Appendix. Proofs

This appendix provides:

- Proofs for Propositions 1, 2, and 3 (Section 3.3)
- Proofs for Theorem 1 and Corollary 1 (Section 4.1)
- An example related to Theorem 1
- A proof for Proposition 4 (Section 5.3).

Proof of Proposition 1. From Equation (4) in Definition 3, we know that $b_\rho(A,B) = \rho\big((E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)\big)$. Then Pereira et al. (2009) prove that

$$(E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B) \subseteq (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C), \tag{32}$$

to show that the triangle inequality holds for boundary distance, because Equation (32) implies

$$\begin{aligned} b_\rho(A,B) &= \rho\left((E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)\right) \\ &\le \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\ &\le \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\ &\le b_\rho(A,C) + b_\rho(B,C). \end{aligned}$$

To prove the backward direction of Proposition 1, suppose $E_A \cap E_B \subseteq_\rho E_C \subseteq_\rho E_A \cup E_B$. It suffices to show that the assumptions imply $b_\rho(A,B) \ge b_\rho(A,C) + b_\rho(B,C)$. This proof relies on the following two claims:

Claim 1: $(E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C) \subseteq_\rho (E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)$.

Claim 2: $\rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right)$
$= \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right)$.

If Claims 1 and 2 hold, then we have

$$\begin{aligned} b_\rho(A,C) + b_\rho(B,C) &= \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\ &= \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\ &\le \rho\left((E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)\right) \\ &= b_\rho(A,B), \end{aligned}$$

as desired. We now prove Claims 1 and 2.

Proof of Claim 1. Let $e \in (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)$ with $\rho(e) > 0$. Then, based on the assumptions, at least one of the following four cases holds:

1. $e \in E_A \cap E_C^{\mathsf{c}} \subseteq_\rho E_A \cap (E_A \cap E_B)^{\mathsf{c}} = E_A \cap E_B^{\mathsf{c}}$,
2. $e \in E_A^{\mathsf{c}} \cap E_C \subseteq_\rho E_A^{\mathsf{c}} \cap (E_A \cup E_B) = E_A^{\mathsf{c}} \cap E_B$,
3. $e \in E_B \cap E_C^{\mathsf{c}} \subseteq_\rho E_B \cap (E_A \cap E_B)^{\mathsf{c}} = E_A^{\mathsf{c}} \cap E_B$,
4. $e \in E_B^{\mathsf{c}} \cap E_C \subseteq_\rho E_B^{\mathsf{c}} \cap (E_A \cup E_B) = E_A \cap E_B^{\mathsf{c}}$.

Therefore, in all cases, $e \in (E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)$. Hence, Claim 1 holds.

Proof of Claim 2. Suppose $e \in [(E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)] \cap [(E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)]$ and $\rho(e) > 0$. Then, based on the assumptions, at least one of the following four cases holds:

1. $e \in E_A \cap E_C^{\mathsf{c}}$ and $e \in E_B \cap E_C^{\mathsf{c}}$, which implies that $e \in E_A \cap E_B$ but $e \notin E_C$. This contradicts the assumption that $E_A \cap E_B \subseteq_\rho E_C$.
2. $e \in E_A \cap E_C^{\mathsf{c}}$ and $e \in E_B^{\mathsf{c}} \cap E_C$, which is a contradiction because it implies that $e \in \varnothing$.
3. $e \in E_A^{\mathsf{c}} \cap E_C$ and $e \in E_B \cap E_C^{\mathsf{c}}$, which is a contradiction because it implies that $e \in \varnothing$.
4. $e \in E_A^{\mathsf{c}} \cap E_C$ and $e \in E_B^{\mathsf{c}} \cap E_C$, which implies that $e \in E_C$ but $e \notin E_A \cup E_B$. This contradicts the assumption that $E_C \subseteq_\rho E_A \cup E_B$.

Because all cases lead to a contradiction, we must have

$$\rho\left([(E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)] \cap [(E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)]\right) = 0.$$

Therefore, it follows that

$$\begin{aligned}
&\rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\
&\qquad = \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\
&\qquad\qquad - \rho\left([(E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)] \cap [(E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)]\right) \\
&\qquad = \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right).
\end{aligned}$$

Hence, Claim 2 holds. Because Claims 1 and 2 hold, we have $b_\rho(A,B) = b_\rho(A,C) + b_\rho(B,C)$.

To prove the forward direction of Proposition 1, we proceed by contrapositive, i.e., we show that if $E_A \cap E_B \subseteq_\rho E_C \subseteq_\rho E_A \cup E_B$ does not hold, then $b_\rho(A,B) < b_\rho(A,C) + b_\rho(B,C)$. Suppose it is not the case that $E_A \cap E_B \subseteq_\rho E_C \subseteq_\rho E_A \cup E_B$. Then at least one of the following two cases holds:

1. If $E_A \cap E_B \nsubseteq_\rho E_C$, then there exists some $e \in E_A \cap E_B$ with $\rho(e) > 0$ such that $e \notin E_C$. Then $e \notin (E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)$, but $e \in E_A \cap E_C^{\mathsf{c}}$ and $e \in E_B \cap E_C^{\mathsf{c}}$. Therefore,
$$e \in (E_A \cap E_C^{\mathsf{c}}) \cap (E_B \cap E_C^{\mathsf{c}}) \subseteq (E_A \cap E_C^{\mathsf{c}}) \cup (E_B \cap E_C^{\mathsf{c}}) \subseteq (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C).$$
2. If $E_C \nsubseteq_\rho E_A \cup E_B$, then there exists some $e \in E_C$ with $\rho(e) > 0$ such that $e \notin E_A \cup E_B$. Then $e \notin (E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)$, but $e \in E_A^{\mathsf{c}} \cap E_C$ and $e \in E_B^{\mathsf{c}} \cap E_C$. Therefore,
$$e \in (E_A^{\mathsf{c}} \cap E_C) \cap (E_B^{\mathsf{c}} \cap E_C) \subseteq (E_A^{\mathsf{c}} \cap E_C) \cup (E_B^{\mathsf{c}} \cap E_C) \subseteq (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C).$$

In both cases, there exists some $e \in (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)$ with $\rho(e) > 0$ such that $e \notin (E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)$. Combined with Equation (32), this result implies

$$(E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B) \subsetneq_\rho (E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C),$$

which gives

$$\begin{aligned}
b_\rho(A,B) &= \rho\left((E_A \cap E_B^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_B)\right) \\
&< \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C) \cup (E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\
&\leq \rho\left((E_A \cap E_C^{\mathsf{c}}) \cup (E_A^{\mathsf{c}} \cap E_C)\right) + \rho\left((E_B \cap E_C^{\mathsf{c}}) \cup (E_B^{\mathsf{c}} \cap E_C)\right) \\
&= b_\rho(A,C) + b_\rho(B,C),
\end{aligned}$$

as desired. $\square$

Proof of Proposition 2. We omit the proof of Proposition 2 because it is nearly identical to the proof of Proposition 1. The sets of cut edges (e.g., E_A) are simply replaced with sets of separated vertex pairs (e.g., Q_A) and the edge weight ρ is replaced with $\hat{\omega}$. □

Proof of Proposition 3. First, we re-prove that the triangle inequality holds for the transfer distance; although previous authors have proven this property (e.g., Pinto da Costa and Rao (2004) and Cardoso and Corte-Real (2005)), the string of inequalities our proof provides is useful for proving this proposition. Let $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ be optimal (note that such an M_1 and M_2 must always exist). Consider $M_{12} = \{\{A_i, B_j\} \in A \times B : \{A_i, C_l\} \in M_1$ and $\{B_j, C_l\} \in M_2$ for some $C_l \in C\}$. Since M_1 and M_2 are perfect matchings on H_1 and H_2, respectively, M_{12} is a perfect matching on H_3. If we show that

$$R^{AB}(M_{12}) \subseteq R^{AC}(M_1) \cup R^{BC}(M_2), \tag{33}$$

then we have

$$\begin{aligned} t_\omega(A,B) &= \min_{M \in \mathcal{M}_3} \left\{\omega\left(R^{AB}(M)\right)\right\} \\ &\le \omega\left(R^{AB}(M_{12})\right) \\ &\le \omega\left(R^{AC}(M_1) \cup R^{BC}(M_2)\right) \\ &\le \omega\left(R^{AC}(M_1)\right) + \omega\left(R^{BC}(M_2)\right) \\ &= t_\omega(A,C) + t_\omega(B,C). \end{aligned} \tag{34}$$

We now show that $R^{AB}(M_{12}) \subseteq R^{AC}(M_1) \cup R^{BC}(M_2)$. Let $v \in R^{AB}(M_{12})$. We know that $v \in A_i \cap B_j \cap C_l$ for some $A_i \in A$, $B_j \in B$, and $C_l \in C$. By way of contradiction, suppose that $v \notin R^{AC}(M_1) \cup R^{BC}(M_2)$. Then $v \notin R^{AC}(M_1)$ and $v \notin R^{BC}(M_2)$. It follows that $\{A_i, C_l\} \in M_1$ and $\{B_j, C_l\} \in M_2$, meaning $\{A_i, B_j\} \in M_{12}$. This implies that $v \in F^{AB}(M_{12})$, which is a contradiction because $v \in R^{AB}(M_{12})$. Hence, we must have $v \in R^{AC}(M_1) \cup R^{BC}(M_2)$. So $R^{AB}(M_{12}) \subseteq R^{AC}(M_1) \cup R^{BC}(M_2)$, as desired.

Now we prove Proposition 3. To show the backward direction, suppose there exist optimal $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$, and $M_3 \in \mathcal{M}_3$ such that $R^{AB}(M_3) =_\omega R^{AC}(M_1) \cup R^{BC}(M_2)$ and $R^{AC}(M_1) \cap R^{BC}(M_2) =_\omega \varnothing$. Then we have

$$t_\omega(A,B) = \omega\left(R^{AB}(M_3)\right) = \omega\left(R^{AC}(M_1) \cup R^{BC}(M_2)\right) = \omega\left(R^{AC}(M_1)\right) + \omega\left(R^{BC}(M_2)\right) = t_\omega(A,C) + t_\omega(B,C),$$

which gives $t_\omega(A,B) = t_\omega(A,C) + t_\omega(B,C)$.

To prove the forward direction, suppose $t_\omega(A,B) = t_\omega(A,C) + t_\omega(B,C)$. Then Equation (34) has equalities throughout, meaning

$$t_\omega(A,B) = \omega\left(R^{AB}(M_{12})\right) = \omega\left(R^{AC}(M_1) \cup R^{BC}(M_2)\right) = \omega\left(R^{AC}(M_1)\right) + \omega\left(R^{BC}(M_2)\right) \tag{35}$$

for all optimal $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$. The first equality in Equation (35) implies that $M_{12} \in \mathcal{M}_3$ is optimal. Equation (33), the second equality in Equation (35), and Definition 1 imply that $R^{AB}(M_{12}) =_\omega R^{AC}(M_1) \cup R^{BC}(M_2)$. Lastly, the third equality in Equation (35) implies that $R^{AC}(M_1) \cap R^{BC}(M_2) =_\omega \varnothing$. □

Proof of Theorem 1. First, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the sets of perfect matchings on the parts auxiliary graphs $H_1 = H_\omega(G,A,C)$ and $H_2 = H_\omega(G,B,C)$, respectively.

To prove the statement 1, represent C with decision variables $\{x_{ki}\}$, i.e., for all $i \in V$ and $k \in [K]$, set $x_{ki} = 1$ if and only if $i \in C_k$. By assumption, these $\{x_{ki}\}$ variables satisfy the constraints in (21) and (22). Consider $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ such that

$$M_1 = \{\{A_k, C_k\} : k \in [K]\}$$

and

$$M_2 = \{\{B_k, C_k\} : k \in [K]\}.$$

By the constraints in (21), we have $S_{kk} = A_k \cap B_k \subseteq_\omega C_k$ for all $k \in [K]$. More intuitively, each fixed core of M^* is contained within a different part of C. It follows that $A_k \cap B_k =_\omega A_k \cap B_k \cap C_k$ for all $k \in [K]$. Therefore, we have

$$\begin{aligned}
F^{AC}(M_1) \cap F^{BC}(M_2) &= \left(\bigcup_{\{A_k, C_j\} \in M_1} (A_k \cap C_j) \right) \cap \left(\bigcup_{\{B_k, C_j\} \in M_2} (B_k \cap C_j) \right) \\
&= \left(\bigcup_{k \in [K]} (A_k \cap C_k) \right) \cap \left(\bigcup_{k \in [K]} (B_k \cap C_k) \right) \\
&= \bigcup_{j \in [K]} \left[\left(\bigcup_{k \in [K]} (A_k \cap C_k) \right) \cap (B_j \cap C_j) \right] \\
&= \bigcup_{j \in [K]} \left(\bigcup_{k \in [K]} (A_k \cap C_k \cap B_j \cap C_j) \right) \\
&= \bigcup_{k \in [K]} (A_k \cap B_k \cap C_k) \\
&=_\omega \bigcup_{k \in [K]} (A_k \cap B_k) \\
&= \bigcup_{\{A_k, B_j\} \in M^*} (A_k \cap B_j) \\
&= F^{AB}(M^*).
\end{aligned}$$

The first equality follows from Definition 6. The second equality follows from the construction of M_1 and M_2. The third and fourth equalities follow from the distributive property of set intersection/union. The fifth equality follows from the fact that partitions have disjoint parts; more specifically, $A_k \cap C_k \cap B_j \cap C_j = \varnothing$ for all $j \neq k$. The sixth equality follows from the fact that $A_k \cap B_k =_\omega A_k \cap B_k \cap C_k$ for all $k \in [K]$. Then the seventh equality follows from the definition of M^* and the eighth equality follows from Definition 6. Hence, we have

$$F^{AC}(M_1) \cap F^{BC}(M_2) =_\omega F^{AB}(M^*). \tag{36}$$

By Remark 2, we can rewrite Equation (36) as

$$R^{AC}(M_1) \cup R^{BC}(M_2) =_\omega R^{AB}(M^*). \tag{37}$$

Next, for the sake of developing a contradiction, suppose there exists some $i \in R^{AC}(M_1) \cap R^{BC}(M_2)$ with $\omega(i) > 0$. We know that

$$R^{AC}(M_1) = \left(\bigcup_{\{A_h, C_k\} \notin M_1} (A_h \cap C_k) \right)$$

and

$$R^{BC}(M_2) = \left(\bigcup_{\{B_m, C_k\} \notin M_2} (B_m \cap C_k) \right).$$

Because i must belong to some part in each partition, we know that we must have $i \in A_h \cap B_m \cap C_k$ for some $h, m, k \in [K]$. Then because $i \in C_k$, we must have $x_{ki} = 1$. Moreover, because $i \in R^{AC}(M_1)$, it follows that $\{A_h, C_k\} \notin M_1$. Similarly, because $i \in R^{BC}(M_2)$, we also have $\{B_m, C_k\} \notin M_2$. Hence, $h \neq k$ and $m \neq k$ by the construction of M_1 and M_2, respectively. We must also have $h \neq m$; otherwise $i \in S_{hh} = A_h \cap B_h$ with $\omega(i) > 0$, but $x_{hi} = 0$ (by the constraints in (10) and the fact that $x_{ki} = 1$ with $k \neq h$), which is not permitted by the constraints in (21). However, then we have $x_{ki} = 1$ for $i \in S_{hm} = A_h \cap B_m$ with $\omega(i) > 0$ for $h, m, k \in [K]$ with $h \neq m$, $h \neq k$, and $m \neq k$, which contradicts the constraints in (22). Therefore, no such $i \in R^{AC}(M_1) \cap R^{BC}(M_2)$ exists, meaning we have

$$R^{AC}(M_1) \cap R^{BC}(M_2) =_\omega \varnothing. \tag{38}$$

Using Equations (37) and (38), we have

$$\begin{aligned} t_\omega(A, B) &= \omega\left(R^{AB}(M^*)\right) \\ &= \omega\left(R^{AC}(M_1) \cup R^{BC}(M_2)\right) \\ &= \omega\left(R^{AC}(M_1)\right) + \omega\left(R^{BC}(M_2)\right) \\ &\geq t_\omega(A, C) + t_\omega(B, C). \end{aligned} \tag{39}$$

The first equality follows from the optimality of M^*. The second equality follows from Equation (37). The third equality follows from Equation (38). Then the final inequality follows from the definition of transfer distance and Remark 1. Because the triangle inequality holds for the transfer distance (i.e., $t_\omega(A, B) \leq t_\omega(A, C) + t_\omega(B, C)$), Equation (39) must have equalities throughout. Hence, M_1 and M_2 are optimal and C is transfer-tight with respect to A, B, and M^*, as desired. This completes the proof of statement 1.

To prove statement 2, suppose that M^* yields at most one empty fixed core and C is transfer-tight with respect to A, B, and M^*. Then by Proposition 3, there exist optimal $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ such that

$$R^{AC}(M_1) \cup R^{BC}(M_2) =_\omega R^{AB}(M^*) \tag{40}$$

and

$$R^{AC}(M_1) \cap R^{BC}(M_2) =_\omega \varnothing. \tag{41}$$

First, relabel the parts in C such that $\{A_k, C_k\} \in M_1$ for all $k \in [K]$. Then represent C with decision variables $\{x_{ki}\}$, i.e., for all $i \in V$ and $k \in [K]$, set $x_{ki} = 1$ if and only if $i \in C_k$.

Next, Remark 2 allows us to rewrite Equation (40) as

$$F^{AC}(M_1) \cap F^{BC}(M_2) =_\omega F^{AB}(M^*). \tag{42}$$

Then by Definition 6, M^*, and M_1, Equation (42) can be written as

$$\left(\bigcup_{k \in [K]} (A_k \cap C_k) \right) \cap \left(\bigcup_{\{B_k, C_j\} \in M_2} (B_k \cap C_j) \right) =_\omega \left(\bigcup_{k \in [K]} (A_k \cap B_k) \right). \tag{43}$$

Similarly, by Definition 7, Equation (41) can be written as

$$\left(\bigcup_{\{A_h,C_k\}\notin M_1}(A_h\cap C_k)\right)\cap\left(\bigcup_{\{B_m,C_k\}\notin M_2}(B_m\cap C_k)\right)=_\omega\varnothing. \tag{44}$$

We prove statement 2 in two stages:

1. Showing that the constraints in (21) hold
2. Showing that the constraints in (22) hold

For stage 1, consider $i\in V$ with $\omega(i)>0$ such that $i\in S_{kk}=A_k\cap B_k$ for some $k\in[K]$. Then i is in the right-hand side of Equation (43). Because $\omega(i)>0$, it follows that i is also in the left-hand side of Equation (43). Because partitions have disjoint parts and the unions in Equation (43) are unions of disjoint sets, we must have

$$i\in A_k\cap C_k\cap B_k\cap C_j \tag{45}$$

for $j\in[K]$ such that $\{B_k,C_j\}\in M_2$. Suppose $j\neq k$ in Equation (45). Then because C is a partition, we have $C_k\cap C_j=\varnothing$. Hence,

$$i\in A_k\cap C_k\cap B_k\cap C_j=\varnothing,$$

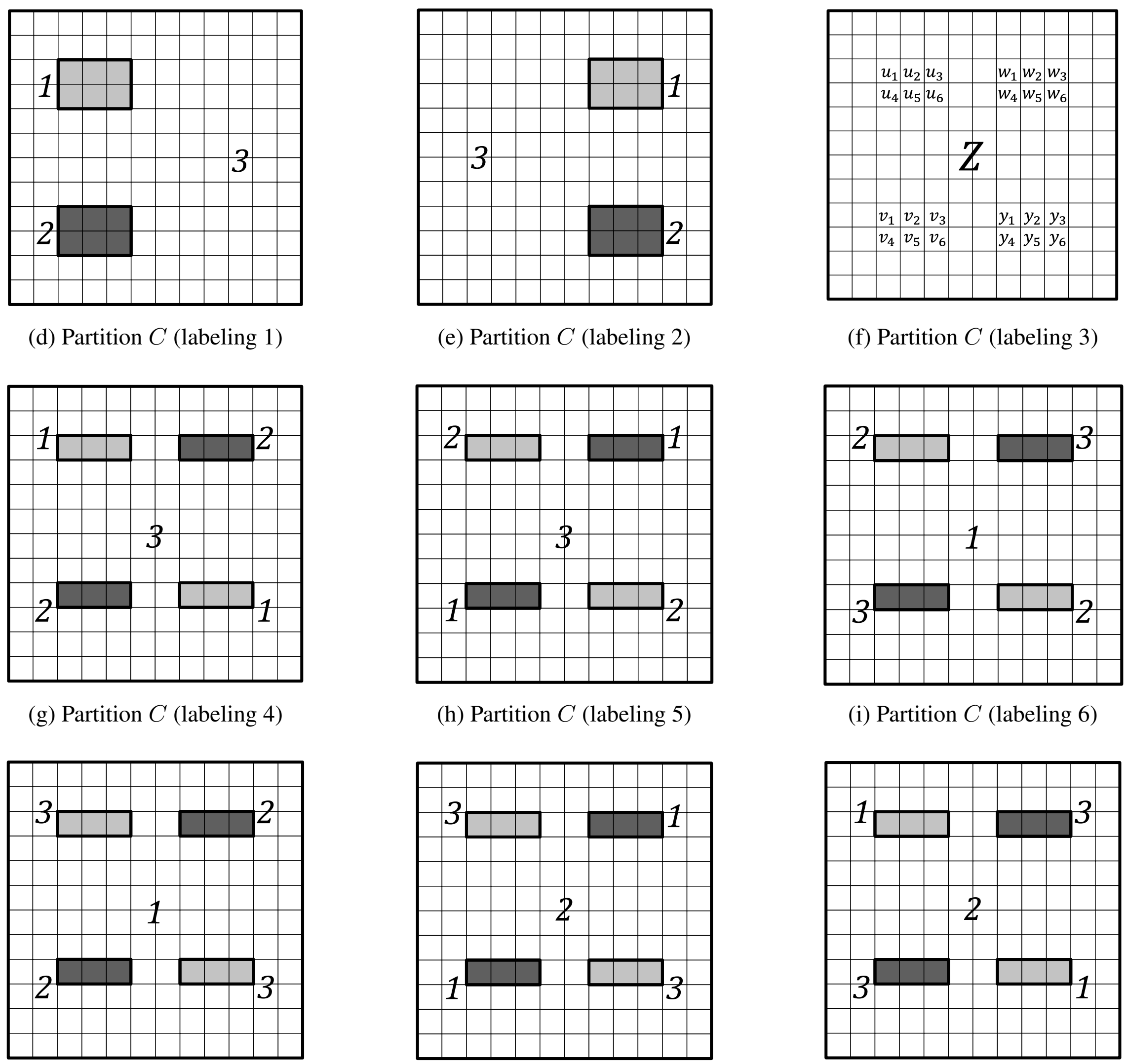
which is a contradiction. So we must have $j=k$ in Equation (45). Therefore, it follows that $\{B_k,C_k\}\in M_2$. Then because $i\in C_k$, we must have $x_{ki}=1$. Because this is the case for all $k\in[K]$ and all $i\in S_{kk}$ with $\omega(i)>0$, the constraints in (21) hold. This proves stage 1 of statement 2.

Observe that the proof of stage 1 implies that we must have $\{B_k,C_k\}\in M_2$ for all $k\in[K]$ such that $\omega(A_k\cap B_k)>0$ (i.e., for all $k\in[K]$ such that the fixed core S_{kk} has non-zero weight) because we have considered all positive-weight vertices that belong to fixed cores of M^*. By assumption, M^* yields either no zero-weight fixed cores or one zero-weight fixed core. If M^* yields no zero-weight fixed cores, then we know $\{B_k,C_k\}\in M_2$ for all $k\in[K]$. If there exists exactly one $j\in[K]$ such that the fixed core S_{jj} has zero weight, then we know $\{B_k,C_k\}\in M_2$ for all $k\in[K]\backslash\{j\}$. However, then we must have $\{B_j,C_j\}\in M_2$ as well, because M_2 is a perfect matching. Therefore, in both cases, $\{B_k,C_k\}\in M_2$ for all $k\in[K]$.

To prove stage 2, consider $i\in V$ with $\omega(i)>0$ such that $i\in S_{hm}=A_h\cap B_m$ for some $h,m\in[K]$ with $h\neq m$. For the sake of developing a contradiction, suppose that $x_{ki}=1$ for some $k\in[K]$ with $k\neq h$ and $k\neq m$. Then we have $i\in C_k$, meaning $i\in A_h\cap C_k$ and $i\in B_m\cap C_k$. Because $\{A_h,C_k\}\notin M_1$ and $\{B_m,C_k\}\notin M_2$ for $h\neq k$ and $m\neq k$, it follows that i is in the left-hand side of Equation (44). This is a contradiction, because $\omega(i)>0$. Therefore, we must have $x_{ki}=0$ for all $k\in[K]$ with $k\neq h$ and $k\neq m$. Because this is the case for all $i\in V$ with $\omega(i)>0$ such that $i\in S_{hm}=A_h\cap B_m$ for some $h,m\in[K]$ with $h\neq m$, the constraints in (22) hold. This completes the proof of statement 2. □

EXAMPLE 1. Consider an unweighted 12×12 grid graph with partitions A and B as shown with part labels in Figures 13a and 13b. Figures 13d-13i depict the six possible part label permutations of a third partition C. Consider an optimal perfect matching $M^*=\{\{A_1,B_1\},\{A_2,B_2\},\{A_3,B_3\}\}$ on $H_1(G,A,B)$. Note that we already have $k=h$ for all $\{A_k,B_h\}\in M^*$ and that M^* yields two zero-weight fixed cores. We will show that C is transfer-tight with respect to A, B, and M^*, but cannot be represented by decision variables $\{x_{ki}\}$ that satisfy Constraints (21) and (22), regardless of the permutation of its part labels.

Figure 13 Partitions A, B, and C on a 12×12 grid graph G; Figure 13c labels some individual vertices and labels the set of remaining vertices Z. Although C is transfer-tight with respect to a given optimal perfect matching $M^* = \{\{A_1, B_1\}, \{A_2, B_2\}, \{A_3, A_3\}\}$ on $H_1(G, A, B)$, C cannot be represented by decision variables $\{x_{ki}\}$ that satisfy Constraints (21) and (22).



(a) Partition A (b) Partition B (c) Vertex labels
(d) Partition C (labeling 1) (e) Partition C (labeling 2) (f) Partition C (labeling 3)
(g) Partition C (labeling 4) (h) Partition C (labeling 5) (i) Partition C (labeling 6)

Regardless of the labeling for parts in C (Figures 13d-13i), the unique optimal perfect matching M_1 on $H_1(G, A, C)$ pairs white with white, gray with gray, and black with black. Similarly, the unique optimal perfect matching M_2 on $H_1(G, B, C)$ pairs white with white, gray with black, and black with gray. Utilizing the vertex labels in Figure 13c, we can write the fixed sets associated with M^*, M_1, and M_2:

$$F^{AB}(M^*) = Z$$

$$F^{AC}(M_1) = Z \cup \{u_1, u_2, u_3, v_1, v_2, v_3, w_4, w_5, w_6, y_4, y_5, y_6\}$$

$$F^{BC}(M_2) = Z \cup \{u_4, u_5, u_6, v_4, v_5, v_6, w_1, w_2, w_3, y_1, y_2, y_3\}.$$

It follows that

$$F^{AB}(M^*) = Z = F^{AC}(M_1) \cap F^{BC}(M_2).$$

Therefore, by Remark 2, we have

$$R^{AB}(M^*) = R^{AC}(M_1) \cup R^{BC}(M_2).$$

We can also write the reassignment sets associated with M_1 and M_2:

$$R^{AC}(M_1) = \{u_4, u_5, u_6, v_4, v_5, v_6, w_1, w_2, w_3, y_1, y_2, y_3\}$$
$$R^{BC}(M_2) = \{u_1, u_2, u_3, v_1, v_2, v_3, w_4, w_5, w_6, y_4, y_5, y_6\}.$$

It follows that

$$R^{AC}(M_1) \cap R^{BC}(M_2) = \varnothing.$$

Therefore, by Proposition 3, C is transfer-tight with respect to A, B, and M^*.

For each labeling of parts in C (Figures 13d-13i), represent C with decision variables $\{x_{ki}\}$, i.e., for $i \in V$ and $k \in [K]$, set $x_{ki} = 1$ if and only if $i \in C_k$. First, consider the $\{x_{ki}\}$ variables for a labeling that labels the white part in C with 1 or 2 (i.e., Figures 13f-13i). Let $z \in Z$ (for the set of vertices Z shown in Figure 13c). Then $z \in S_{33} = A_3 \cap B_3$, but $x_{3z} = 0 \neq 1$; this violates the constraints in (21).

Second, consider the $\{x_{ki}\}$ variables for the labeling in Figure 13d. Consider the vertex w_1 (shown in Figure 13c). We have $w_1 \in S_{31} = A_3 \cap B_1$. However, because $w_1 \in C_2$, we have $x_{2w_1} = 1 \neq 0$; this violates the constraints in (22).

Third, consider the $\{x_{ki}\}$ variables for the labeling in Figure 13e. Consider the vertex v_1 (shown in Figure 13c). We have $v_1 \in S_{23} = A_2 \cap B_3$. However, because $v_1 \in C_1$, we have $x_{1v_1} = 1 \neq 0$; this violates the constraints in (22).

Therefore, C cannot be represented by decision variables $\{x_{ki}\}$ that satisfy Constraints (21) and (22) (even after allowing for permutations of the part labels for C).

Proof of Corollary 1. First, we claim that M^* yields at most one zero-weight fixed core. For the sake of developing a contradiction, suppose there exist $j, m \in [K]$ such that $\omega(S_{jj}) = \omega(A_j \cap B_j) = 0$ and $\omega(S_{mm}) = \omega(A_m \cap B_m) = 0$. It must be the case that $\omega(A_j \cap B_m) = 0$ and $\omega(A_m \cap B_j) = 0$ as well; otherwise M^* would not be optimal. Let $\hat{M}$ be a perfect matching on $H_\omega(G, A, B)$ such that $\{A_j, B_m\} \in \hat{M}$, $\{A_m, B_j\} \in \hat{M}$, and $\{A_k, B_k\} \in \hat{M}$ for all $k \in [K] \backslash \{j, m\}$. Then the weight of $\hat{M}$ equals the weight of M^*, so $\hat{M}$ is also optimal. This contradicts the uniqueness of M^*. Therefore, it must be the case that M^* yields at most one zero-weight fixed core.

To prove the forward direction, suppose C is transfer-tight with respect to A and B. Because M^* is the unique optimal perfect matching on $H_\omega(G, A, B)$, it follows that C is transfer-tight with respect to A, B, and M^*. Because M^* has at most one zero-weight fixed core, Theorem 1 implies that C can be represented by decision variables $\{x_{ki}\}$ that satisfy Constraints (21) and (22) (after a possible permutation of the part labels for C).

To prove the backward direction, represent C with decision variables $\{x_{ki}\}$, i.e., for all $i \in V$ and $k \in [K]$, set $x_{ki} = 1$ if and only if $i \in C_k$. By assumption, these $\{x_{ki}\}$ variables satisfy the constraints in (21) and (22). Then by Theorem 1, C is transfer-tight with respect to A, B, and M^*. Therefore, C is transfer-tight with respect to A and B, as desired. □

Proof of Proposition 4. First, we claim that

$$t_\omega(A,B) = \sum_{x=0}^{n} t_\omega(P_x, P_{x+1}). \tag{46}$$

If this claim holds, then we have

$$\begin{aligned} t_\omega(A,B) &\le \sum_{x=0}^{h-1} t_\omega(P_x,P_{x+1}) + t_\omega(P_h,P_j) + \sum_{x=j}^{n} t_\omega(P_x,P_{x+1}) \\ &\le \sum_{x=0}^{h-1} t_\omega(P_x,P_{x+1}) + t_\omega(P_h,P_i) + t_\omega(P_i,P_j) + \sum_{x=j}^{n} t_\omega(P_x,P_{x+1}) \\ &\le \sum_{x=0}^{n} t_\omega(P_x,P_{x+1}) \\ &= t_\omega(A,B), \end{aligned}$$

which implies equalities throughout. Note that the first three lines follow from the triangle inequality. Therefore, it must be the case that

$$t_\omega(P_h,P_j) = t_\omega(P_h,P_i) + t_\omega(P_i,P_j).$$

We now prove the claim in (46) by strong induction on n. First, consider the base case of $n = 0$. Then we have

$$\sum_{x=0}^{n} t_\omega(P_x,P_{x+1}) = t_\omega(P_0,P_1) = t_\omega(A,B),$$

so the claim holds.

Now consider $n > 0$. We know Algorithm 1 generates $P_{\lceil \frac{n}{2} \rceil}$ first. The TDβP's transfer-tight constraints imply

$$t_\omega(A,B) = t_\omega(P_0, P_{\lceil \frac{n}{2} \rceil}) + t_\omega(P_{\lceil \frac{n}{2} \rceil}, P_{n+1}).$$

Note that for $n > 0$, we have $0 < \lceil \frac{n}{2} \rceil < n+1$. Hence, it follows by the inductive hypothesis that

$$t_\omega(P_0, P_{\lceil \frac{n}{2} \rceil}) = \sum_{x=0}^{\lceil \frac{n}{2} \rceil - 1} t_\omega(P_x,P_{x+1})$$

and

$$t_\omega(P_{\lceil \frac{n}{2} \rceil}, P_{n+1}) = \sum_{x=\lceil \frac{n}{2} \rceil}^{n} t_\omega(P_x,P_{x+1}).$$

Therefore, we have

$$\begin{aligned} t_\omega(A,B) &= t_\omega(P_0, P_{\lceil \frac{n}{2} \rceil}) + t_\omega(P_{\lceil \frac{n}{2} \rceil}, P_{n+1}) \\ &= \sum_{x=0}^{\lceil \frac{n}{2} \rceil - 1} t_\omega(P_x,P_{x+1}) + \sum_{x=\lceil \frac{n}{2} \rceil}^{n} t_\omega(P_x,P_{x+1}) \\ &= \sum_{x=0}^{n} t_\omega(P_x,P_{x+1}), \end{aligned}$$

as desired. $\square$