{I}}^1_j$. Let $y_{ij}=\tilde{y}_{ij}$, for $i\in\mathcal{I}$. Then the feasibility cut (29) becomes $\sum_{i\in\mathcal{I}^1_j}\tilde{y}_{ij}\leq|\mathcal{I}^1_j|-1$, which is decomposed to $\sum_{i\in\mathcal{I}^1_j\cap\tilde{\mathcal{I}}^1_j}\tilde{y}_{ij}+\sum_{i\in\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j}\tilde{y}_{ij}\leq|\mathcal{I}^1_j\cap\tilde{\mathcal{I}}^1_j|+|\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j|-1 \iff \sum_{i\in\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j}\tilde{y}_{ij}\leq|\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j|-1$. If $\mathcal{I}^1_j\subseteq\tilde{\mathcal{I}}^1_j$, $\tilde{\boldsymbol{y}}$ is not a feasible solution, and does not satisfy the feasibility cut. Otherwise, $\sum_{i\in\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j}\tilde{y}_{ij}=0$ and $|\mathcal{I}^1_j\setminus\tilde{\mathcal{I}}^1_j|-1\geq0$. □

Appendix C: Algorithm Details

C.1. Dynamic Programming for Up-lifting Coefficient

For $k = 1,\ldots,|\mathcal{I}\backslash\mathcal{C}|$, $\lambda_1 = 0,\ldots,|\mathcal{C}\backslash\mathcal{D}|-1$, and $\lambda_2 = 0,\ldots,\rho_j - 1 - |\mathcal{D}|$, let $A_{\pi_k}(\lambda_1,\lambda_2) = \underset{\boldsymbol{y}_j\in\{0,1\}^{|(\mathcal{C}\backslash\mathcal{D})\cup\boldsymbol{\pi}(k-1)|}}{\text{minimize}} \{\sum_{i\in\mathcal{C}\backslash\mathcal{D}} \xi_i^\omega y_{ij} + \sum_{i\in\boldsymbol{\pi}(k-1)} \xi_i^\omega y_{ij} | \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\boldsymbol{\pi}(k-1)} \alpha_i y_{ij} \geq \lambda_1, \sum_{i\in\mathcal{C}\backslash\mathcal{D}} y_{ij} + \sum_{i\in\boldsymbol{\pi}(k-1)} y_{ij} \leq \lambda_2\}$ and l_t, $t=0,\ldots,|\mathcal{C}\backslash\mathcal{D}|-1$ be the sum of the t smallest ξ_i^ω, $i\in\mathcal{C}\backslash\mathcal{D}$. Algorithm 1 gives an outline of our dynamic programming framework. Since Algorithm 1 is a dynamic programming based approach, it is easy to see that it has the complexity $O\left(|\mathcal{I}\backslash\mathcal{C}|\cdot(\rho-|\mathcal{D}|)\cdot|\mathcal{C}\backslash\mathcal{D}|\right)$ for calculating the up-lifting coefficients exactly.

Algorithm 1: Dynamic Programming for the Lifting Coefficients

for $\lambda_2 = 0,\ldots,\rho_j - 1 - |\mathcal{D}|$ **do**
    for $\lambda_1 = 0,\ldots,|\mathcal{C}\backslash\mathcal{D}|-1$ **do**
        if $\lambda_1 \leq \lambda_2$ **then**
            $A_{\pi_1}(\lambda_1,\lambda_2) = l_{\lambda_1}$.
        end
        else
            $A_{\pi_1}(\lambda_1,\lambda_2) = +\infty$.
        end
    end
end
for $k = 1,\ldots,|\mathcal{I}\backslash\mathcal{C}|$ **do**
    $obj_{\pi_k} = \max\left\{\lambda_1 : A_{\pi_k}(\lambda_1,\rho_j - 1 - |\mathcal{D}|) \leq m_j^\omega(\omega) - \xi_{\pi_k}^\omega - \sum_{i\in\mathcal{D}} \xi_i^\omega\right\}$, $\alpha_{\pi_k} = |\mathcal{C}\backslash\mathcal{D}| - 1 - obj_{\pi_k}$.
    for $\lambda_2 = 0,\ldots,\rho_j - 1 - |\mathcal{D}|$ **do**
        for $\lambda_1 = 0,\ldots,|\mathcal{C}\backslash\mathcal{D}|-1$ **do**
            if $\lambda_1 \geq \alpha_{\pi_k}$ *and* $\lambda_2 \geq 1$ **then**
                $A_{\pi_{k+1}}(\lambda_1,\lambda_2) = \min\left\{A_{\pi_k}(\lambda_1,\lambda_2), A_{\pi_k}(\lambda_1 - \alpha_{\pi_k},\lambda_2 - 1) + \zeta_{\pi_k}^\omega\right\}$.
            end
            else
                $A_{\pi_{k+1}}(\lambda_1,\lambda_2) = A_{\pi_k}(\lambda_1,\lambda_2)$.
            end
        end
    end
end

C.2. Separation Heuristic for (1)

Algorithm 2 gives an overview of separation heuristic for (1).

C.3. Separation Heuristic for (21)

Algorithm 3 gives an overview of separation heuristic for (21).

Algorithm 2: Separation Heuristic for (1)

---

1 Given the LP relaxation optimal solution $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$.
2 **for** $j = 1, \ldots, |\mathcal{J}|$ **do**
3   **for** $\omega = 1, \ldots, N$ **do**
4   **if** $z_{j\omega} = 1$ **then**
5   Sort $\hat{\boldsymbol{y}}_j$: $\hat{y}_{i_1 j} \geq \ldots \geq \hat{y}_{i_{|\mathcal{I}|} j}$. Let $\mathcal{C} = \{i_1, \ldots, i_o\}$ where $o \leq |\mathcal{I}|$ is a smallest number such that $\mathcal{C}$ is a cover.
6   Delete elements from $\mathcal{C}$ in non-decreasing order of $\hat{y}_j$ to get a minimal cover $\mathcal{C}$.
7   Let $\mathcal{D} = \{i \in \mathcal{C} : \hat{y}_{ij} = 1\}$ and $\mathcal{I}_0 = \{i \in \mathcal{I} \backslash \mathcal{C} | \hat{y}_{ij} = 0\}$. Calculate α_i for $i \in \mathcal{I} \backslash (\mathcal{C} \cup \mathcal{I}_0)$.
8   **if** $\sum_{i \in \mathcal{C} \backslash \mathcal{D}} \hat{y}_{ij} + \sum_{i \in \mathcal{I} \backslash (\mathcal{C} \cup \mathcal{I}_0)} \alpha_i \hat{y}_{ij} > |\mathcal{C} \backslash \mathcal{D}| - 1$ **then**
9   Calculate β_i for $i \in \mathcal{D}$, and α_i for $i \in \mathcal{I}_0$.
10   Calculate δ_k, $k \in \Omega \backslash \omega$, set $\gamma = \delta_{\bar{k}_q 1}$ for (CAP), $\gamma = \delta_{\bar{k}_{\bar{q}} 1}$ for (DR-CAP). Obtain the inequality (1).
11   **end**
12   **end**
13   **end**
14 **end**

Algorithm 3: Separation Heuristic for (21)

---

1 Given the LP relaxation optimal solution $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$.
2 **for** $j = 1, \ldots, |\mathcal{J}|$ **do**
3   Let $\Omega_1 = \{\omega \in \Omega | \hat{z}_{j\omega} = 1\}$.
4   **if** $\sum_{\omega \in \Omega_1} p_\omega \hat{z}_{j\omega} \geq 1 - \varepsilon$ *(for (CAP)) or* $\inf_{\boldsymbol{p} \in \mathscr{P}} \sum_{\omega \in \Omega_1} p_\omega \hat{z}_{j\omega} \geq 1 - \varepsilon$ *(for (DR-CAP))* **then**
5   Sort $\hat{\boldsymbol{y}}_j$ in non-increasing order: $\hat{y}_{i_1 j} \geq \ldots \geq \hat{y}_{i_{|\mathcal{I}|} j}$.
6   **for** $\omega \in \Omega_1$ **do**
7   Let $\mathcal{C} = \{i_1, \ldots, i_o\}$ where $o \leq |\mathcal{I}|$ is a smallest number such that $\mathcal{C}$ is a cover for ω.
8   Delete elements from $\mathcal{C}$ in non-decreasing order of $\hat{y}_j$ to get a minimal cover $\mathcal{C}$.
9   Let set $\mathcal{D} = \{i \in \mathcal{C} | \hat{y}_{ij} = 1\}$ and $\mathcal{I}_0 = \{i \in \mathcal{I} \backslash \mathcal{C} | \hat{y}_{ij} = 0\}$. Calculate $\bar{\alpha}_i$ for $i \in \mathcal{I} \backslash (\mathcal{C} \cup \mathcal{I}_0)$.
10   **if** $\sum_{i \in \mathcal{C} \backslash \mathcal{D}} \hat{y}_{ij} + \sum_{i \in \mathcal{I} \backslash (\mathcal{C} \cup \mathcal{I}_0)} \bar{\alpha}_i \hat{y}_{ij} > |\mathcal{C} \backslash \mathcal{D}| - 1$ **then**
11   Calculate $\bar{\beta}_i$ for $i \in \mathcal{D}$, $\bar{\alpha}_i$ for $i \in \mathcal{I}_0$, and γ_ω for $\omega \in \Omega_1$. Obtain the violated inequality (21).
12   **end**
13   If (21) is obtained, go to step 2.
14   **end**
15   **end**
16 **end**

C.4. Branch-and-Cut Algorithm

The branch-and-cut algorithm is provided in Algorithm 4.

Algorithm 4: Branch-and-Cut Implementation

1 **Initialize** UB $=+\infty$, LB $=-\infty$, $k=0$. Node list $\mathcal{N}=\{o\}$, o is a branching node without constraints.
2 **while** *($\mathcal{N}$ is nonempty)* **do**
3   Select a node $o \in \mathcal{N}$, $\mathcal{N} \leftarrow \mathcal{N}/\{o\}$.
4   At the node o, solve the LP relaxation problem of (IP). $k=k+1$.
5   Obtain an optimal solution $(\boldsymbol{y}^k, \boldsymbol{z}^k)$ and objective value obj^k.
6   **if** $obj^k < UB$ **then**
7   **if** $(\boldsymbol{y}^k, \boldsymbol{z}^k)$ *is fractional* **then**
8   **if** *Violated inequalities are found* **then**
9   Add the violated inequalities to the LP relaxation problem. Go to line 5.
10   **end**
11   **else**
12   Branch, resulting in nodes o^* and o^{**}, $\mathcal{N} \leftarrow \mathcal{N} \cup \{o^*, o^{**}\}$.
13   **end**
14   **end**
15   **else**
16   Update UB, UB $= obj^k$, $(\boldsymbol{y}^*, \boldsymbol{z}^*)=(\boldsymbol{y}^k, \boldsymbol{z}^k)$.
17   **end**
18   **end**
19 **end**
20 **return** UB and its corresponding optimal solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$.

C.5. Branch-and-Cut with Probability Cuts Algorithm for (DR-JCAP)

Algorithm 5 gives an overview of the branch-and-cut with probability cuts algorithm for (DR-JCAP).

Appendix D: Dynamic Programming Approach for Computing Big-M values

In this appendix, we use the *dynamic programming* approach proposed by **?** to compute the Big-M values in the model reformulation. For $j \in J$, let $D(|\mathcal{I}|, t_j, \rho_j)$ represents (5), where $|\mathcal{I}|$ denotes the $|\mathcal{I}|$ variables of $\boldsymbol{y}_j$. Let $D(n, t_j, \rho_j)$ be a subproblem of $D(|\mathcal{I}|, t_j, \rho_j)$, where n denotes the first n variables of $\boldsymbol{y}_j$ in (5). Let $S(n, t_j, \rho_j)$ be the optimal objective value of $D(n, t_j, \rho_j)$. If $D(n, t_j, \rho_j)$ is infeasible, we set $S(n, t_j, \rho_j) = -\infty$. Note that if $y_{nj} = 0$, $S(n, t_j, \rho_j)$ is equal to $S(n-1, t_j, \rho_j)$. If $y_{nj} = 1$, $S(n, t_j, \rho_j)$ is equal to $S(n-1, t_j - \xi_n^k, \rho_j - 1) + \xi_n^\omega$, Thus, we have

$$S(n, t_j, \rho_j) = \max\{S(n-1, t_j, \rho_j), S(n-1, t_j - \xi_n^k, \rho_j - 1) + \xi_n^\omega\},$$

where $n = 2, \ldots, |\mathcal{I}|$, with an initial condition $S(1, t_j, \rho_j)$. Hence,

$$m_j^\omega(k) = S(|\mathcal{I}|, t_j, \rho_j).$$

Appendix E: Statistics of Surgery Duration

Table 1 presents the statistics of surgery duration for the real-life data, i.e. mean, standard deviation and the percentage for each surgery type.

Appendix F: Computational Results using Weaker Big-M in (CAP)

Table 2 reports computational results for the weaker big-M of (CAP)

Algorithm 5: Branch-and-Cut Algorithm with Probability Cuts for (DR-JCAP)

1 ***Initialize*** $\mathbb{P}^0 \in \mathscr{P}$, the number of iteration $k = 0$, $\mathrm{UB} = +\infty$, $\mathrm{LB} = -\infty$, $\mathcal{N} = \{o\}$, o has no branching constraints.

2 Initialize the root node with the LP relaxation of (MP). Let the LP relaxation of (MP) be denoted by (LMP).

$$\text{(MP)} \qquad \underset{(\boldsymbol{y},\boldsymbol{z'}) \in \left\{\{0,1\}^{|I||J|} \times \{0,1\}^N\right\} \cap \mathcal{X}}{\text{minimize}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} y_{ij}$$
$$\text{subject to } (1b), (1c), (33b),$$

3 **while** *($\mathcal{N}$ is nonempty)* **do**
4   Select a node $o \in \mathcal{N}$, $\mathcal{N} \leftarrow \mathcal{N}/\{o\}$.
5   Solve (LMP) at the node o. $k = k+1$.
6   Obtain the optimal solution (y^k, z'^k) and the optimal objective $lobj^k$ of (LMP).
7   **if** $lobj^k < UB$ **then**
8   **if** (y^k, z'^k) *is an integer* **then**
9   Solve (SP), and obtain an optimal solution (p^k) and objective value $uobj^k$
10   **if** $uobj^k < 1 - \varepsilon$ **then**
11   Add the cuts (34) and $\sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}_j^1} y_{ij} \leq |\mathcal{I}| - 1$ to (LMP).
12   **end**
13   **if** *The cuts in Step 11 are found* **then**
14   Go to step 5.
15   **end**
16   **else**
17                 $\mathrm{UB} = lobj^k$, $(y^*, z'^*) = (y^k, z'^k)$.
18   **end**
19   **end**
20   **if** $(\boldsymbol{y}^k, \boldsymbol{z'}^k)$ *is fractional* **then**
21   Use the algorithms that are similar to Algorithm 2 and 3 to find the violated inequalities (35) and (38).
22   **if** *Violated inequalities are found* **then**
23   Add the violated inequalities to (LMP). Go to line 5.
24   **end**
25   **else**
26   Branch, resulting in nodes o^* and o^{**}, $\mathcal{N} \leftarrow \mathcal{N} \cup \{o^*, o^{**}\}$.
27   **end**
28   **end**
29   **end**
30 **end**
31 **return** UB and its corresponding optimal solution (y^*, z'^*).

Table 1 For each surgery type, the mean (mean), standard deviation (std) in hours, and the percentage for each surgery type (percentage) are reported

surgery type	mean (hrs)	std (hrs)	percentage
Gynaecology	1.1	1.3	0.29
Galactophore	1.6	1.0	0.15
Lymphatic	3.2	1.1	0.14
Ear	2.8	1.7	0.13
Urology	2.3	1.7	0.07
Vascular	2.6	1.5	0.07
Obstetrics	1.5	0.5	0.06
Joint	2.8	1.3	0.06
Orthopeadic	3.2	1.8	0.03

Table 2 The average time (in seconds) for the weaker big-M computations (AvT-M), the branch-and-cut algorithm (AvT-B&C), the average number of nodes (# of nodes), and the number of instances solved to optimality (solved).

ε	N	AvT-M	AvT-B&C	# of nodes	solved
0.12	500	11.4	122.6	1,798	5/5
	1000	43.8	219.7	2,088	5/5
	1500	98.7	771.0	5,090	5/5
0.1	500	11.4	164.9	3,914	5/5
	1000	43.8	604.7	7,192	5/5
	1500	98.7	2,298.8	11,049	5/5
0.08	500	11.4	1,290.8	42,876	5/5
	1000	43.8	2,777.8	25,874	5/5
	1500	98.7	8,459.9[0.03]	103,689	4/5
0.06	500	11.4	[0.11]	2,232,748	0/5
	1000	43.8	[0.21]	632,822	0/5
	1500	98.7	[0.28]	362,215	0/5

"[·]" in column of *AvT-B&C* means the average sub-optimality gap for instances that cannot be solved to optimality within 10 hours time limit.