Online Supplement to "Ex-Ante Information and the Design of Keyword Auctions"

De Liu
University of Kentucky
de.liu@uky.edu

Jianqing Chen
University of Calgary
jiachen@ucalgary.ca

Andrew B. Whinston
University of Texas at Austin
abw@uts.cc.utexas.edu

1 Proof of the Monotonicity Requirement on Bidding Functions

Proof. Take l-type advertisers as an example. The incentive compatibility condition requires that for any $v'' > v'$.

$$\begin{aligned}\left[v' - b_l\left(v'\right)\right]\rho_l\left(v'\right) &\geq \left[v' - b_l\left(v''\right)\right]\rho_l\left(v''\right) && (1)\\ \left[v'' - b_l\left(v''\right)\right]\rho_l\left(v''\right) &\geq \left[v'' - \rho_l\left(v'\right)\right]\rho_l\left(v'\right) && (2)\end{aligned}$$

Combining (1) and (2), we get $\rho_l\left(v''\right) \geq \rho_l\left(v'\right)$, which implies $b_l\left(v''\right) \geq b_l\left(v'\right)$. Next we show $b_l\left(v'\right) \neq b_l\left(v''\right)$. Actually, if $b_l\left(v'\right) = b_l\left(v''\right) \equiv b$, then $b_l(v) = b$ for all $v \in [v', v'']$. An l-type advertiser with valuation-per-click v'' is better off by bidding $b + \epsilon$ (ϵ is an infinitesimal positive number) since the advertiser loses only ϵ for per-unit resource awarded, yet improves the probability of beating another advertiser by a significant amount. This contradicts the equilibrium condition. Therefore $b_l\left(v\right)$ must be strictly increasing. ■

2 Extension to Multiple CTR-types

We now consider multiple CTR types, indexed by $\theta = 1, 2, ..., k$. Assume the probability of being a CTR-type θ is α_θ. Denote the expected CTR, the distribution of valuation-per-click, and the weighting factor for CTR-type θ as Q_θ, $F_\theta\left(v\right)$, and w_θ, respectively. We assume $Q_1 > Q_2 > ... > Q_k$.

Similar to Lemma 1, we can establish the relationship between the bidding functions for any two CTR-types.

Lemma 1 *For any two CTR-types t and r,*

$$w_t b_t(v) = w_r b_r(\frac{w_t}{w_r} v), \forall v, \frac{w_t}{w_r} v \in [0,1] \tag{3}$$

Proof. Consider an r-type advertiser with valuation-per-click $\frac{w_t}{w_r} v$ who bids $\frac{w_t}{w_r} b$ and a t-type advertiser with v who bids b. Both advertisers get a score $w_t b$, and their payoff functions are

$$U_t(v,b) = Q_t(v-b) \sum\nolimits_{j=1}^{m} \delta_j \Pr(w_t b \text{ ranks } j\text{th}) \tag{4}$$

$$U_r\left(\frac{w_t}{w_r} v, \frac{w_t}{w_r} b\right) = Q_r\left(\frac{w_t}{w_r} v - \frac{w_t}{w_r} b\right) \sum\nolimits_{j=1}^{m} \delta_j \Pr(w_t b \text{ ranks } j\text{th}) \tag{5}$$

It is easy to establish that

$$U_r\left(\frac{w_t}{w_r} v, \frac{w_t}{w_r} b\right) = \frac{w_t}{w_r} \frac{Q_r}{Q_t} U_t(v,b). \tag{6}$$

For $b_t(v)$ and $b_r(v)$ to be equilibrium bidding functions, at any v, $b_t(v)$ must maximize $U_t(v,b)$ and $b_r(v)$ must maximize $U_r(v,b)$. So, (6) suggests that if bidding b is the best for a t-type advertiser with valuation-per-click v, bidding $\frac{w_t}{w_r} b$ must be the best for an r-type advertiser with valuation-per-click $\frac{w_r}{w_t} v$, which implies $b_r(\frac{w_t}{w_r} v)$ equals $\frac{w_t}{w_r} b_t(v)$. ■

The above lemma says a t-type advertiser with valuation-per-click v matches an r-type advertiser with valuation-per-click $\frac{w_t}{w_r} v$. Similar to the two-type case, we can then formulate a t-type advertiser's one-on-one winning probability as

$$G_t(v) = \sum_{\theta=1}^{k} \alpha_\theta F_\theta \left(\frac{w_t}{w_\theta} v\right).$$

The probability of winning the jth slot, $P_\theta^j(v) = \binom{n-1}{n-j} G_\theta(v)^{n-j} \left[1 - G_\theta(v)\right]^{j-1}$, is defined as before. For notational simplicity, we denote $\rho_\theta(v) \equiv \sum_{j=1}^{m} \delta_j P_\theta^j(v)$ as before.

Proposition 1 *The equilibrium bidding functions are given by*

$$b_\theta(v) = v - \frac{\int_0^v \rho_\theta(x)dx}{\rho_\theta(v)}, \theta \in \{1, 2, ..., k\}, v \in [0, 1]$$

Proof. Denote the inverse bidding functions as $b_\theta^{-1}(b)$, which are strictly increasing given the monotonicity of the bidding functions. Substituting this into (4), we can uniformly write the payoff functions as

$$U_\theta(v, b) = Q_\theta(v - b)\rho_\theta\left(b_\theta^{-1}(b)\right)$$

We denote

$$V_\theta(v) \equiv U_\theta(v, b_\theta(v)) = Q_\theta(v - b_\theta(v))\rho_\theta\left(b_\theta^{-1}(b_\theta(v))\right) \tag{7}$$

as the equilibrium payoff of an advertiser with valuation-per-click v.

$$\begin{aligned} \frac{dV_\theta(v)}{dv} &= \frac{\partial U_\theta(v, b_\theta(v))}{\partial v} + \frac{\partial U_\theta(v, b_\theta(v))}{\partial b}\frac{db_\theta(v)}{dv} \\ &= \frac{\partial U_\theta(v, b_\theta(v))}{\partial v} = Q_\theta \rho_\theta(v) \end{aligned}$$

where the second equality is due to $\frac{\partial U_\theta(v, b_\theta(v))}{\partial b} = 0$ (the first-order condition). Applying the boundary condition $V_\theta(0) = 0$, we get

$$V_\theta(v) = Q_\theta \int_0^v \rho_\theta(x)dx \tag{8}$$

Combining (7) (note $b_\theta^{-1}(b_\theta(v)) = v$) and (8), we can solve the equilibrium bidding function as specified in the proposition. ■

We can also similarly analyze the efficient weighting scheme. The assignment of an advertiser with valuation-per-click v and CTR-signal θ to slot j will generate an expected valuation of $v\delta_j Q_\theta$. Given that the probability of assigning an advertiser to slot j is $P_\theta^j(v)$, the total expected valuation generated by all advertisers is,

$$W = n\sum_{\theta=1}^{k} \alpha_\theta Q_\theta \int_0^1 v\rho_\theta(v) f_\theta(v)\, dv. \tag{9}$$

The following proposition shows that, as in the two CTR-type case, it is efficient to weigh advertisers' bids by their expected CTRs.

Proposition 2 *The efficient weighting factors satisfy* $\frac{Q_1}{w_1} = \frac{Q_2}{w_2} = ... = \frac{Q_k}{w_k}$.

Proof. First notice

$$\frac{dG_t\left(v\right)}{dw_t} = \sum_{\theta \neq t} \alpha_\theta f_\theta \left(\frac{w_t}{w_\theta} v \right) \frac{1}{w_\theta} v \tag{10}$$

and

$$\frac{dG_r\left(v\right)}{dw_t} = -\alpha_t f_t \left(\frac{w_r}{w_t} v \right) \frac{w_r}{w_t^2} v. \tag{11}$$

Taking the first-order derivative of (9) with respect to w_t yields

$$\begin{aligned}
& \alpha_t Q_t \int_0^1 v \frac{d\rho_t(v)}{dw_t} f_t\left(v\right) dv + \sum_{\theta \neq t} \alpha_\theta Q_\theta \int_0^1 v \frac{d\rho_\theta(v)}{dw_t} f_\theta\left(v\right) dv \\
= \; & \alpha_t Q_t \int_0^1 v \frac{d\rho_t(v)}{dG_t\left(v\right)} \sum_{\theta \neq t} \alpha_\theta f_\theta \left(\frac{w_t}{w_\theta} v \right) \frac{1}{w_\theta} v f_t\left(v\right) dv \\
& - \sum_{\theta \neq t} \alpha_\theta Q_\theta \int_0^1 v \frac{d\rho_\theta(v)}{dG_\theta(v)} \alpha_t f_t \left(\frac{w_\theta}{w_t} v \right) \frac{w_\theta}{w_t^2} v f_\theta\left(v\right) dv \\
= \; & \alpha_t \sum_{\theta \neq t, w_\theta \geq w_t} \alpha_\theta \left[Q_t \int_0^1 \frac{d\rho_t(v)}{dG_t\left(v\right)} f_\theta \left(\frac{w_t}{w_\theta} v \right) \frac{1}{w_\theta} v^2 f_t\left(v\right) dv \right. \\
& \qquad \left. - Q_\theta \int_0^{\frac{w_t}{w_\theta}} \frac{d\rho_\theta(v)}{dG_\theta(v)} f_t \left(\frac{w_\theta}{w_t} v \right) \frac{w_\theta}{w_t^2} v^2 f_\theta\left(v\right) dv \right] \\
& + \alpha_t \sum_{\theta \neq t, w_\theta < w_t} \alpha_\theta \left[Q_t \int_0^{\frac{w_\theta}{w_t}} \frac{d\rho_t(v)}{dG_t\left(v\right)} f_\theta \left(\frac{w_t}{w_\theta} v \right) \frac{1}{w_\theta} v^2 f_t\left(v\right) dv \right. \\
& \qquad \left. - Q_\theta \int_0^1 \frac{d\rho_\theta(v)}{dG_\theta(v)} f_t \left(\frac{w_\theta}{w_t} v \right) \frac{w_\theta}{w_t^2} v^2 f_\theta\left(v\right) dv \right] \\
= \; & \alpha_t w_t \sum_{\theta \neq t, w_\theta \geq w_t} \frac{\alpha_\theta}{w_\theta} \left(\frac{Q_t}{w_t} - \frac{Q_\theta}{w_\theta} \right) \int_0^1 \frac{d\rho_t(v)}{dG_t\left(v\right)} f_t\left(v\right) f_\theta \left(\frac{w_t}{w_\theta} v \right) v^2 dv \\
& + \frac{\alpha_t}{w_t^2} \sum_{\theta \neq t, w_\theta < w_t} \alpha_\theta w_\theta^2 \left(\frac{Q_t}{w_t} - \frac{Q_\theta}{w_\theta} \right) \int_0^1 \frac{d\rho_\theta(v)}{dG_\theta(v)} f_\theta\left(v\right) f_t \left(\frac{w_\theta}{w_t} v \right) v^2 dv
\end{aligned} \tag{12}$$

where the first step is by substituting (10) and (11), the second step is because $f_t(v) = 0$ for $v > 1$ and $f_\theta(v) = 0$ for $v > 1$, the final step is due to integration by substitution and the fact that

$\frac{d\rho_t(v)}{dG_t(v)} = \frac{d\rho_\theta(v)}{dG_\theta(v)}|_{\frac{w_t}{w_\theta}v}$ (noting $G_t(v) = G_\theta(\frac{w_t}{w_\theta}v)$) and $\frac{d\rho_t(v)}{dG_t(v)}|_{\frac{w_\theta}{w_t}v} = \frac{d\rho_\theta(v)}{dG_\theta(v)}$.

Because $\frac{d\rho_t(v)}{dG_t(v)} > 0$ and $\frac{d\rho_\theta(v)}{dG_\theta(v)} > 0$, the integrals in (12) are positive. (12) implies that the efficient weighting factors must satisfy $\frac{Q_1}{w_1} = \frac{Q_2}{w_2} = ... = \frac{Q_k}{w_k}$. Otherwise, assume $\frac{Q_h}{w_h}$ is the largest one, and $\frac{Q_\theta}{w_\theta}$ strictly less than $\frac{Q_h}{w_h}$. According to (12), the first-order derivative of the total valuation W with respect to w_h is strictly positive, which means that one can increase the total valuation W by slightly increasing w_h, a contradiction. ■

Similar to the two-type case, we can formulate the expected revenue as follows.

$$\pi = n \sum_{\theta=1}^{k} \alpha_\theta Q_\theta \int_0^1 \rho_\theta(v) \left[v - \frac{1 - F_\theta(v)}{f_\theta(v)} \right] f_\theta(v) dv. \tag{13}$$

The advertising provider should choose the k weighting factors (fixing one of them to 1) to maximize (13).

The following proposition shows that when valuation-per-click and CTR signals are independent, the advertising provider should promote the lowest type and handicap the highest type to maximize the revenue. This result is in line with our main intuition developed in the two CTR-type case (Proposition 3). However, we do not have general conclusions on whether the auctioneer should promote/handicap other (middle) CTR-types because, unlike the two-type case, the advertising provider must consider which low types, among many candidates, to promote.

Proposition 3 *If the valuation-per-click and CTR signals are independent, and $F(v)$ is IHR, then the optimal weighting factors for the highest type and the lowest type satisfy $\frac{Q_1}{w_1} > \min_{\theta>1}\left\{\frac{Q_\theta}{w_\theta}\right\}$ and $\frac{Q_k}{w_k} < \max_{\theta<k}\left\{\frac{Q_\theta}{w_\theta}\right\}$, respectively.*

Proof. Taking the first-order derivative of (13) with respect to w_t yields

$$
\begin{aligned}
& \alpha_t Q_t \int_0^1 \frac{d\rho_t(v)}{dw_t}\left(v - \frac{1-F_t(v)}{f_t(v)}\right) f_t\left(v\right) dv + \sum_{\theta\neq t} \alpha_\theta Q_\theta \int_0^1 \frac{d\rho_\theta(v)}{dw_t}\left(v - \frac{1-F_\theta(v)}{f_\theta(v)}\right) f_\theta\left(v\right) dv \\
= \; & \alpha_t Q_t \int_0^1 \frac{d\rho_t(v)}{dG_t\left(v\right)} \sum_{\theta\neq t} \alpha_\theta f_\theta\left(\frac{w_t}{w_\theta}v\right) \frac{1}{w_\theta} v \left(v - \frac{1-F_t(v)}{f_t(v)}\right) f_t\left(v\right) dv \\
& - \sum_{\theta\neq t} \alpha_\theta Q_\theta \int_0^1 \frac{d\rho_\theta(v)}{dG_\theta(v)} \alpha_t f_t\left(\frac{w_\theta}{w_t}v\right) \frac{w_\theta}{w_t^2} v \left(v - \frac{1-F_\theta(v)}{f_\theta(v)}\right) f_\theta\left(v\right) dv \\
= \; & \alpha_t \sum_{\theta\neq t, w_\theta \geq w_t} \alpha_\theta \left[Q_t \int_0^1 \frac{d\rho_t(v)}{dG_t\left(v\right)} f_\theta\left(\frac{w_t}{w_\theta}v\right) \frac{1}{w_\theta} v \left(v - \frac{1-F_t(v)}{f_t(v)}\right) f_t\left(v\right) dv \right. \\
& \qquad \left. - Q_\theta \int_0^{\frac{w_t}{w_\theta}} \frac{d\rho_\theta(v)}{dG_\theta(v)} f_t\left(\frac{w_\theta}{w_t}v\right) \frac{w_\theta}{w_t^2} v \left(v - \frac{1-F_\theta(v)}{f_\theta(v)}\right) f_\theta\left(v\right) dv \right] \\
& + \alpha_t \sum_{\theta\neq t, w_\theta < w_t} \alpha_\theta \left[Q_t \int_0^{\frac{w_\theta}{w_t}} \frac{d\rho_t(v)}{dG_t\left(v\right)} f_\theta\left(\frac{w_t}{w_\theta}v\right) \frac{1}{w_\theta} v \left(v - \frac{1-F_t(v)}{f_t(v)}\right) f_t\left(v\right) dv \right. \\
& \qquad \left. - Q_\theta \int_0^1 \frac{d\rho_\theta(v)}{dG_\theta(v)} f_t\left(\frac{w_\theta}{w_t}v\right) \frac{w_\theta}{w_t^2} v \left(v - \frac{1-F_\theta(v)}{f_\theta(v)}\right) f_\theta\left(v\right) dv \right] \\
= \; & \alpha_t \sum_{\theta\neq t, w_\theta \geq w_t} \frac{\alpha_\theta}{w_\theta} \int_0^1 \frac{d\rho_t(v)}{dG_t\left(v\right)} f_t\left(v\right) f_\theta\left(\frac{w_t}{w_\theta}v\right) v \\
& \qquad \times \left[\left(Q_t - \frac{w_t}{w_\theta}Q_\theta\right) v + Q_\theta \frac{1-F_\theta(\frac{w_t}{w_\theta}v)}{f_\theta(\frac{w_t}{w_\theta}v)} - Q_t \frac{1-F_t(v)}{f_t(v)} \right] dv \\
& + \frac{\alpha_t}{w_t^2} \sum_{\theta\neq t, w_\theta < w_t} \alpha_\theta w_\theta \int_0^1 \frac{d\rho_\theta(v)}{dG_\theta(v)} f_\theta\left(v\right) f_t\left(\frac{w_\theta}{w_t}v\right) v \\
& \qquad \times \left[\left(\frac{w_\theta}{w_t}Q_t - Q_\theta\right) v + Q_\theta \frac{1-F_\theta(v)}{f_\theta(v)} - Q_t \frac{1-F_t(\frac{w_\theta}{w_t}v)}{f_t(\frac{w_\theta}{w_t}v)} \right] dv
\end{aligned}
\tag{14}
$$

where the first step is by substituting (10) and (11), the second step is because $f_t(v) = 0$ for $v > 1$ and $f_\theta(v) = 0$ for $v > 1$, and the final step is due to integration by substitution and the fact that $\frac{d\rho_t(v)}{dG_t(v)} = \frac{d\rho_\theta(v)}{dG_\theta(v)}|_{\frac{w_t}{w_\theta}v}$ and $\frac{d\rho_t(v)}{dG_t(v)}|_{\frac{w_\theta}{w_t}v} = \frac{d\rho_\theta(v)}{dG_\theta(v)}$. Then (14) implies that $\frac{Q_1}{w_1} > \min_{\theta>1}\left\{\frac{Q_\theta}{w_\theta}\right\}$. Otherwise, we have $\frac{Q_1}{w_1} \leq \frac{Q_\theta}{w_\theta}$ for all $\theta > 1$. This implies that $\left(Q_1 - \frac{w_1}{w_\theta}Q_\theta\right)v + Q_\theta\frac{1-F(\frac{w_1}{w_\theta}v)}{f(\frac{w_1}{w_\theta}v)} - Q_1\frac{1-F(v)}{f(v)} < 0$ (noting the IHR property of $F(v)$) and $\left(\frac{w_\theta}{w_1}Q_1 - Q_\theta\right)v + Q_\theta\frac{1-F(v)}{f(v)} - Q_1\frac{1-F(\frac{w_\theta}{w_1}v)}{f(\frac{w_\theta}{w_1}v)} < 0$. Therefore, the first-order derivative of (13) with respect to w_1 is negative, contradicting the optimality condition. Similarly, we can argue that $\frac{Q_k}{w_k} < \max_{\theta<k}\left\{\frac{Q_\theta}{w_\theta}\right\}$. ■