Appendix

Proof of Lemma 1

Let there be a segment of customers that exercise option 1. We consider three cases to cover all possibilities:

Case 1:

Let $p_i < f_u - f$ and $p_u < f_u - f$. In this case, surplus of all customers is greater if they exercise option 2 than if the exercise option 1. Hence, it cannot be correct that some customers exercise option 1.

Case 2:

Let $p_i \geq f_u - f$ and $p_u < f_u - f$. In this case, surplus of all customers is greater if they exercise option 2 than if the exercise option 1. Hence, it cannot be correct that some customers exercise option 1.

Case 3:

Let $p_i \geq f_u - f$ and $p_u \geq f_u - f$. In this case, note that customers who exercise option 1 do not buy in period 2. Hence the incumbent can improve its profits by setting $p_u < f_u - f$, since by doing so, the customers exercising option 1 will switch to option 2 while the decisions of all other customers are unaffected as their surpluses do not change. Hence, again, it cannot be correct that some customers exercise option 1.

Proof of Lemma 2

From the discussion in Section 4.1, the second period equilibrium price possibilities are given by the following:

$$p_u = \text{Min}\{f_u - f, p_d + s + d - 2M_1 d\} \tag{9}$$

$$p_d = (f_u - f) - d + 2M_2 \cdot d - s, \tag{10}$$

$$p_i = \text{Min}\{f_u - M_2 \cdot d, p_e + d - 2M_2 d\} \tag{11}$$

$$p_e = \text{Min}\{f_u - d + M_3 \cdot d, p_i - d + 2M_3 d\} \tag{12}$$

This gives us 8 systems of equations to solve for which gives us second period prices in terms of the first period cut-offs: M_1, M_2, M_3.

Writing out the conditions for the indifference points in the first period and simplifying:

$$M_1 \cdot d = f - p_1 - p_u + p_i \tag{13}$$

$$M_2 \cdot d = d + p_1 + p_d + s - f - p_i \tag{14}$$

$$M_3 \cdot d = f - p_1 - p_d + p_e - s \tag{15}$$

Simultaneously solving each of the eight systems of equations (from(9) through (12)) along with equations (13) through (15), we never get a solution where $M_1 < M_2 < M_3$. This shows that an equilibrium with all four market segments A, B, C, D cannot exist.

Proof of Proposition 1

Note that for the market structure ABC, we must have $0 < M_1 < M_2 < M_3 = 1$. This violates Lemma 2. Hence ABC cannot exist in equilibrium.

For the market structure BCD, we must have $M_1 = 0$ and $M_1 < M_2 < M_3 < 1$. This again violates Lemma 2. Hence BCD cannot exist in equilibrium.

For the market structure BC, we must have $M_1 = 0$, $M_3 = 1$ and $0 < M_2 < 1$. This requirement also violates Lemma 2. Hence BC cannot exist in equilibrium.

The market structure CD implies that the incumbent is not active in period 2 and this violates Assumption 2. Hence CD cannot exist in equilibrium.

The market structure AC implies that the incumbent covers the entire market in period 1 and this violates Assumption 1. Hence AC cannot exist in equilibrium.

The market structure C violates both assumptions 1 and 2. Hence it cannot exist in equilibrium.

This leaves us with only the market structure ACD.

Proof of Theorem 1

1) **Proof that there exists a unique set of second period prices in ACD Equilibrium**

We divide this proof into several parts that we then put together for the final results.

To show that there exist unique second period prices for first period customers of the incumbent's product:

Let θ be the indifference point between segments A and C. For any prices p_d and p_u, θ is given by comparing the corresponding consumer surpluses:

$$\theta = \frac{p_d + s + d - p_u}{2d} \tag{16}$$

Writing the second period profits as functions of first period market size M_3 (equations (5) and (6)) and applying first order conditions provides us with the following implicit functions for best response prices:

$$\left(\frac{p_u}{2d}\right) \cdot \left(\frac{g(\theta)}{G(\theta)}\right) = 1 \tag{17}$$

$$\left(\frac{p_d}{2d}\right) \cdot \left(\frac{g(\theta)}{G(M_3) - G(\theta)}\right) = 1 \tag{18}$$

For $t \sim IFR$, the best response prices in equations (17) and (18) are unique as shown in Kim 2007. Let t^* be the value of θ at equilibrium. The value of t^* is obtained implicitly by substituting the equilibrium values p_u^* and p_d^* and using $\theta = t^*$ in Equation (16):

$$t^* = \left(\frac{1}{2} + \frac{s}{2d}\right) + \left(\frac{G(M_3) - 2G(t^*)}{g(t^*)}\right) \tag{19}$$

The left hand side (lhs) of equation (19) is a monotonically increasing function of t^* while the right hand side(rhs) is a monotonically decreasing function of t^* for $t \sim IFR$. At $t^* = 0$, lhs = 0 and rhs $= \left(\frac{1}{2} + \frac{s}{2d}\right) + \left(\frac{G(M_3)}{g(0)}\right) > 0$. Hence to ensure existence and uniqueness of t^*, the left hand side (lhs) of the above equation must be greater than the right hand side (rhs) at $t^* = 1$. At $t^* = 1$, the rhs is:

$$rhs = \left(\frac{1}{2} + \frac{s}{2d}\right) + \left(\frac{G(M_3) - 2}{g(1)}\right)$$

A necessary condition for existence and uniqueness is the rhs to be less than 1. Hence:

$$s < d\left(1 + 2\left(\frac{2 - G(M_3)}{g(1)}\right)\right)$$

If the above condition is obeyed, we have a unique t^* and unique prices p_u^* and p_d^* obtained by substituting $\theta = t^*$ in equations (17), (18) and (19).

To show that the entrant's second period price for non-first period customers is unique:

From the discussion in Section 4.2 we know that $p_e = Min[p_i - d + 2dM_3, f_u - d + dM_3]$. Suppose that the incumbent sets p_i so low that the incentive compatibility constraint binds. Then we have:

$$p_e^* = p_i - d + 2dM_3 \tag{20}$$

We analyze the implications of this assumption on p_i in sub-parts:

i) To show that the incumbent's second period profit is increasing in M_3:

To show this, we evaluate the total derivative of incumbent's second period profit with respect to M_3 using chain rule:

$$\begin{aligned}\frac{d\pi_{i2}^*}{dM_3} &= \frac{\partial \pi_{i2}^*}{\partial M_3} + \frac{\partial \pi_{i2}^*}{\partial p_u^*} \cdot \frac{dp_u^*}{dM_3} + \frac{\partial \pi_{i2}^*}{\partial p_d^*} \cdot \frac{dp_d^*}{dM_3} \\ &= 0 + 0 + \frac{\partial \pi_{i2}^*}{\partial p_d^*} \cdot \frac{dp_d^*}{dM_3} \\ &= \left(\frac{p_u^*}{2d}\right) g(\theta^*) \cdot \frac{dp_d^*}{dM_3}\end{aligned} \tag{21}$$

Thus, the sign of $\frac{d\pi_{i2}^*}{dM_3}$ depends on the sign of $\frac{dp_d^*}{dM_3}$ since the other terms in the expression are positive. The total derivative $\frac{dp_d^*}{dM_3}$ can be written as:

$$\frac{dp_d^*}{dM_3} = \frac{\partial p_d^*}{\partial M_3} + \frac{\partial p_d^*}{\partial p_u^*} \cdot \frac{dp_u^*}{dM_3} \tag{22}$$

We need to examine the sign of each term in the above expression. We begin by examining the sign of $\frac{\partial p_d^*}{\partial M_3}$. Rearranging equation (18), we get the following fixed point equation:

$$p_d = 2d \cdot \left(\frac{G(M_3) - G(\theta)}{g(\theta)} \right) = \phi(p_d) \tag{23}$$

Set $M_3 = M_3^1$ to get $\phi_1(p_d)$ and $M_3 = M_3^2$ to get $\phi_2(p_d)$. If $M_3^2 \geq M_3^1$, then $\phi_2(p_d) \geq \phi_1(p_d)$. Then, using the Proposition 1 from Villas-Boas 1997 and the uniqueness of best response p_d for $t \sim IFR$, we get that best response p_d is increasing in M_3. This implies that $\frac{\partial p_d^*}{\partial M_3} \geq 0$.

We now examine the sign of $\frac{\partial p_d^*}{\partial p_u^*}$. For $t \sim IFR$, we can show from equation (23) that:

$$\frac{\partial p_d^*}{\partial p_u^*} \geq \frac{1 - G(M_3)}{1 - G(\theta)} \geq 0$$

Finally, we examine the sign of $\frac{dp_u^*}{dM_3}$ by rearranging equation (17) to provide:

$$p_u^* = 2d \cdot \left(\frac{G(t^*)}{g(t^*)} \right) \tag{24}$$

where t^* is the value of θ at equilibrium. For $t \sim IFR$ and $g(t)$ symmetric in the range [0,1], $\frac{G(t^*)}{g(t^*)}$ is increasing in t^*. Consider equation (19) once again:

$$t^* = \left(\frac{1}{2} + \frac{s}{2d} \right) + \left(\frac{G(M_3) - 2G(t^*)}{g(t^*)} \right) = \beta(t^*)$$

Set $M_3 = M_3^1$ to get $\beta_1(t^*)$ and $M_3 = M_3^2$ to get $\beta_2(t^*)$. If $M_3^2 \geq M_3^1$, then $\beta_2(t^*) \geq \beta_1(t^*)$. Further, $\beta(t^*)$ is decreasing in t^* for $t \sim IFR$. Then, using Proposition 3 from Villas-Boas 1997, we get that t^* is increasing in M_3. Since $\frac{G(t^*)}{g(t^*)}$ is increasing in t^* and t^* is increasing in M_3, we have p_u^* increasing in M_3 and consequently: $\frac{dp_u^*}{dM_3} \geq 0$.

Observe that we have shown that all terms in the right had side of equation (22) are positive. Consequently, $\frac{dp_d^*}{dM_3} \geq 0$. Using this in equation (21), we get $\frac{d\pi_{i2}^*}{dM_3} \geq 0$ and hence π_{i2}^* is increasing in M_3.

ii) To show that M_3 is increasing in p_i.

While p_i does not directly influence second period profit for the incumbent, there may be an indirect influence through first period cut-off, M_3. The indifference condition between segments C and D (Option 3) can be rewritten as (using equation (20)):

$$M_3 = \frac{f - p_1 - p_d^* + p_e^* - s}{d} = \frac{f - p_1 - p_d^* + p_i - d + 2dM_3 - s}{d} = \eta(M_3) \tag{25}$$

Set $p_i = p_i^1$ to get $\eta_1(M_3)$ and $p_i = p_i^2$ to get $\eta_2(M_3)$. If $p_i^2 \geq p_i^1$, then $\eta_2(M_3) \geq \eta_1(M_3)$. The incumbent's second period profit π_{i2}^* is increasing in M_3 from part i). Also, for a given p_1 the incumbent can always increase first period profit $p_1 \cdot M_3$ by increasing M_3. Thus, for a given p_i, the incumbent's focus is on the highest fixed point of equation (25) since this would increase the incumbent's overall profit over two periods. Then, using Proposition 1 from Villas-Boas 1997, we get that M_3 is increasing in p_i.

iii) To show that $p_e^* = f_u - d + Md_3$

Since π_{i2}^* is increasing in M_3 from part i) and M_3 is increasing in p_i from part ii), π_{i2}^* is also increasing in p_i. Thus, in the second period, the incumbent will increase p_i till IR constraint binds on p_e at which point $p_e^* = f_u - d + dM_3$. In this situation, M_3 is no longer a function of p_i and hence π_{i2}^* becomes independent of p_i.

Thus, for a given M_3, we get a unique p_e^*. Since second period prices are unique for both first and non-first period customers for a given M_3, a unique set of second period prices enforces the ACD market structure.

2) **Proof that there exists some first period price in ACD equilibrium** Equation (24) provides us with the equilibrium value of p_u^* while the indifference condition between segments A and C provides us with the value of p_1 as a function of M_3:

$$p_1 = f + f_u - d - s - \left(2d\left(\frac{G(M_3) - G(t^*)}{g(t^*)}\right)\right)$$

The two period profit function for the incumbent π_i is:

$$\begin{aligned}\pi_i &= p_1 \cdot G(M_3) + p_u \cdot G(t^*) \\ &= \left[f + f_u - d - s - \left(2d\left(\frac{G(M_3) - G(t^*)}{g(t^*)}\right)\right)\right] \cdot G(M_3) + 2d \cdot \left(\frac{G(t^*)^2}{g(t^*)}\right)\end{aligned}$$

From part 1), we know that t^* is a continuous, increasing function of M_3. Thus, the profit π_i is a continuous function in M_3 over a compact set [0,1]. Hence, there exists an optimal M_3^* that maximizes the incumbent's overall profit function.

Proof of Proposition 2

From Lemma 3, we know that:

$$p_e = f_u - d + dM_3 \tag{26}$$

Using the second period profit functions from equations (5) and (6) and optimizing on second period prices using the uniform distribution assumption on t yields:

$$p_u = \frac{d(2M_3 + 1) + s}{3} \tag{27}$$

$$p_d = \frac{d(4M_3 - 1) - s}{3} \tag{28}$$

Substituting these prices into the profit function for the incumbent (given by Equation (5)), we have:

$$\pi_{i2} = \frac{(s + d + 2dM_3)^2}{18d}$$

For further analysis of the equilibrium with market structure A, C, D, we use the second period prices from equations (26), (27) and (28) and the indifference condition between segment C and D given by Equation 25. We solve these equations to obtain values of p_e, p_u, p_d and M_3 purely in terms of first period price p_1.

$$p_u = \frac{f + f_u - p_1}{2}$$

$$p_d = f + f_u - s - d - p_1$$

$$M_3 = \frac{3(f + f_u) - 2(s + d) - 3p_1}{4d}$$

$$p_e = \frac{3f + 7f_u - 2s - 6d - 3p_1}{4}$$

The profit function for the incumbent inclusive of periods 1 and 2 is:

$$\pi_i = p_1 \cdot M_3 + p_u \cdot \left(\frac{p_d + s + d - p_u}{2d} \right) \tag{29}$$

where $\pi_{i1} = p_1 M_3$ and $\pi_{i2} = p_u \cdot \left(\frac{p_d+s+d-p_u}{2d}\right)$ (see Equation 4). Note that π_i is concave in p_1. Using the first order condition, we get the equilibrium value of first period price, p_1^*:

$$p_1^* = \frac{2(f + f_u - s - d)}{5}$$

Using p_1^*, we can now completely determine this equilibrium.

$$p_1^* = \frac{2(f + f_u - s - d)}{5}$$

$$p_u^* = \frac{3(f + f_u) + 2(s + d)}{10}$$

$$p_d^* = \frac{3(f + f_u - s - d)}{5}$$

$$p_e^* = \frac{9f + 29f_u - 4s - 24d}{20}$$

$$M_1^* = \frac{3(f + f_u) + 2(s + d)}{20d}$$

$$M_3^* = \frac{9(f + f_u) - 4(s + d)}{20d} \tag{30}$$

Now, we check for the various constraints. These checks will yield any potential conditions on the parameter space.

1. For $p_d^* < p_e^*$:

$$s > \frac{3f - 17f_u + 12d}{8} \tag{31}$$

2. For $M_1^* < M_3^*$:

$$s < f + f_u - d \tag{32}$$

3. For $M_3^* < 1$:

$$s > \frac{9(f + f_u) - 24d}{4} \tag{33}$$

4. One condition for market structure ACD to be preserved requires that the customer indifferent between segments A and C in a competitive equilibrium (M_1^* in this case) has positive second period surplus when compared with her reservation utility. Thus, customer M_1^* receives positive surplus if:

$$(f_u - M_1^* \cdot d - p_u) - (f - M_1^* \cdot d) \geq 0$$

This simplifies to:

$$p_u^* \leq f_u - f$$

Substituting the value of p_u^*, we get:

$$s < \frac{7f_u - 13f - 2d}{2} \tag{34}$$

5. We also need to check whether segment C customers receive positive surplus over the two periods. This requires substitution of profit maximizing prices in the expression $f + f_u - d - p_1^* - p_d^* - s$ and we find this to be true.

Putting together the conditions in inequalities (31) through (34):

$$\max\left\{\frac{3f - 17f_u + 12d}{8}, \frac{9(f + f_u) - 24d}{4}\right\} < s < \min\left\{f + f_u - d, \frac{7f_u - 13f - 2d}{2}\right\}$$

For this range to exist, we need the following condition on f_u:

$$\max\left\{\frac{4d - f}{5}, \frac{4d + 11f}{9}, 7f - 4d\right\} < f_u < 4d - f$$

For the above range to exist, we need:

$$f < d$$

This completes the necessary parameter set required to implement the competitive upgrade discount pricing equilibrium.

Proof of Proposition 3

We consider one-by-one the incentives for the entrant and the incumbent to deviate from their ACD equilibrium prices in the second period while holding the prices of the other player fixed at their equilibrium values.

Entrant's Unilateral Deviation The entrant sets price p_d^* for the incumbent's first period customers and p_e^* for the non-first period customers. Hence, it can deviate by changing either p_d^* or p_e^* in the second period.

First consider p_e^*. Suppose that the entrant deviates by changing this price to p_e^D. It cannot be that $p_e^D < p_e^*$ since the entrant already covers the non-first period customer segment and there are no new customers to be gained in this segment by lowering prices. Hence, the entrant profits will reduce if p_e^D is set lower than p_e^*. Now, suppose that $p_e^D > p_e^*$. Note that increasing the prices in the non-first period customer segment will not effect its profits from the first period customers. The index of the non-first period customer indifferent between buying and not buying the entrant's product is obtained by solving $f_u - (1-t)d - p_e^D = 0$ for t. This gives us $t = \frac{d - f_u + p_e^D}{d}$, which we refer to as t^D. In writing the indifference condition we used the fact (from Theorem 1) that the incumbent's price for the non-first period customers is set so high so that no customer gets a positive surplus by buying its product. Hence the non-first period customers must choose between buying from the entrant or not buying at all. Also note that since $p_e^D > p_e^*$, it must be that $t^D > M_3^*$. The consequent profits of the entrant from the non-first period customer segment is then $\pi_e^D = p_e^D\left(1 - \frac{d - f_u + p_e^D}{d}\right)$. Maximizing these profits with respect to p_e^D yields $p_e^{D*} = \frac{f_u}{2}$, i.e., the monopoly prices and consequent profits from the non-customer segment. The necessary and sufficient condition for the entrant to not deviate from p_e^* is therefore $p_e^{D*} < p_e^*$. This comparison yields the parameter condition:

$$s < \frac{19f_u + 9f - 24d}{4} \tag{35}$$

Next consider the entrant's deviation from p_d^* in the first period customer segment. From proof of Proposition **??** note that the entrant's prices are the best response to the incumbent's price p_u^* and M_3^*. Hence, the entrant will not deviate.

Incumbent's Unilateral Deviation The incumbent may deviate by either changing its price p_u^* for the first period customers, or by changing its price p_i^* for the non-first period customers.

First consider p_u^*. Like in the case of the entrant, this price is the best response to the entrant's price p_d^* and M_3^*. Hence the incumbent will not deviate by changing this price.

Next consider p_i^*. From Theorem 1, note that the incumbent's price for the non-first period customers is so high that the marginal customer with index M_3^* gets a zero surplus from buying the incumbent's product. Hence, if the incumbent deviates, it must do so by lowering its prices and consequently acquiring some customers from this segment. Suppose the incumbent deviates by setting a price $p_i^D < p_i^*$. The index of the customer indifferent between the incumbent and the entrant's product is obtained by solving $f_u - td - p_i^D = f_u - (1-t)d - p_e^*$ for t, which gives $t = \frac{d+p_e^*-p_i^D}{2d}$. The consequent second period profits of the incumbent from the non-first period customer segment are $\Pi_N^D = p_i^D(\frac{d+p_e^*-p_i^D}{2d} - M_3^*)$. Taking the first order condition of Π_N^D with respect to p_i^D and solving we obtain $p_i^L = \frac{4d-9f+11f_u+4s}{40}$. Note that p_i^L is the lowest price to which the incumbent will deviate to. However, note that $p_u^* > p_i^L$. Hence the pricing constraint is violated and the entrant must reduce its price from p_u^* to p_i^L in the first period customer segment implying a reduction in profit from that segment (since p_u^* is the optimal price for that segment). Consequently, the incumbent may not find it profit maximizing to set its price while deviating to be as low as p_i^L. The maximum price p_i^H at which the incumbent acquires any non-first period customers (and a positive profit from this segment) is given by solving $f_u - M_3^* d - p_i^H = 0$ for p_i^H, which gives $p_i^H = \frac{4d-9f+11f_u+4s}{20}$. Note that if $p_i^H > p_u^*$, then the incumbent can set $p_i^D = p_u^*$ and make positive profits from the non-first period customer segment while not impacting profits from the first period customer segment. Clearly, if this happens, the incumbent will deviate. A necessary condition for the incumbent to not deviate is therefore obtained by $p_i^H < p_u^*$. This results in the parameter condition:

$$f_u < 3f \tag{36}$$

Note, however, that the above parameter condition is not sufficient since the incumbent may deviate by setting a price $p_i^L \le p_i^D \le p_i^H$ even when $p_i^H < p_u^*$. To obtain sufficient conditions, we must show that the gain in profit from the non-first period customer segment by deviating is more than offset by the loss in profit in the first period customer segment. Now, the profit from the first period customers without deviating is $\pi_{i2} = p_u^* M_1^*$. If the incumbent deviates by setting price p_i^D, the price p_u^* must be reduced to p_i^D to meet the pricing constraint. Hence the index of the indifferent customer is obtained by solving $f_u - td - p_i^D = f_u - (1-t)d - s - p_d^*$ for t, which gives $t = \frac{d+p_d^*-p_i^D+s}{2d}$. The profits on deviating from the first period customers is $\Pi_F^D = p_i^D \frac{d+p_d^*-p_i^D+s}{2d}$. Thus the loss in profit from the first period customers is $\pi_{i2} - \Pi_F^D$ and the gain in profit from the non-first period customers is Π_N^D due to deviation from p_i^* to p_i^D. The sufficient conditions for the incumbent to not deviate can therefore be obtained from $\pi_{i2} - \Pi_F^D - \Pi_N^D > 0$.

We now evaluate the expression $\Delta = \pi_{i2} - \Pi_F^D - \Pi_N^D$. In particular, we derive necessary and sufficient conditions for this expression to be positive. First note that Δ is a convex function of p_i^D since $\frac{d^2\Delta}{dp_i^{D2}} = \frac{2}{d} > 0$. Next, we evaluate the value of Δ at the two end-points of the range $[p_i^L, p_i^H]$:

$$\Delta\,|_{p_i^D = p_i^L} = \frac{(3f - f_u)^2}{32d} > 0$$

$$\Delta\left.\right|_{p_i^D=p_i^H}=\frac{(3f-f_u)(2d+3f+3f_u+2s)}{80d}>0$$

The above expression is positive because the condition $f_u<3f$ is a necessary condition for $p_i^H<p_u^*$. Thus, the function Δ is convex and positive at the two end-points of the interval. The function Δ is minimized at $p_i^D(opt)$ which is obtained by setting the first derivative of Δ with respect to p_i^D equal to zero:

$$p_i^D(opt)=\frac{12d+3f+23f_u+12s}{80}$$

If $p_i^D(opt)$ is less than p_i^L or greater than p_i^H, then the convexity of Δ ensures that Δ is positive over the entire interval $[p_i^L,p_i^H]$. To evaluate this, we calculate the following:

$$p_i^L-p_i^D(opt)=-\frac{4d+21f+f_u+4s}{80}<0$$

$$p_i^D(opt)-p_i^H=\frac{-4d+39f-21f_u-4s}{80}$$

If $s>\frac{39f-21f_u-4d}{4}$, then $p_i^D(opt)>p_i^H$ and hence the incumbent has no incentive to deviate. Thus, a sufficient parameter condition for no deviation is:

$$s<\frac{39f-21f_u-4d}{4} \tag{37}$$

Adding conditions given by (35), (36) and (37) to the necessary condition set given in Proposition (2), we get **Condition Set 1** for the existence of a subgame perfect equilibrium with the ACD market structure involving competitive upgrade pricing:

$$0.7d<f<0.77d$$

$$\max\left\{\frac{11f+4d}{9},7f-4d,\frac{12d-3f}{11}\right\}<f_u<\min\left\{\frac{3f+2d}{3},\frac{15f-4d}{5}\right\}$$

$$\max\left\{\begin{array}{c}\frac{3f-17f_u+12d}{8},\\ \frac{9(f+f_u)-24d}{4}\end{array}\right\}<s<\min\left\{\begin{array}{c}\frac{7f_u-13f-2d}{2},\\ \frac{19f_u+9f-24d}{4},\\ \frac{39f-21f_u-4d}{2}\end{array}\right\}$$

Another condition set may also result in the required equilibrium. This is possible when:

$$s\geq\frac{39f-21f_u-4d}{4} \tag{38}$$

If $s\geq\frac{39f-21f_u-4d}{4}$, then $p_i^D(opt)\in[p_i^L,p_i^H]$ and we need to ensure that the value of Δ at $p_i^D(opt)$ is positive. Let $K=\Delta\left.\right|_{p_i^D=p_i^D(opt)}$. We evaluate the derivative of K with respect to s:

$$\frac{dK}{ds}=\frac{-4d+39f-21f_u-4s}{800d}$$

Since $s \geq \frac{39f-21f_u-4d}{4}$, we must have $\frac{dK}{ds} < 0$. Consequently, K is minimized by setting s to its upper limit in the parameter conditions given in Proposition 2. For those parameter conditions, we know that:

$$s < \min\left\{f + f_u - d, \frac{7f_u - 13f - 2d}{2}\right\}$$

However, $\frac{7f_u-13f-2d}{2} < f + f_u - d$ for the condition given by (36) and using condition given by (35), we get:

$$s < \min\left\{\frac{7f_u - 13f - 2d}{2}, \frac{19f_u + 9f - 24d}{4}\right\}$$

We consider two cases.

Case 1: $\frac{19f_u+9f-24d}{4} < \frac{7f_u-13f-2d}{2}$

This requires:

$$f_u < 4d - 7f \tag{39}$$

To minimize K, we evaluate K at the highest possible value of $s = \frac{19f_u+9f-24d}{4}$:

$$K\,|_{s=\frac{19f_u+9f-24d}{4}} = \frac{-4d^2 + 9f^2 + 12f \cdot f_u - 14f_u^2 + 4d\,(-3f + 4f_u)}{64d}$$

First, observe that $K\,|_{s=\frac{19f_u+9f-24d}{4}}$ is concave in f_u. Consequently, $K\,|_{s=\frac{19f_u+9f-24d}{4}} > 0$ when:

$$\frac{\left(8+2\sqrt{2}\right)d + \left(6-9\sqrt{2}\right)f}{14} < f_u < \frac{\left(8-2\sqrt{2}\right)d + \left(6+9\sqrt{2}\right)f}{14} \tag{40}$$

Putting together parameter conditions in Proposition 2 and those given by (35), (36), (38), (39) and (40), we find that there are no parameters d, f, f_u and s that satisfy them simultaneously. Consequently, a subgame perfect equilibrium with ACD market structure and competitive upgrade pricing does not exist for this case.

Case 2: $\frac{19f_u+9f-24d}{4} \geq \frac{7f_u-13f-2d}{2}$

This requires:

$$f_u \geq 4d - 7f \tag{41}$$

To minimize K, we evaluate K at the highest possible value of $s = \frac{7f_u-13f-2d}{2}$:

$$K\,|_{s=\frac{7f_u-13f-2d}{2}} = \frac{134f \cdot f_u - 97f^2 - 41f_u^2}{256d}$$

First, observe that $K\,|_{s=\frac{7f_u-13f-2d}{2}}$ is concave in f_u. Consequently, $K\,|_{s=\frac{7f_u-13f-2d}{2}} > 0$ when:

$$1.08f < f_u < 2.19f \tag{42}$$

Putting together parameter conditions in Proposition 2 and those given by (35), (36), (38), (41) and (42), we get **Condition Set 2** for the existence of a subgame perfect equilibrium with the ACD market structure involving competitive upgrade pricing:

$$0.46d < f < 0.83d$$

$$\max\left\{\frac{11f+4d}{9}, 7f-4d, 4d-7f, \frac{12d-3f}{11}, \frac{13f}{7}, \frac{3f+2d}{4}\right\} < f_u < 2.19f$$

$$\max\left\{\frac{3f-17f_u+12d}{8}, \frac{9\left(f+f_u\right)-24d}{4}, \frac{39f-21f_u-4d}{2}\right\} < s < \frac{7f_u-13f-2d}{2}$$

Entrant and Incumbent Deviate Simultaneously Finally, it is also possible that both the incumbent and the entrant may *simultaneously* set prices in the second period that are different from the ACD market structure prices given the cut-off M_3^* between the first period and non-first period customers. However, under the parameter conditions that ensure unilateral deviation, simultaneous deviation in prices is also ruled out. This is because the incumbent's and the entrant's prices in the first period customer segment are their best response prices. Further, the entrant enjoys monopoly profits in the non-first period customer segment (see proof for Entrant's Unilateral Deviation part). Hence, it has no incentives to deviate from these prices. Clearly, simultaneous deviation in prices by both firms is ruled out.

Proof of Proposition 4

The expression for $\pi_i^*\left(A, C, D\right)$ is easily obtainable from Equation (7). Differentiating this expression with respect to s gives us:

$$\frac{d\pi_i^*}{ds} = -\left(\frac{f+f_u-s-d}{5d}\right)$$

$\frac{d\pi_i^*}{ds} < 0$ given the parameter space for the competitive upgrade discount equilibrium. Hence the incumbent's profit is decreasing.

Proof of Proposition 5

The expression for $\pi_e^*\left(A, C, D\right)$ is easily obtainable from Equation (8). Differentiating this expression with respect to s gives us:

$$\frac{d\pi_e^*}{ds} = \frac{-6d-9f+f_u+14s}{50d}$$

We show that the entrant's profit is decreasing in s by contradiction. Suppose that the entrant's profit is non-decreasing in s in the competitive upgrade discount equilibrium. This implies that $\frac{d\pi_e^*}{ds} \geq 0$ and this requires:

$$s \geq \frac{9f+6d-f_u}{14}$$

Combining this condition with either condition set 1 or condition set 2 required for the competitive upgrade discount equilibrium reveals that there exist no parameter values for d, f, f_u and s such that these conditions are satisfied. Thus, the entrant's profit function in the competitive upgrade discount equilibrium can never be non-decreasing in s. Hence, it must be increasing in s in this equilibrium.

Proof of Proposition 6

The expressions for consumer surplus for each of the three segments A, C and D are different. The overall consumer surplus is written taking into account these expressions and the cutoffs for the segments, M_1^* and M_3^*, as in proof of Proposition 2. This is:

$$CS = \int_0^{M_1^*} (f + f_u - 2td - p_1^* - p_u^*)\, dt + \int_{M_1^*}^{M_3^*} (f + f_u - s - d - p_1^* - p_d^*)\, dt + \int_{M_3^*}^{1} (f_u - (1-t)\, d - p_e^*)\, dt$$

Substituting for prices and market shares as in Proposition **??**, we get:

$$CS = \frac{584d^2 + 8\,(26s - 51\,(f + f_u))\, d + 99\,(f + f_u)^2 + 24s^2 - 48\,(f + f_u)\, s}{800d}$$

The second derivative of this expression with respect to s is $\frac{3}{50d} > 0$ implying that CS is convex in s. CS attains a minimum value as per the first order condition at $s = f + f_u - \frac{13d}{3}$. To show that CS is increasing in s given either of the two condition sets for a competitive upgrade discount equilibrium, we need to show that the condition set implies $s > f + f_u - \frac{13d}{3}$ and given the convexity of CS in s further implies that CS is increasing in s. The condition $s > f + f_u - \frac{13d}{3}$ always holds for condition set 1 or condition 2 if:

$$f + f_u - \frac{13d}{3} < \frac{9\,(f + f_u) - 24d}{4}$$

This simplifies to:

$$f_u > \frac{4d - 3f}{3}$$

However, $\frac{4d-3f}{3} < \frac{11f+4d}{9}$ since this only requires $f > \frac{d}{5}$ which is true given both condition sets and since $f_u > \frac{11f+4d}{9}$ in both condition sets, $f_u > \frac{4d-3f}{3}$ and consequently $s > f + f_u - \frac{13d}{3}$ is always true in the competitive upgrade discount pricing equilibrium. Thus, CS must always increase in s in the competitive upgrade discount pricing equilibrium.

Derivation and market shares for Section 6.1 (Discount factor extension)

First period options for the customer with the discount factor are given in the main body of the paper in the extension section. For the market structure ABCD to be plausible, we need the coefficient of t in option 4 (segment B) to be less than the coefficient of t in option 3 (segment C). This provides the following condition:

$$-\delta < -1 + \delta$$

This simplifies to:

$$\delta > \frac{1}{2}$$

This condition is always satisfied in the original problem without a discount factor since $\delta = 1$ in that case.

The second period pricing constraints with a discount factor are the same as in the original problem since decision making in the second period once the first period has elapsed does not require use of the discount factor. Using the first period indifference points with the second period pricing constraints (assuming $\delta > \frac{1}{2}$) provides the same ABCD non-existence result as in Lemma 2. Applying assumptions 1 and 2 along with the result in Lemma 2 implies a result similar to Proposition 1 that the only possible market structure with competitive upgrade discount pricing is ACD. The second period profit functions for the incumbent and entrant are the same as in equations (5) and (6). Consequently, second period prices p_u and p_d and the incumbent's second period profit are the same as in the original problem:

$$p_u = \frac{d(2M_3 + 1) + s}{3}$$
$$p_d = \frac{d(4M_3 - 1) - s}{3}$$
$$\pi_{i2} = \frac{(s + d + 2dM_3)^2}{18d}$$

The first period indifference condition between segments C and D provides an expression for M_3:

$$M_3 = \frac{f - p_1 - \delta p_d + \delta p_e - \delta s}{d}$$

First we assume that p_e is bound by the IC rather than the IR constraint. Substituting for the expressions for p_d and p_e and rearranging to obtain M_3 solely as a function of p_1:

$$M_3 = \frac{3f - 2\delta d - 2\delta s + 3\delta p_i - 3p_1}{(3 - 2\delta)\, d}$$

Thus, the second period profit for the incumbent, π_{i2} is increasing in M_3 and further, M_3 is increasing in p_i. Hence, the incumbent will increase p_i high enough such that IR constraint binds on p_e. Consequently, the first period indifference point M_3 as a function of first period price p_1 is given by:

$$M_3 = \frac{3f + 3\delta f_u - 2\delta d - 2\delta s - 3p_1}{(3 + \delta)\, d}$$

Similarly, expressions for M_1, p_u, p_d and p_e can be computed as a function of first period incumbent price p_1. We use these expressions in the incumbent's profit function for the overall game:

$$\pi_i = p_1 \cdot M_3 + p_u \cdot M_1$$

Using this expression, we can show that π_i is concave in p_1 since:

$$\frac{d^2\pi_i}{dp_1^2} = -\frac{2(7+3\delta)}{d(3+\delta)^2} < 0$$

Thus, the first order condition provides the optimal p_1^*:

$$p_1^* = \frac{5f - 2s - 2d(1+\delta)^2 + \delta(3f - 2s(2+\delta) + f_u(5+3\delta))}{14+6\delta}$$

Based on this price, all other equilibrium prices and market shares can be computed as function of the primitive parameters.

Derivation of prices and market shares for Section 6.2 (Network externality extension)
First period options for the customer with network externality are given in the main body of the paper in the extension section. Similarly, second period pricing constraints with network externality effects are also provided in the same section. As in the original problem, the most general market structure with competitive upgrade discount pricing is ABCD. However, a combination of the first and second period customer conditions provides a result similar to Lemma 2 that the market structure ABCD cannot exist at equilibrium. Applying assumptions 1 and 2 along with the result in Lemma 2 implies a result similar to Proposition 1 that the only possible market structure with competitive upgrade discount pricing is ACD. The second period profit functions for the incumbent and entrant can be written and optimal prices p_u and p_d as a function of first period market cut-off M_3.

$$p_u = \frac{(d - \gamma - k \cdot f_u)(2M_3 + 1) + s}{3}$$
$$p_d = \frac{(d - \gamma - k \cdot f_u)(4M_3 - 1) - s}{3}$$

The incumbent's second period profit as a function of M_3 is given by substituting the above value of p_u:

$$\pi_{i2} = \frac{(s + d - \gamma - k \cdot f_u + 2(d - \gamma - k \cdot f_u) M_3)^2}{18(d - \gamma - k \cdot f_u)}$$

A necessary condition to ensure positive prices is $\gamma < d$ implying that: for the old customers in the second period, network externality drives down differentiation. The first period market cut-off is obtained by using the indifference condition between segments C and D:

$$M_3 = \frac{f - p_1 - p_d + p_e - s}{d - \gamma - k \cdot f}$$

Assuming IC condition binds on p_e and using the expression above for p_d, we rearrange the above equation to obtain M_3 purely as a function of p_1. If $\gamma + \frac{3}{5}k \cdot f + \frac{2}{5}k \cdot f_u < \frac{d}{5}$, then M_3 is increasing in p_i and since π_{i2} is increasing in M_3, the incumbent will raise p_i till the IR constraint binds on p_e. Consequently, the first period indifference point M_3 as a function of first period price p_1 can be obtained. Similarly, expressions for M_1, p_u, p_d and p_e can be computed as a function of first period incumbent price p_1. We use these expressions in the incumbent's profit function for the overall game:

$$\pi_i = p_1 \cdot M_3 + p_u \cdot M_1$$

Using this expression, we can show that π_i is concave in p_1 since:

$$\frac{d^2\pi_i}{dp_1^2} = -\left(\frac{20d - 32\gamma - 18k \cdot f - 14k \cdot f_u}{(2d - 3\gamma)^2}\right)$$

$\frac{d^2\pi_i}{dp_1^2} < 0$ when $\gamma < \frac{5}{8}d - \frac{9}{16}k \cdot f - \frac{7}{16}k \cdot f_u$. Thus, the first order condition provides the optimal p_1^* when:

$$\gamma < \min\left\{\frac{d}{5} - \frac{3}{5}k \cdot f - \frac{2}{5}k \cdot f_u, \frac{5}{8}d - \frac{9}{16}k \cdot f - \frac{7}{16}k \cdot f_u\right\}$$

When $k = 0$, this condition reduces to $\gamma < \frac{d}{5}$. Based on the price p_1^*, all other equilibrium prices and market shares can be computed as function of the primitive parameters. The prices with $k = 0$ is given in the main body of the paper.

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