E-companion: “Algorithmic Approaches for Identifying the Trade-off between Pessimism and Optimism in a Stochastic Fixed Charge Facility Location Problem”

EC.1. Mathematical Proofs

EC.1.1. Proof of Theorem 1

Proof. For part (a), note that $v^\star(\theta) = \min_{\boldsymbol{o}\in\mathcal{O}}\{f(\boldsymbol{o};\theta)\}$, where $f(\boldsymbol{o};\theta) = \sum_{i\in I} c^{\mathrm{f}} o_i + \mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\boldsymbol{o},\boldsymbol{d})] + \theta\big\{\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})] - \mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\boldsymbol{o},\boldsymbol{d})]\big\}$. It is easy to see that for any $\boldsymbol{o}\in\mathcal{O}$, the function $f(\boldsymbol{o};\theta)$ is linear in θ. Since the feasible set $\mathcal{O}\subseteq\{0,1\}^{|I|}$ is finite, $v^\star(\theta)$ is the pointwise minimum of a finite number of linear functions. It follows that $v^\star$ is piecewise linear and concave. Next, for part (b). Since $\widehat{\mathbb{P}}_N\in\mathcal{F}$, we have $\mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\boldsymbol{o},\boldsymbol{d})] \le \sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})]$. Thus, $f(\boldsymbol{o};\theta)$ is non-decreasing in θ for any $\boldsymbol{o}\in\mathcal{O}$. This shows that $v^\star$ is non-decreasing. $\square$

EC.1.2. Proof of Proposition 1

Proof. Note that $\sup_{\mathbb{P}\in\mathcal{F}'(\theta)}\mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d}) = (1-\theta)\mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\boldsymbol{o},\boldsymbol{d})] + \theta\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})]$. The equivalent reformulation (4) follows from dualizing $\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})]$ using standard duality results of moment problems; see, e.g., Shapiro (2001). $\square$

EC.1.3. Proof of Proposition 2

Proof. We first prove part (a). We start by proving that $\widehat{v}$ is the function linearly interpolating the extreme points of $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$. Since $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$ is closed, it follows from Theorem 5.36 of Güler (2010) that $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$ is the (Minkowski) sum of the convex hull of its extreme points and extreme directions, i.e., $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big) = \operatorname{conv}\Big(\mathcal{E}\big(\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)\big)\Big) + \{t\cdot(0,-1)\mid t\ge 0\}$, where $\mathcal{E}\big(\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)\big)$ is the set of extreme points of $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$. Therefore, we can rewrite $\widehat{v}$ as $\widehat{v}(\theta) = \sup\Big\{z \;\Big|\; (\theta,z)\in\operatorname{conv}\Big(\mathcal{E}\big(\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)\big)\Big)\Big\}$. Since the number of extreme points of $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$ is finite, $\widehat{v}$ is the function linearly interpolating the extreme points of $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$. In particular, $\widehat{v}$ is piecewise linear and concave. Next, we show $\widehat{v}\le v^\star$. Suppose, on the contrary, that there exists $\theta'\in(0,1)$ such that $v^\star(\theta') < \widehat{v}(\theta')$. Let $\{(\theta_{m_r},\underline{v}_{m_r})\}_{r=1}^R$ be the set of extreme points of $\operatorname{conv}\big(\operatorname{hyp}_{[0,1]}(\breve{v})\big)$ with $0=\theta_{m_1}<\theta_{m_2}<\cdots<\theta_{m_R}=1$ and $r'\in[R]$ be such that $\theta'\in(\theta_{m_{r'}},\theta_{m_{r'+1}})$. Writing $\theta' = (1-\alpha)\theta_{m_{r'}} + \alpha\theta_{m_{r'+1}}$ with $\alpha = (\theta'-\theta_{m_{r'}})/(\theta_{m_{r'+1}}-\theta_{m_{r'}})\in(0,1)$, we have $v^\star(\theta') < \widehat{v}(\theta') = (1-\alpha)\widehat{v}(\theta_{m_{r'}}) + \alpha\widehat{v}(\theta_{m_{r'+1}}) \le (1-\alpha)v^\star(\theta_{m_{r'}}) + \alpha v^\star(\theta_{m_{r'+1}})$. This contradicts the concavity of $v^\star$ in Theorem 1. This completes the proof of part (a).

Now, we prove part (b). Since $\widehat{\mathbb{P}}_N \in \mathcal{F}$, if follows from Theorem 1 that $\upsilon^\star$ is non-decreasing. Therefore, for any fixed $m \in [M]$, we have $\underline{\upsilon}_{m'} \le \upsilon^\star(\theta_{m'}) \le \upsilon^\star(\theta_m)$ for all $m' \le m$. This shows that $\underline{\upsilon}'_m = \max_{m' \le m} \underline{\upsilon}_{m'}$ is a valid lower bound on $\upsilon^\star(\theta_m)$. It follows from part (a) that the function $\widehat{\upsilon}(\theta)$ defined in (6) constructed based on $\{(\theta_m, \underline{\upsilon}'_m)\}_{m=1}^M$ is piecewise linear and concave with $\widehat{\upsilon}^\varepsilon \le \upsilon^\star$. Moreover, since $\{\underline{\upsilon}'_m\}_{m=1}^M$ is non-decreasing by construction, $\widehat{\upsilon}(\theta)$ is also non-decreasing. □

EC.1.4. Proof of Proposition 3

Proof. For part (a), by definition, $\widetilde{\upsilon}(\theta)$ is the pointwise minimum of the linear functions $\widehat{f}(\widehat{\boldsymbol{o}}_m;\theta) = \sum_{i\in I} c^{\mathrm{f}}(\widehat{\boldsymbol{o}}_m)_i + (1-\theta)\widehat{\varphi}_m^{\mathrm{SP}} + \theta\acute{\varphi}_m^{\mathrm{DRO}}$ for $m \in [M]$. This shows that $\widetilde{\upsilon}$ is piecewise linear and concave. Moreover, since $\acute{\varphi}_m^{\mathrm{DRO}} \ge \widehat{\varphi}_m^{\mathrm{DRO}}$ for all $m \in [M]$, we have $\widetilde{\upsilon} \ge \widetilde{\upsilon}^{\mathrm{exact}} \ge \upsilon^\star$. For part (b), since $\widehat{\mathbb{P}}_N \in \mathcal{F}$, we have $\acute{\varphi}_m^{\mathrm{DRO}} - \widehat{\varphi}_m^{\mathrm{SP}} \ge \widehat{\varphi}_m^{\mathrm{DRO}} - \widehat{\varphi}_m^{\mathrm{SP}} = \sup_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[Q(\widehat{\boldsymbol{o}}_m, \boldsymbol{d})] - \mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\widehat{\boldsymbol{o}}_m, \boldsymbol{d})] \ge 0$ for all $m \in [M]$. Thus, $\widehat{f}(\widehat{\boldsymbol{o}}_m;\theta)$ is non-decreasing in θ. This shows that $\widetilde{\upsilon}$ is non-decreasing. □

EC.1.5. Proof of Proposition 5

Proof. The reformulation of the primal optimality cut (9c) has been well established in the literature; see, e.g., Zeng and Zhao (2013). Thus, we focus on reformulating the new dual optimality cut (9d). Note that

$$\sup_{\boldsymbol{d}\in\mathcal{S}} \left\{ \sum_{j\in J} w_j^e d_j - \sum_{k\in[K]} \rho_k f_k(\boldsymbol{d}) - \sum_{k'\in[K']} \gamma_{k'} g_{k'}(\boldsymbol{d}) \right\}$$

$$= \sup_{\boldsymbol{d}\in\mathcal{S}} \inf_{\boldsymbol{\kappa},\boldsymbol{\tau}} \left\{ \sum_{j\in J} w_j^e d_j - \sum_{k\in[K]} \rho_k \Big[\boldsymbol{d}^\top \boldsymbol{\kappa}_k - f_k^*(\boldsymbol{\kappa}_k)\Big] - \sum_{k'\in[K']} \gamma_{k'} \Big[\boldsymbol{d}^\top \boldsymbol{\tau}_{k'} - g_{k'}^*(\boldsymbol{\tau}_{k'})\Big] \right\} \tag{EC.1a}$$

$$= \inf_{\boldsymbol{\kappa},\boldsymbol{\tau}} \left\{ \sum_{k\in[K]} \rho_k f_k^*(\boldsymbol{\kappa}_k) + \sum_{k'\in[K']} \gamma_{k'} g_{k'}^*(\boldsymbol{\tau}_{k'}) + \sup_{\boldsymbol{d}\in\mathcal{S}} \left\{ \sum_{j\in J} w_j^e d_j - \sum_{k\in[K]} \rho_k (\boldsymbol{\kappa}_k^\top \boldsymbol{d}) - \sum_{k'\in[K']} \gamma_{k'} (\boldsymbol{\tau}_{k'}^\top \boldsymbol{d}) \right\} \right\} \tag{EC.1b}$$

$$= \inf_{\boldsymbol{\kappa},\boldsymbol{\tau}} \left\{ \sum_{k\in[K]} \rho_k f_k^*(\boldsymbol{\kappa}_k) + \sum_{k'\in[K']} \gamma_{k'} g_{k'}^*(\boldsymbol{\tau}_{k'}) + \min_{\boldsymbol{\pi}\in\mathcal{D}(\boldsymbol{\kappa},\boldsymbol{\tau})} \left\{ \sum_{j\in J} (\overline{\pi}_j \overline{d}_j - \underline{\pi}_j \underline{d}_j) \right\} \right\}, \tag{EC.1c}$$

where $\mathcal{D}(\boldsymbol{\kappa},\boldsymbol{\tau}) = \big\{(\overline{\boldsymbol{\pi}}, \underline{\boldsymbol{\pi}}) \in \mathbb{R}^{|J|} \times \mathbb{R}^{|J|} \mid \boldsymbol{w}^e - \sum_{k\in[K]} \rho_k \boldsymbol{\kappa}_k - \sum_{k'\in[K']} \gamma_{k'} \boldsymbol{\tau}_{k'} = \overline{\boldsymbol{\pi}} - \underline{\boldsymbol{\pi}}\big\}$. Here, (EC.1a) follows from the conjugacy theorem for proper, closed, convex functions $\{f_k\}_{k\in[K]}$ and $\{g_{k'}\}_{k'\in[K']}$ (see Proposition 1.6.1 of Bertsekas 2009), (EC.1b) follows from the Sion min-max theorem (Sion 1958), and (EC.1c) follows from the LP duality of the inner maximization problem in (EC.1b). Replacing problem $\sup_{\boldsymbol{d}\in\mathcal{S}}\{\cdot\}$ with its equivalent reformulation (EC.1c), we obtain the desired reformulation (12). □

EC.1.6. Proof of Proposition EC.1

PROPOSITION EC.1. *Subproblem (10) is equivalent to the following MINLP:*

$$\underset{\substack{\boldsymbol{x}\geq 0, \boldsymbol{u}\geq 0, \boldsymbol{w},\\ \boldsymbol{v}\leq 0, \boldsymbol{s}, \boldsymbol{d}}}{\text{maximize}} \quad \sum_{i\in I}\sum_{j\in J} t_{i,j}x_{i,j} + \sum_{j\in J} c^{\text{u}}_j u_j - \sum_{k\in K}\rho^h_k f_k(\boldsymbol{d}) - \sum_{k'\in[K']}\gamma^h_{k'} g_{k'}(\boldsymbol{d}) \tag{EC.2a}$$

$$\text{subject to} \quad \sum_{j\in J} x_{i,j} \leq C_i o^h_i, \ \sum_{i\in I} x_{i,j} + u_j = d_j, \quad \forall i\in I,\, j\in J, \tag{EC.2b}$$

$$w_j + v_i \leq t_{i,j}, \ w_j \leq c^{\text{u}}_j, \quad \forall i\in I,\, j\in J, \tag{EC.2c}$$

$$C_i o_i - \sum_{j\in J} x_{i,j} \leq M^{1,1}_i s^1_i, \ -v_i \leq M^{1,2}_i(1-s^1_i), \quad \forall i\in I, \tag{EC.2d}$$

$$t_{i,j} - w_j - v_i \leq M^{2,1}_{i,j} s^2_{i,j}, \ x_{i,j} \leq M^{2,2}_{i,j}(1-s^2_{i,j}), \quad \forall i\in I,\, j\in J, \tag{EC.2e}$$

$$c^{\text{u}}_j - w_j \leq M^{3,1}_j s^3_j, \ u_j \leq M^{3,2}_j(1-s^3_j), \quad \forall j\in J, \tag{EC.2f}$$

$$\underline{d}_j \leq d_j \leq \overline{d}_j, \ s^1_j \in\{0,1\}, \, s^2_{i,j}\in\{0,1\}, \, s^3_j\in\{0,1\}, \quad \forall i\in I,\, j\in J. \tag{EC.2g}$$

Here, the big-M parameters can be chosen as: $M^{1,1}_i = C_i$, $M^{1,2}_i = \max_{j\in J}\{c^{\text{u}}_j\} - \min_{j\in J}\{t_{i,j}\}$, $M^{2,1}_{i,j} = t_{i,j} + M^{1,2}_i$, $M^{2,2}_{i,j} = C_i$, $M^{3,1}_j = c^{\text{u}}_j$, $M^{3,2}_j = \overline{d}_j$ *for all* $i\in I$ *and* $j\in J$.

Proof. We first derive an equivalent reformulation of $Q(\boldsymbol{o},\boldsymbol{d})$. Note that the second set of inequality constraints in (1b) always achieve equality at optimality (since we are minimizing the traveling cost and unmet demand penalty). Thus, we can write $Q(\boldsymbol{o},\boldsymbol{d}) = \min_{(\boldsymbol{x},\boldsymbol{u})\in\mathcal{A}_{\text{P}}}\big\{\sum_{i\in I}\sum_{j\in J} t_{i,j}x_{i,j} + \sum_{j\in J} c^{\text{u}}_j u_j\big\}$ with $\mathcal{A}_{\text{P}} = \big\{(\boldsymbol{x},\boldsymbol{u}) \bigm| \sum_{j\in J} x_{i,j} \leq C_i o_i,\, \sum_{i\in I} x_{i,j} + u_j = d_j,\, x_{i,j}\geq 0,\, u_j \geq 0,\, \forall i\in I,\, j\in J\big\}$, which is equivalent to its dual problem $Q(\boldsymbol{o},\boldsymbol{d}) = \max_{(\boldsymbol{w},\boldsymbol{u})\in\mathcal{A}_{\text{D}}}\big\{\sum_{i\in I} C_i v_i o_i + \sum_{j\in J} w_j d_j\big\}$ with $\mathcal{A}_{\text{D}} = \big\{(\boldsymbol{w},\boldsymbol{v}) \bigm| w_j + v_i \leq t_{i,j},\, w_j \leq c^{\text{u}}_j,\, v_i \leq 0,\, \forall i\in I,\, j\in J\big\}$. Let $(\boldsymbol{x},\boldsymbol{u})\in\mathcal{A}_{\text{P}}$ and $(\boldsymbol{w},\boldsymbol{v})\in\mathcal{A}_{\text{D}}$ be feasible solutions solution to the primal and dual problems respectively. The pair $(\boldsymbol{x},\boldsymbol{u})$ is primal optimal and $(\boldsymbol{w},\boldsymbol{v})$ is dual optimal if and only if they satisfy the following complementary slackness conditions: $\mathcal{A}_{\text{C}} = \big\{(\boldsymbol{x},\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \bigm| \big(C_i o_i - \sum_{j\in J} x_{i,j}\big)(-v_i) = 0,\, (t_{i,j} - w_j - v_i)x_{i,j} = 0,\, (c^{\text{u}}_j - w_j)u_j = 0,\, \forall i\in I,\, j\in J\big\}$. Then, the set $\{(\boldsymbol{u},\boldsymbol{x},\boldsymbol{w},\boldsymbol{v}) \mid (\boldsymbol{x},\boldsymbol{u})\in\mathcal{A}_{\text{P}},\, (\boldsymbol{w},\boldsymbol{v})\in\mathcal{A}_{\text{D}},\, (\boldsymbol{u},\boldsymbol{x},\boldsymbol{w},\boldsymbol{v})\in\mathcal{A}_{\text{C}}\}$ characterizes the set of primal and dual optimal solutions. Finally, one can introduce binary variables $\boldsymbol{s}^1$, $\boldsymbol{s}^2$, and $\boldsymbol{s}^3$ with constraints (EC.2d)–(EC.2f) to linearize the non-linear constraints in $\mathcal{A}_{\text{C}}$. Therefore, we can reformulate $Q(\boldsymbol{o},\boldsymbol{d})$ as

$$\underset{\boldsymbol{u},\boldsymbol{x},\boldsymbol{w},\boldsymbol{v},\boldsymbol{s}}{\text{maximize}} \quad \sum_{i\in I}\sum_{j\in J} t_{i,j}x_{i,j} + \sum_{j\in J} c^{\text{u}}_j u_j \tag{EC.3a}$$

$$\text{subject to} \quad (\boldsymbol{x},\boldsymbol{u})\in\mathcal{A}_{\text{P}},\, (\boldsymbol{w},\boldsymbol{v})\in\mathcal{A}_{\text{D}},\, \{(\boldsymbol{u},\boldsymbol{x},\boldsymbol{w},\boldsymbol{v},\boldsymbol{s}) \mid \text{(EC.2d)–(EC.2f)}\}. \tag{EC.3b}$$

Using the maximization reformulation of $Q(\boldsymbol{o},\boldsymbol{d})$ in (EC.3), we can reformulate the subproblem (10), which is equivalent to $\sup_{\boldsymbol{d}\in\mathcal{S}}\big\{Q(\boldsymbol{o},\boldsymbol{d})-\sum_{k\in[K]}\rho^h_k f_k(\boldsymbol{d})-\sum_{k'\in[K']}\gamma^h_{k'}g_{k'}(\boldsymbol{d})\big\}$, as the desired MINLP (EC.2).

To complete the proof, we next derive tight values for the big-M parameters. (i) For $M^{1,1}_i$, note that $C_io_i\le C_i$ and $\sum_{j\in J}x_{i,j}\ge 0$. Thus, we can set $M^{1,1}_i=C_i$. (ii) For $M^{1,2}_i$, it follows from Lemma EC.1 (presented below) that $-v_i\le\max_{j\in J}\{c^{\text{u}}_j\}-\min_{j\in J}\{t_{i,j}\}$. Thus, we can set $M^{1,2}_i=\max_{j\in J}\{c^{\text{u}}_j\}-\min_{j\in J}\{t_{i,j}\}$. (iii) For $M^{2,1}_{i,j}$, note that $w_j\ge 0$ and $v_i\ge-\max_{j\in J}\{c^{\text{u}}_j\}+\min_{j\in J}\{t_{i,j}\}$. Thus, we can set $M^{2,1}_{i,j}=t_{i,j}+M^{1,2}_i$. (iv) For $M^{2,2}_{i,j}$, since $x_{i,j}\le C_i$, we can set $M^{2,2}_{i,j}=C_i$. (v) For $M^{3,1}_j$, since $w_j\ge 0$, we can set $M^{3,1}_j=c^{\text{u}}_j$. (vi) For $M^{3,2}_j$, note that $u_j\le d_j\le\overline{d}_j$. Thus, we can set $M^{3,2}_j=\overline{d}_j$. This completes the proof. $\square$

LEMMA EC.1. *Consider the dual problem* $\max_{(\boldsymbol{w},\boldsymbol{u})\in\mathcal{A}_D}\big\{\sum_{i\in I}C_iv_io_i+\sum_{j\in J}w_jd_j\big\}$. *The bound* $v_i\ge-\max_{j\in J}\{c^{\text{u}}_j\}+\min_{j\in J}\{t_{i,j}\}$ *is a valid lower bound on* v_i *for all* $i\in I$.

Proof of Lemma EC.1. For notational simplicity, we use $a\wedge b$ to denote $\min\{a,b\}$ for any $\{a,b\}\subset\mathbb{R}$. We first claim that for any extreme point $(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})$ of the dual feasible set $\Pi=\{(\boldsymbol{w},\boldsymbol{v})\mid w_j+v_i\le t_{i,j},\ 0\le w_j\le c^{\text{u}}_j,\ v_i\le 0,\ \forall i\in I,\ j\in J\}$, either $\overline{v}_i=0$ or $\overline{v}_i=t_{i,j}-\overline{w}_j$ for some $j\in J$. Suppose, on the contrary, that there exists $i\in I$ such that $\overline{v}_i<\min_{j\in J}\big\{t_{i,j}-\overline{w}_j\big\}\wedge 0$. Define $(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^1)$ and $(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^2)$ as follows: $\tilde{v}^1_{i'}=\overline{v}_{i'}$ if $i'\ne i$ and $\tilde{v}^1_{i'}=v'$ if $i'=i$ for some $v'<\overline{v}_i$; $\tilde{v}^2_{i'}=\overline{v}_{i'}$ if $i'\ne i$ and $\tilde{v}^2_{i'}=\min_{j\in J}\big\{t_{i,j}-\overline{w}_j\big\}\wedge 0$ if $i'=1$. By construction, both $(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^1)$ and $(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^2)$ are feasible. Indeed, the dual constraints $w_j+v_i\le t_{i,j}$ on $(\boldsymbol{w},\boldsymbol{v})$ trivially holds for all $j\in J$ if $i'\ne i$. Moreover, when $i'=i$, we have $\overline{w}_j+\tilde{v}^1_i=\overline{w}_j+v'<\overline{w}_j+\overline{v}_i\le t_{i,j}$ and $\overline{w}_j+\tilde{v}^2_i=\overline{w}_j+\big(\min_{j'\in J}\big\{t_{i,j'}-\overline{w}_{j'}\big\}\wedge 0\big)\le\overline{w}_j+\big(t_{i,j}-\overline{w}_j\big)=t_{i,j}$ for all $j\in J$. Since $(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})=(1-\lambda)(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^1)+\lambda(\overline{\boldsymbol{w}},\tilde{\boldsymbol{v}}^2)$ with $\lambda=(\overline{v}_i-v')/\big[\big(\min_{j\in J}\big\{t_{i,j}-\overline{w}_j\big\}\wedge 0\big)-v'\big]\in(0,1)$, this contradicts that $(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})$ is an extreme point of Π. Thus, we must have either $\overline{v}_i=0$ or $\overline{v}_i=t_{i,j}-\overline{w}_j$ for some $j\in J$.

Now, we show that the desired lower bound $v_i\ge-\max_{j\in J}\{c^{\text{u}}_j\}+\min_{j\in J}\{t_{i,j}\}$ is not violated by any extreme points of Π. Let us consider the following two cases. First, when $\overline{v}_i=0$, we have $-\max_{j'\in J}\{c^{\text{u}}_{j'}\}+\min_{j'\in J}\{t_{i,j'}\}<0=\overline{v}_i$, where the strict inequality follows from $c^{\text{u}}_j>t_{i,j}$ for all $i\in I$ and $j\in J$. Second, when $\overline{v}_i=t_{i,j}-\overline{w}_j$ for some $j\in J$, we have $-\max_{j'\in J}\{c^{\text{u}}_{j'}\}+\min_{j'\in J}\{t_{i,j'}\}\le-c^{\text{u}}_j+t_{i,j}\le-\overline{w}_j+t_{i,j}=\overline{v}_i$, where the last inequality follows from $\overline{w}_j\le c^{\text{u}}_j$ since $(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})\in\Pi$. Thus, any extreme point $(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})$ of Π satisfies the desired lower bounds. $\square$

Algorithm 4: A gift wrapping algorithm to construct the lower bounding function $\widehat{\upsilon}$

Initialization: $\mathcal{E} \leftarrow \{(\theta_1, \underline{\upsilon}_1)\}$, $\ell \leftarrow 1$

while $\ell < M$ **do**

- Compute $\texttt{slope}_{m'} = (\underline{\upsilon}_{m'} - \underline{\upsilon}_{\ell})/(\theta_{m'} - \theta_{\ell})$ for all $m' > \ell$.
- Identify the index $m_{\max} = \max\{m' > \ell \mid \texttt{slope}_{m'} \geq \texttt{slope}_{m''}, \forall m'' > \ell\}$.
- If $m_{\max} > \ell + 1$, update the lower bounding values $\underline{\upsilon}_m = \underline{\upsilon}_{\ell} + \texttt{slope}_{m_{\max}}(\theta_m - \theta_{\ell})$ for all $m \in \{\ell+1, \ldots, m_{\max} - 1\}$.
- Set $\ell \leftarrow m_{\max}$ and update $\mathcal{E} \leftarrow \mathcal{E} \cup \{(\theta_{\ell}, \underline{\upsilon}_{\ell})\}$.

Construct $\widehat{\upsilon}$ by linearly interpolating the updated points $\{(\theta_m, \underline{\upsilon}_m)\}_{m=1}^{M}$.

EC.1.7. Proof of Lemma 1

Proof. Note that $\sup_{\boldsymbol{o}\in\mathcal{O}} \sup_{\boldsymbol{d}\in\mathcal{S}} |Q(\boldsymbol{o},\boldsymbol{d})| < \infty$ since $\mathcal{O}$ is finite, $\mathcal{S}$ is compact, and $|Q(\boldsymbol{o},\boldsymbol{d})| < \infty$ for all $\boldsymbol{d} \in \mathcal{S}$. Moreover, by Remark 2.1 in Zhang et al. (2016), the Slater-type condition (14) implies that $\left\{\boldsymbol{\rho} \in \mathbb{R}^K, \boldsymbol{\gamma} \in \mathbb{R}^K_+ \;\middle|\; -\sum_{k\in[K]} \rho_k f_k(\boldsymbol{d}) - \sum_{k'\in[K']} \gamma_{k'} g_{k'}(\boldsymbol{d}) \leq 0,\ \forall \boldsymbol{d} \in \mathcal{S}\right\} = \{\boldsymbol{0}\}$. Therefore, it follows from Proposition 2.4 of Xu et al. (2018) that the set of optimal dual solutions $(\boldsymbol{\rho}, \boldsymbol{\gamma})$ is uniformly bounded. $\square$

EC.2. A Gift Wrapping Algorithm for Finding Two-Dimensional Convex Hulls

In Algorithm 4, we present a gift-wrapping algorithm tailored to our context to obtain the extreme points of $\mathrm{conv}\left(\mathrm{hyp}_{[0,1]}(\breve{\upsilon})\right) \subseteq [0,1] \times \mathbb{R}$ (Jarvis 1973), and thus the lower bounding function $\widehat{\upsilon}$.

EC.3. Tailored Reformulations for Some Specific Ambiguity Sets

In this section, we provide tailored reformulations of the TRO-CFLP model and valid inequalities for two popular choices of ambiguity sets: the mean-support (MS) ambiguity set and the mean-absolute-deviation (MAD) ambiguity set. Recall that the MS ambiguity set is defined by letting $K = |J|$ with $f_k(\boldsymbol{d}) = d_k$ for $k \in J$ and $K' = 0$ in (3), whereas the MAD ambiguity set is defined by letting $K = |J|$ with $f_k(\boldsymbol{d}) = d_k$ for $k \in J$ and $K' = |J|$ with $g_{k'}(\boldsymbol{d}) = |d_{k'} - \mu_{k'}|$ for $k' \in J$ in (3). We call the TRO-CFLP model equipped with the MS and MAD ambiguity sets as the TRO-CFLP-MS and TRO-CFLP-MAD models, respectively.

EC.3.1. Reformulations of the TRO-CFLP Model

COROLLARY EC.1. *Consider the TRO-CFLP problem in (4).*

(a) The TRO-CFLP-MS is equivalent to

$$\underset{\boldsymbol{o}\in\mathcal{O},\boldsymbol{x},\boldsymbol{u},\boldsymbol{\rho},\delta}{\text{minimize}} \quad \sum_{i\in I} c^{\mathrm{f}}_i o_i + \frac{1-\theta}{N}\sum_{n=1}^{N}\left(\sum_{i\in I}\sum_{j\in J} t_{i,j}x^n_{i,j} + \sum_{j\in J} c^{\mathrm{u}}_j u^n_j\right) + \theta\big(\boldsymbol{\mu}^\top\boldsymbol{\rho}+\delta\big) \tag{EC.4a}$$

$$\text{subject to} \quad \text{(4b)}, \quad \delta \geq Q(\boldsymbol{o},\boldsymbol{d}) - \sum_{j\in J}\rho_j d_j, \quad \forall \boldsymbol{d}\in\mathcal{S}. \tag{EC.4b}$$

(b) The TRO-CFLP-MAD problem (4) is equivalent to

$$\underset{\substack{\boldsymbol{o}\in\mathcal{O},\boldsymbol{x},\boldsymbol{u},\\ \boldsymbol{\rho},\boldsymbol{\gamma}\geq\boldsymbol{0},\delta}}{\text{minimize}} \quad \sum_{i\in I} c^{\mathrm{f}}_i o_i + \frac{1-\theta}{N}\sum_{n=1}^{N}\left(\sum_{i\in I}\sum_{j\in J} t_{i,j}x^n_{i,j} + \sum_{j\in J} c^{\mathrm{u}}_j u^n_j\right) + \theta\big(\boldsymbol{\mu}^\top\boldsymbol{\rho}+\boldsymbol{\sigma}^\top\boldsymbol{\gamma}+\delta\big) \tag{EC.5a}$$

$$\text{subject to} \quad \text{(4b)}, \quad \delta \geq Q(\boldsymbol{o},\boldsymbol{d}) - \sum_{j\in J}\rho_j d_j - \sum_{j\in J}\gamma_j|d_j-\mu_j|, \quad \forall \boldsymbol{d}\in\mathcal{S}. \tag{EC.5b}$$

EC.3.2. Linear Reformulations of the Optimality Cut (9d)

In Proposition EC.2, we show that if the TRO set is defined using the MS and MAD ambiguity sets, the optimality cut (9d) is equivalent to a set of linear constraints ((EC.6) and (EC.7), respectively). Thus, in such cases, the master problem (9) in Algorithm 3 is linear.

PROPOSITION EC.2. *Consider optimality cut (9d).*

(a) For the TRO-CFLP-MS model, optimality cut (9d) is equivalent to

$$\delta \geq \sum_{i\in I} C_i v^e_i o_i + \sum_{j\in J}\left(\overline{\pi}^e_j\overline{d}_j - \underline{\pi}^e_j\underline{d}_j\right), \tag{EC.6a}$$

$$\underline{\pi}^e_j - \overline{\pi}^e_j = \rho_j - w^e_j, \;\; \underline{\pi}^e_j \geq 0, \;\; \overline{\pi}^e_j \geq 0 \quad \forall j\in J. \tag{EC.6b}$$

(b) For the TRO-CFLP-MAD model, optimality cut (9d) is equivalent to

$$\delta \geq \sum_{i\in I} C_i v^e_i o_i + \sum_{j\in J}\left(\overline{\pi}^e_j\overline{d}_j - \underline{\pi}^e_j\underline{d}_j + \overline{\theta}^e_j\mu_j - \underline{\theta}^e_j\mu_j\right), \tag{EC.7a}$$

$$\underline{\pi}^e_j - \overline{\pi}^e_j + \underline{\theta}^e_j - \overline{\theta}^e_j = \rho_j - w^e_j, \quad \forall j\in J, \tag{EC.7b}$$

$$\overline{\theta}^e_j + \underline{\theta}^e_j = \gamma_j, \;\; \underline{\pi}^e_j \geq 0, \;\; \overline{\pi}^e_j \geq 0, \;\; \underline{\theta}^e_j \geq 0, \;\; \overline{\theta}^e_j \geq 0, \quad \forall j\in J. \tag{EC.7c}$$

Proof. Consider the maximization problem over $\boldsymbol{d}\in\mathcal{S}$ in optimality cut (9d). The objective function in $\boldsymbol{d}$ is linear in the TRO-CFLP-MS model and concave piecewise linear in the TRO-CFLP-MAD model. Thus, we can apply LP duality to reformulate the maximization problem $\boldsymbol{d}\in\mathcal{S}$ to an equivalent minimization problem, and the desired reformulations follow □

EC.3.3. MILP Reformulations of the Subproblem (10)

In Proposition EC.3, we present equivalent MILP reformulations of the subproblem (10) in Algorithm 3 for the TRO-CFLP-MS and TRO-CFLP-MAD models. Since these reformulations are standard, we omit the proof here and refer to Tsang et al. (2023, 2025) for detailed derivations of similar reformulations. Recall that $\Pi = \{(\boldsymbol{w},\boldsymbol{v}) \mid w_j + v_i \le t_{i,j},\ 0 \le w_j \le c^{\mathrm{u}}_j,\ v_i \le 0,\ \forall i \in I,\ j \in J\}$ is the dual feasible set of the second-stage problem.

PROPOSITION EC.3. *Consider subproblem (10).*

(a) The subproblem (10) associated with the TRO-CFLP-MS model is equivalent to

$$\underset{\boldsymbol{w},\boldsymbol{v},\boldsymbol{a},\boldsymbol{\beta}}{\text{maximize}} \quad \sum_{i\in I} C_i o_i^h v_i + \sum_{j\in J}\Big[\underline{d}_j\big(w_j - \rho_j^h\big) + \Delta d_j\big(\beta_j - a_j\rho_j^h\big)\Big] \tag{EC.8a}$$

$$\text{subject to} \quad \beta_j \ge 0,\ \beta_j \ge w_j + c^{\mathrm{u}}_j(a_j - 1),\ \beta_j \le w_j,\ \beta_j \le c^{\mathrm{u}}_j a_j, \quad \forall j \in J, \tag{EC.8b}$$

$$(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\ a_j \in \{0,1\}, \quad \forall j \in J, \tag{EC.8c}$$

where $\Delta d_j = \overline{d}_j - \underline{d}_j$ *for* $j \in J$.

(b) The subproblem (10) associated with the TRO-CFLP-MAD model is equivalent to

$$\underset{\boldsymbol{w},\boldsymbol{v},\boldsymbol{a},\boldsymbol{b},\boldsymbol{\beta},\boldsymbol{\iota}}{\text{maximize}} \quad \sum_{i\in I} C_i o_i^h v_i + \sum_{j\in J}\Big[\mu_j(w_j - \rho_j) - \underline{\Delta}_j(\beta_j - a_j\rho_j^h + a_j\gamma_j^h) + \overline{\Delta}_j(\iota_j - b_j\rho_j^h - b_j\gamma_j^h)\Big] \tag{EC.9a}$$

$$\text{subject to} \quad \beta_j \ge 0,\ \beta_j \ge w_j + c^{\mathrm{u}}_j(a_j - 1),\ \beta_j \le w_j,\ \beta_j \le c^{\mathrm{u}}_j a_j, \quad \forall j \in J, \tag{EC.9b}$$

$$\iota_j \ge 0,\ \iota_j \ge w_j + c^{\mathrm{u}}_j(b_j - 1),\ \iota_j \le w_j,\ \iota_j \le c^{\mathrm{u}}_j b_j, \quad \forall j \in J, \tag{EC.9c}$$

$$(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\ a_j \in \{0,1\},\ b_j \in \{0,1\},\ a_j + b_j \le 1, \quad \forall j \in J, \tag{EC.9d}$$

where $\overline{\Delta}_j = \overline{d}_j - \mu_j$ *and* $\underline{\Delta}_j = \mu_j - \underline{d}_j$ *for* $j \in J$.

EC.3.4. Additional Valid Inequalities

In this section, we derive valid inequalities related to the dual variables $(\boldsymbol{\rho},\boldsymbol{\gamma})$ in the reformulation (4). First, in Proposition EC.4, we derive valid inequalities for the reformulation (EC.4) of the TRO-CFLP-MS model.

PROPOSITION EC.4. *The inequalities* $0 \le \rho_j \le c^{\mathrm{u}}_j$ *for all* $j \in J$ *are valid for* (EC.4).

Proof. Using the strong duality of moment problem and the dual reformulation of $Q(\boldsymbol{o},\boldsymbol{d})$, we can reformulate the DRO problem $\sup_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})]$ as follows:

$$\underset{\boldsymbol{\rho}}{\text{minimize}} \quad h(\boldsymbol{\rho};\boldsymbol{o}) := \boldsymbol{\mu}^\top\boldsymbol{\rho} + \sup_{(\boldsymbol{w},\boldsymbol{v})\in\Pi,\, \boldsymbol{d}\in\mathcal{S}} \Big\{\sum_{i\in I} C_i o_i v_i + \sum_{j\in J}(w_j - \rho_j)d_j\Big\}. \tag{EC.10}$$

We first derive the upper bound on $\rho_{j'}$ for a fixed $j' \in J$. Note that for any $\rho_{j'} \geq c^{\mathrm{u}}_{j'}$, an optimal solution $\boldsymbol{d}^\star_{j'}$ to the inner supremum problem in (EC.10) is $\boldsymbol{d}^\star_{j'} = \underline{d}_{j'}$. Indeed, if $\rho_{j'} \geq c^{\mathrm{u}}_{j'}$, then it follows from the bound on w_j (i.e., $0 \leq w_j \leq c^{\mathrm{u}}_j$) that $w_{j'} - \rho_{j'} \leq c^{\mathrm{u}}_{j'} - \rho_{j'} \leq 0$. Since $\underline{d}_{j'} \leq d_{j'} \leq \overline{d}_{j'}$, an optimal solution $\boldsymbol{d}^\star_{j'}$ to the inner supremum problem in (EC.10) is $\boldsymbol{d}^\star_{j'} = \underline{d}_{j'}$. To show the upper bound on $\rho_{j'}$, for any fixed $\boldsymbol{o} \in \mathcal{O}$, let $\boldsymbol{\rho}^\star$ be an optimal solution to (EC.10) with $\rho^\star_{j'} > c^{\mathrm{u}}_{j'}$. Consider another feasible solution $\widetilde{\boldsymbol{\rho}}$ defined as follows: $\widetilde{\rho}_j = \rho^\star_j$ for $j \in J \setminus \{j'\}$ and $\widetilde{\rho}_{j'} = \rho^\star_j - \varepsilon \geq c^{\mathrm{u}}_{j'}$ with $\varepsilon \in (0, \rho^\star_{j'} - c^{\mathrm{u}}_{j'}]$. Note that

$$
\begin{aligned}
& h(\widetilde{\boldsymbol{\rho}}; \boldsymbol{o}) \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\, \boldsymbol{d} \in \mathcal{S}} \left\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J \setminus \{j'\}} (w_j - \rho^\star_j) d_j + (w_{j'} - \rho^\star_{j'} + \varepsilon) d_{j'} \right\} && \text{(EC.11a)} \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\, \boldsymbol{d} \in \mathcal{S}} \left\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J \setminus \{j'\}} (w_j - \rho^\star_j) d_j \right\} + (w_{j'} - \rho^\star_{j'} + \varepsilon) \underline{d}_{j'} && \text{(EC.11b)} \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\, \boldsymbol{d} \in \mathcal{S}} \left\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J} (w_j - \rho^\star_j) d_j \right\} + \varepsilon \underline{d}_{j'} && \text{(EC.11c)} \\
&= h(\boldsymbol{\rho}^\star; \boldsymbol{o}) + \varepsilon (\underline{d}_{j'} - \mu_{j'}) \leq h(\boldsymbol{\rho}^\star; \boldsymbol{o}).
\end{aligned}
$$

Here, (EC.11a) follows from the definition of $\widetilde{\boldsymbol{\rho}}$, and (EC.11b)–(EC.11c) follow from the fact that $\underline{d}_{j'}$ is an optimal solution to the supremum problem over $d_{j'}$. Since $\widetilde{\boldsymbol{\rho}}$ yields an optimal value no greater than $\boldsymbol{\rho}^\star$, without loss of optimality, one can impose the constraint $\rho_{j'} \leq c^{\mathrm{u}}_{j'}$. One can apply a similar argument to derive the lower bound on $\rho_{j'}$ for a fixed $j' \in J$. $\square$

Next, in Proposition EC.5, we derive valid inequalities for the reformulation (EC.5) of the TRO-CFLP-MAD model.

PROPOSITION EC.5. *The inequalities $-\gamma_j \leq \rho_j \leq c^{\mathrm{u}}_j + \gamma_j$ for all $j \in J$ are valid for* (EC.5).

Proof. Using the strong duality of moment problem and the dual reformulation of $Q(\boldsymbol{o}, \boldsymbol{d})$, we can reformulate the DRO problem $\sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o}, \boldsymbol{d})]$ as follows:

$$
\underset{\boldsymbol{\rho}, \boldsymbol{\gamma}}{\text{minimize}} \quad h(\boldsymbol{\rho}, \boldsymbol{\gamma}; \boldsymbol{o}) := \boldsymbol{\mu}^\top \boldsymbol{\rho} + \boldsymbol{\sigma}^\top \boldsymbol{\gamma} + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi,\, \boldsymbol{d} \in \mathcal{S}} \left\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J} \left[(w_j - \rho_j) d_j + \gamma_j |d_j - \mu_j| \right] \right\}. \tag{EC.12}
$$

In particular, define $g(d_j) := (w_j - \rho_j) d_j + \gamma_j |d_j - \mu_j|$. Note that (a) if $w_j - \rho_j + \gamma_j \leq 0$, then a maximizer $d^\star_j$ of g over $\underline{d}_j \leq d_j \leq \overline{d}_j$ is $d^\star_j = \underline{d}_j$; (b) if $w_j - \rho_j - \gamma_j \geq 0$, then a maximizer $d^\star_j$ of g over $\underline{d}_j \leq d_j \leq \overline{d}_j$ is $d^\star_j = \overline{d}_j$. With this observation, we first derive the upper bound on $\rho_{j'}$ for a fixed $j' \in J$. Note that for any $\rho_{j'} - \gamma_{j'} \geq c^{\mathrm{u}}_{j'}$, an optimal solution $\boldsymbol{d}^\star_{j'}$ to the inner supremum problem in (EC.12) is $\boldsymbol{d}^\star_{j'} = \underline{d}_{j'}$. Indeed, if $\rho_{j'} - \gamma_{j'} \geq c^{\text{u}}_{j'}$, then it follows from the bound on w_j (i.e., $0 \leq w_j \leq c^{\text{u}}_j$) that $w_{j'} - \rho_{j'} + \gamma_{j'} \leq c^{\text{u}}_{j'} - \rho_{j'} + \gamma_{j'} \leq 0$. Since $\underline{d}_{j'} \leq d_{j'} \leq \overline{d}_{j'}$, an optimal solution $\boldsymbol{d}^\star_{j'}$ to the inner supremum problem in (EC.10) is $\boldsymbol{d}^\star_{j'} = \underline{d}_{j'}$. To show the upper bound on $\rho_{j'}$, for any fixed $\boldsymbol{o} \in \mathcal{O}$, let $(\boldsymbol{\rho}^\star, \boldsymbol{\gamma}^\star)$ be an optimal solution to (EC.10) with $\rho^\star_{j'} - \gamma^\star_{j'} > c^{\text{u}}_{j'}$. Consider another feasible solution $(\widetilde{\boldsymbol{\rho}}, \boldsymbol{\gamma}^\star)$ defined as follows: $\widetilde{\rho}_j = \rho^\star_j$ for $j \in J \setminus \{j'\}$ and $\widetilde{\rho}_{j'} = \rho^\star_j - \varepsilon$ with $\varepsilon \in \big(0, [\rho^\star_{j'} - \gamma^\star_{j'}] - c^{\text{u}}_{j'}\big]$. Note that

$$
\begin{aligned}
&h(\widetilde{\boldsymbol{\rho}}, \boldsymbol{\gamma}^\star; \boldsymbol{o}) \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \boldsymbol{\sigma}^\top \boldsymbol{\gamma}^\star + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi, \boldsymbol{d} \in \mathcal{S}} \Bigg\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J \setminus \{j'\}} \big[(w_j - \rho^\star_j) d_j + \gamma^\star_j |d_j - \mu_j|\big] \\
&\qquad + \big[(w_{j'} - \rho^\star_{j'} + \varepsilon) d_{j'} + \gamma^\star_{j'} |d_{j'} - \mu_{j'}|\big] \Bigg\} \qquad \text{(EC.13a)} \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \boldsymbol{\sigma}^\top \boldsymbol{\gamma}^\star + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi, \boldsymbol{d} \in \mathcal{S}} \Bigg\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J \setminus \{j'\}} \big[(w_j - \rho^\star_j) d_j + \gamma^\star_j |d_j - \mu_j|\big] \Bigg\} \\
&\qquad + \big[(w_{j'} - \rho^\star_{j'} + \varepsilon) \underline{d}_{j'} + \gamma^\star_{j'} |\underline{d}_{j'} - \mu_{j'}|\big] \qquad \text{(EC.13b)} \\
&= \boldsymbol{\mu}^\top \boldsymbol{\rho}^\star - \varepsilon \mu_{j'} + \boldsymbol{\sigma}^\top \boldsymbol{\gamma}^\star + \sup_{(\boldsymbol{w},\boldsymbol{v}) \in \Pi, \boldsymbol{d} \in \mathcal{S}} \Bigg\{ \sum_{i \in I} C_i o_i v_i + \sum_{j \in J} \big[(w_j - \rho^\star_j) d_j + \gamma^\star_j |d_j - \mu_j|\big] \Bigg\} + \varepsilon \underline{d}_{j'} \\
&\qquad \text{(EC.13c)} \\
&= h(\boldsymbol{\rho}^\star, \boldsymbol{\gamma}^\star; \boldsymbol{o}) + \varepsilon(\underline{d}_{j'} - \mu_{j'}) \leq h(\boldsymbol{\rho}^\star, \boldsymbol{\gamma}^\star; \boldsymbol{o}). \qquad \text{(EC.13d)}
\end{aligned}
$$

Here, (EC.13a) follows from the definition of $\widetilde{\boldsymbol{\rho}}$, and (EC.13b)–(EC.13c) follow from the fact that $\underline{d}_{j'}$ is an optimal solution to the supremum problem over $d_{j'}$. Since $(\widetilde{\boldsymbol{\rho}}, \boldsymbol{\gamma}^\star)$ yields an optimal value no greater than $(\boldsymbol{\rho}^\star, \boldsymbol{\gamma}^\star)$, without loss of optimality, one can impose the constraint $\rho_{j'} \leq c^{\text{u}}_{j'} + \gamma_{j'}$. One can apply a similar argument to derive the lower bound on $\rho_{j'}$ for a fixed $j' \in J$. □

EC.4. Additional Computational Results

In this section, we provide additional results comparing the computational performance of the proposed hC&CG algorithm with the BD, pC&CG, and dC&CG algorithms for the TRO model with a TRO set constructed using the MAD ambiguity set. We again observe that BD solution times are significantly longer than the C&CG-based algorithms. Thus, we only present results for pC&CG, dC&CG, and hC&CG algorithms. For illustrative purposes, Figure EC.1 shows the time performance profiles of the three C&CG algorithms for solving instances 8–9. Clearly, hC&CG has the best computational performance among the three algorithms, while pC&CG has the worst. In particular, hC&CG can solve instances 8–9 under nearly all parameter combinations within

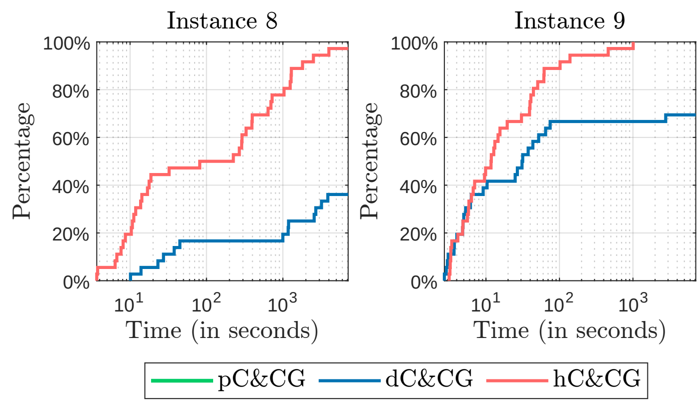

Figure EC.1 Time performance profile for pC&CG, dC&CG, and hC&CG when solving instances 8–9 with the MAD ambiguity set and $\varepsilon = 2\%$

two hours, while pC&CG cannot solve any. Moreover, hC&CG solution times are considerably shorter than that of dC&CG. These results further demonstrate the superior performance of our new hC&CG algorithm.

EC.5. Comparison with Wasserstein-Based Models

Per a reviewer's suggestion, in this section, we present additional results comparing the spectra of solutions from our TRO-CFLP models with solutions obtained from Wasserstein-based DRO models. We define the Wasserstein distance as $\mathsf{d}_W(\mathbb{P}_1, \mathbb{P}_2) = \min_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int_{\mathcal{S}\times\mathcal{S}} \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_1 \, \mathrm{d}\pi(\boldsymbol{d}_1, \boldsymbol{d}_2)$ for any probability measures $\{\mathbb{P}_1, \mathbb{P}_2\} \subseteq \mathcal{P}(\mathcal{S})$, where $\Pi(\mathbb{P}_1, \mathbb{P}_2)$ is the set of all joint distributions with marginal distributions $\mathbb{P}_1$ and $\mathbb{P}_2$. Accordingly, we define the Wasserstein ambiguity set $\mathcal{F}_{\mathrm{W}}(r) = \{\mathbb{P} \in \mathcal{P}(\mathcal{S}) \mid \mathsf{d}_W(\mathbb{P}, \widehat{\mathbb{P}}_N) \leq r\}$. In this experiment, we consider our TRO-CFLP model with the mean-support (MS) ambiguity set (discussed in Sections 6.4 and 6.5) and the following two Wasserstein-based models:

- The first model is the classical Wasserstein DRO model (Mohajerin Esfahani and Kuhn 2018), i.e., $\min_{\boldsymbol{o}\in\mathcal{O}} \left\{ \sum_{i\in I} c_i^{\mathrm{f}} o_i + \sup_{\mathbb{P}\in\mathcal{F}_{\mathrm{W}}(r)} \mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o}, \boldsymbol{d})] \right\}$, with the radius selected via the K-fold cross-validation (CV) procedure. Specifically, we partition the entire data set $\mathcal{D} = \{\widehat{\boldsymbol{d}}_n\}_{n=1}^N$ into K (roughly) equal subsets $\{\mathcal{D}_k\}_{k=1}^K$ and form the training and testing data sets as follows: $\{(\mathcal{D}_k^{\text{train}} = \mathcal{D} \setminus \mathcal{D}_k, \mathcal{D}_k^{\text{test}} = \mathcal{D}_k)\}_{k=1}^K$. For each $k \in [K]$, we formulate the Wasserstein DRO model using the training data $\mathcal{D}_k^{\text{train}}$ and then solve the model with P different radii $\{r_p\}_{p=1}^P = \{c \cdot 10^{-b} \mid c \in \{1, 2, \ldots, 9\}, b \in \{-2, -1, 0, 1\}\} \cup \{0\}$. Then, we compute the CV loss (the average objective value) of the optimal solutions obtained from the training step using the testing data $\mathcal{D}_k^{\text{test}}$ for all $k \in [K]$. Let $\{r_k^\star\}_{k=1}^K$ be the values of r that leads to the minimum CV loss. We set r^{CV} as

Table EC.1 **The best average out-of-sample total cost among all solutions and the heatmap of of percentage differences over the seven solutions: the Wasserstein DRO solution, four distinct solutions from the TRO-CFLP model with MS ambiguity set, and three distinct solutions from the TRO-CFLP model with Wasserstein ambiguity set. The blue color indicates a small difference, followed by the yellow and red colors for larger deviations.**

Δ	Best Cost	Wass DRO	TRO-CFLP (MS)				TRO-CFLP (Wass)		
			Sol-1	Sol-2	Sol-3	Sol-4	Sol-1	Sol-2	Sol-3
-5	68255	0.1%	0.0%	5.6%	11.6%	17.9%	0.0%	7.3%	17.6%
-4	68873	0.1%	0.0%	5.3%	11.2%	17.4%	0.0%	7.0%	17.1%
-3	69398	0.2%	0.0%	5.2%	11.0%	17.1%	0.0%	6.9%	16.8%
-2	69975	0.2%	0.0%	5.0%	10.7%	16.7%	0.0%	6.7%	16.5%
-1	70625	0.3%	0.0%	4.7%	10.3%	16.1%	0.0%	6.4%	16.0%
0	71471	0.4%	0.0%	4.2%	9.7%	15.4%	0.0%	5.9%	15.3%
1	73472	0.5%	0.0%	2.1%	7.4%	12.8%	0.0%	3.8%	12.8%
2	75654	5.7%	5.0%	0.0%	5.0%	10.3%	5.0%	1.6%	10.2%
3	76225	16.8%	15.9%	0.0%	4.9%	10.0%	15.9%	1.7%	10.0%
4	76805	29.1%	28.1%	0.0%	4.8%	9.7%	28.1%	1.7%	9.8%
5	77745	40.5%	39.3%	0.0%	4.0%	8.9%	39.3%	1.6%	9.0%
6	79411	51.6%	50.5%	0.0%	2.5%	7.2%	50.5%	1.5%	7.3%
7	82001	62.6%	61.5%	3.6%	0.0%	4.4%	61.5%	4.8%	4.6%
8	82588	75.6%	74.6%	13.9%	0.0%	4.3%	74.6%	14.9%	4.5%
9	83259	87.7%	86.8%	24.8%	0.0%	4.0%	86.8%	25.7%	4.4%
10	84032	100.1%	99.2%	37.6%	0.0%	3.6%	99.2%	38.3%	4.1%
11	87397	107.4%	106.7%	47.2%	0.0%	0.3%	106.7%	47.8%	1.0%
12	88091	117.2%	116.5%	57.5%	4.0%	0.0%	116.5%	58.0%	1.0%
13	88734	129.5%	128.9%	70.2%	12.6%	0.0%	128.9%	70.5%	1.3%
14	89437	142.0%	141.4%	83.1%	24.7%	0.0%	141.4%	83.4%	1.9%
15	90516	151.6%	151.1%	93.4%	35.3%	0.0%	151.1%	93.7%	2.5%
16	93058	156.9%	156.4%	100.3%	43.6%	0.0%	156.4%	100.5%	2.9%
17	98312	155.8%	155.4%	102.3%	48.4%	0.0%	155.4%	102.4%	3.2%
18	107748	145.3%	145.0%	96.4%	47.1%	0.0%	145.0%	96.5%	3.0%
19	118292	133.4%	133.2%	88.9%	43.9%	0.0%	133.2%	88.9%	2.7%
20	127416	124.2%	124.0%	82.9%	41.1%	0.0%	124.0%	82.9%	2.5%

the average of $\{r_k^\star\}_{k=1}^K$. We then solve the Wasserstein DRO model formulated using the entire data set $\mathcal{D}$ with radius r^{CV}.

- The second model is our TRO-CFLP model with $\mathcal{F}$ being the Wasserstein ambiguity set $\mathcal{F}_{\text{W}}(r)$, i.e., $\min_{\boldsymbol{o}\in\mathcal{O}}\left\{\sum_{i\in I} c_i^{\text{f}} o_i + (1-\theta)\mathbb{E}_{\widehat{\mathbb{P}}_N}[Q(\boldsymbol{o},\boldsymbol{d})] + \theta \sup_{\mathbb{P}\in\mathcal{F}_{\text{W}}(r)} \mathbb{E}_{\mathbb{P}}[Q(\boldsymbol{o},\boldsymbol{d})]\right\}$. To ensure we obtain a spectrum of solutions with a wide range of conservatism, we choose a relatively large radius, $r = 100$. We then apply our Spectrum Search Algorithm to generate the spectrum of optimal solutions to our TRO-CFLP model with $\mathcal{F} = \mathcal{F}_{\text{W}}(r)$.

As in Sections 6.4 and 6.5, we use instance 8 with $(c^{\text{f}}, C, c^{\text{u}}) = (5000, 200, 300)$ for illustrative purposes and brevity. For this instance, we obtain seven solutions in total: the Wasserstein DRO solution, four distinct solutions from the TRO-CFLP model with MS ambiguity set (denoted as Sol-1 through Sol-4, where Sol-1 is the SAA solution and Sol-4 is the DRO solution), and three distinct solutions from the TRO-CFLP model with Wasserstein ambiguity set (solutions labeled similarly). Note that Sol-1 in both TRO-CFLP models is the same, as they correspond to the SAA solution. Following the same procedure in Section 6.5, we analyze these solutions using $N' = 10,000$ out-of-sample scenarios with different demand shifts $\Delta \in \{-5, -4, \ldots, 20\}$. Let $\texttt{best\_cost}$ denote the best (i.e., minimum) average out-of-sample total cost among all solutions. For each solution, we compute the percentage difference between the average out-of-sample total cost of the solution, denoted as $\texttt{sol\_cost}$, and the best total cost as $(\texttt{sol\_cost} - \texttt{best\_cost})/\texttt{best\_cost} \times 100\% \geq 0$.

Table EC.1 reports the best average out-of-sample total cost among all solutions and the heatmap of these percentage differences, where the blue color indicates a small difference, followed by the yellow and red colors for larger deviations. These results show that the spectrum of solutions obtained from the TRO-CFLP model with the MS ambiguity set achieves the best (0%) out-of-sample performance across all values of Δ. The classical Wasserstein DRO solution performs well for small and negative demand shifts (with deviations from best performance $\leq 0.5\%$) but deteriorates rapidly as demand deviation becomes larger (i.e., when Δ increases). The spectrum of solutions to the TRO-CFLP model with Wasserstein ambiguity set performs reasonably well in different regions of Δ: Sol-1, Sol-2, and Sol-3 achieve the best performance when $\Delta \in [-5, 1]$, $\Delta \in [2, 6]$, and $\Delta \in [7, 20]$, respectively. However, these solutions (except for Sol-1) are 1.0%–4.6% worse than the best solution. Notably, the spectrum of solutions obtained using the MS ambiguity set dominates that of the Wasserstein ambiguity set. These results demonstrate the advantage of our TRO-CFLP model by allowing for a flexible choice of the ambiguity set $\mathcal{F}$, such as moment-based ambiguity sets.

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