
Online Supplement for
“Underlying Consumer Heterogeneity in Markets for
Subscription-Based IT Services with Network Effects”

A. Proofs of Main and Supporting Results

Lemma A1. Adoption at the origin. *Let us define $N_F(0^+) \triangleq \lim_{t \downarrow 0} N_F(t)$ and $Q(0^+) \triangleq \lim_{t \downarrow 0} Q(t)$. If $0 < \alpha \leq 1$ and condition (RC) is satisfied, the following results hold:*

- (a) *if $p(0) > \bar{\theta}$, then $N_F(0^+) = N_F(0) = Q(0^+) = Q(0) = 0$, but adoption does not start at $t = 0$;*
- (b) *if $p(0) = \bar{\theta}$, then $N_F(0^+) = N_F(0) = Q(0^+) = Q(0) = 0$ and $\theta(0) = \bar{\theta}$; moreover, $N_F(t)$ and $Q(t)$ are strictly increasing in the immediate vicinity of the origin if and only if $\dot{p}(0) < 0$;*
- (c) *if $p(0) < \bar{\theta}$, then $N_F(0^+) > 0$, $Q(0^+) > 0$, and $\theta(0) < \bar{\theta}$; i.e., there is a jump in both qualified customers and adopters at the origin.*

Proof. In the proof below, note that when $\alpha = 1$ we have $N_F(t) = Q(t)$.

(a), (b). We prove (a) and (b) simultaneously. Suppose that adoption starts at $t = 0$. Around origin, regardless of whether there is a jump in the installed base or not, no customer adopts unless it benefits her. Thus, $N_F(0^+) \leq Q(0^+) \leq m$. Since the price schedule is continuous, we have $p(0) = p(0^+) \geq \bar{\theta}$. From (RC.ii), we have $\underline{\theta} < \bar{\theta} - \nu m \leq p(0^+) - \nu N_F(0^+)$. Therefore, we cannot have instantaneous full adoption at the very beginning. Once the product is introduced to the market, its adoption path follows the laws dictated by equations (5) and (6). At $t > 0$, only a fraction of the qualified people who have not adopted yet will adopt. Thus, it must be the case that the marginal adopter has utility rate 0. Let

$$h(\theta) \triangleq \theta + \nu m \bar{F}(\theta). \quad (\text{A.1})$$

Since $m \bar{F}(\theta(0^+)) = Q(0^+) \geq N_F(0^+)$ we obtain:

$$\theta(0^+) = p(0^+) - \nu N_F(0^+) \geq \bar{\theta} - \nu m \bar{F}(\theta(0^+)), \quad (\text{A.2})$$

which can be rewritten as $h(\theta(0^+)) \geq \bar{\theta} = h(\bar{\theta})$, with equality being possible solely in the case when $p(0) = \bar{\theta}$. From (RC.ii) we see that $\theta(0^+) = \bar{\theta}$ can be the only solution, and that solution occurs solely if $h(\theta(0^+)) = \bar{\theta}$. Therefore, if $p(0) > \bar{\theta}$, adoption does not start and $Q(0) = Q(0^+) = N_F(0) = N_F(0^+) = 0$. On the other hand, if $p(0) = \bar{\theta}$, the highest type becomes qualified and adoption starts (in any small interval around the origin there will be a mass of qualified adopters, and a fraction of them shall adopt immediately). Therefore $\theta(0^+)$ is defined. From the above, it then follows that $\theta(0^+) = \bar{\theta}$ and that $Q(0) = Q(0^+) = N_F(0) = N_F(0^+) = 0$. There is no jump that can keep the system stable around the origin.

Let us now look at the case when $p(0) = \bar{\theta}$. Note that, in continuous time, $\theta(0) = \bar{\theta}$ means that there are basically a negligible number of qualified customers in the beginning. Both the installed

base and the number of qualified customers remain at negligible levels as long as the price equals $\bar{\theta}$ level. Suppose that $p(t) = \bar{\theta}$ for all $t \in [0, \epsilon]$, with $\epsilon > 0$. Pick $\tilde{t} \in [0, \epsilon]$. Then, similar to the above approach in equation (A.2), we can prove that $h(\theta(\tilde{t})) \geq \bar{\theta}$. Thus, $\theta(\tilde{t}) = \bar{\theta}$ and $Q(\tilde{t}) = N_F(\tilde{t}) = 0$. This argument can be used for the reverse direction of the proof as well.

(c) If $p(0) < \bar{\theta}$, then all customers with type $\theta \geq \max\{p(0), \underline{\theta}\}$ adopt even in the absence of network effects. Therefore $\theta(0^+) \leq \max\{p(0), \underline{\theta}\} < \bar{\theta}$. Then, given condition (RC), $Q(0^+) = m[1 - F(\theta(0^+))] > m[1 - F(\bar{\theta})] = 0$. As soon as the product is introduced on the market, there is an instantaneous mass of qualified customers. Among them, a positive fraction adopt immediately. Therefore $Q(0^+) > 0$ and $N_F(0^+) > 0$ and $\theta(0^+) < \bar{\theta}$. \square

Lemma A2. Full adoption when $\alpha = 1$. *Assume F satisfies (RC), customers exhibit full product awareness ($\alpha = 1$), and $\dot{p}(\cdot) < 0$. Then, the following hold:*

- (a) *Full adoption occurs if and only if $h(\underline{\theta}) \geq p(\infty)$;*
- (b) *Full adoption occurs in finite time if and only if $h(\underline{\theta}) > p(\infty)$.*

Proof. (a) (\Rightarrow) Trivial, since the lowest type must adopt.

(\Leftarrow) Suppose that $\underline{\theta} + \nu m = h(\underline{\theta}) \geq p(\infty)$, but we only have partial adoption, stopping at $\theta(\infty) > \underline{\theta}$. When $\alpha = 1$, from condition (RC.i) it immediately follows that h is strictly increasing. Consequently, $h(\theta(\infty)) > h(\underline{\theta})$. Therefore, we have $h(\theta(\infty)) > p(\infty)$. Using (RC.i) and the fact that $\int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\theta = 1$, it follows that $\bar{\theta} > \underline{\theta} + \nu m$. Therefore, $\bar{\theta} > p(\infty)$. Since $p(t)$ is continuous, there exists $0 < \bar{t} < \infty$ such that adoption starts at \bar{t} (even if $\dot{p}(0) = 0$). Since full adoption does not occur for $t \geq \bar{t}$, we have $h(\theta(t)) = p(t)$. Taking the limit $t \rightarrow \infty$, we obtain $h(\theta(\infty)) = p(\infty)$, which is a contradiction.

(b) Proof is similar to that of part (a). \square

Proof of Theorem 1. (a) *Existence.* We first propose a construction for F and then prove that it satisfies all the desired properties, including uniqueness. Let the distribution F on $[\underline{\theta}, \bar{\theta}]$ be defined as

$$F(\theta) = 1 - G(\sigma^{-1}(\theta)), \quad (\text{A.3})$$

where $\sigma(t) = p(t) - \nu m G(t)$. Since $\dot{p}(t) < 0$, $g(t) = \dot{G}(t) > 0$, and $\nu > 0$, we have $\sigma(t)$ strictly decreasing in t (and therefore invertible as well). Moreover, $G(t)$ increases from 0 to 1, $p(0) = \bar{\theta}$ and $p(\infty) = \underline{\theta} + \nu m$. Thus, $\sigma(t)$ completely spans the interval $(\underline{\theta}, \bar{\theta}]$. Furthermore, since σ^{-1} is a strictly decreasing bijection mapping $(\underline{\theta}, \bar{\theta}]$ to $[0, \infty)$, F is a properly defined cumulative distribution function. Moreover, F is strictly increasing. Next, substituting θ by $\sigma(t)$ in (A.3), we obtain

$$\bar{F}(\sigma(t)) = G(t). \quad (\text{A.4})$$

Differentiating the above equation with respect to time t , we obtain $f(\sigma(t))\dot{\sigma}(t) = -g(t)$. From the definition of function σ , we get $\dot{\sigma}(t) = \dot{p}(t) - \nu m g(t) < 0$. Therefore:

$$f(\sigma(t)) = -\frac{g(t)}{\dot{p}(t) - \nu m g(t)} > 0. \quad (\text{A.5})$$

Using (A.5), it follows that for all t , $f(\sigma(t)) < \frac{1}{\nu m}$. Since σ is bijective, we have $f(\theta) < \frac{1}{\nu m}$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. Thus, condition (RC.i) is satisfied. Note that when $\alpha = 1$, condition (RC.i) implies condition (RC.ii). Hence (RC) is satisfied. Using Lemma A2, we see that full adoption occurs at infinity, and at every finite moment in time we have $p(t) = \theta(t) + \nu m \bar{F}(\theta(t)) = h(\theta(t))$. Since we defined $\sigma(t) = p(t) - \nu m G(t) = p(t) - \nu m \bar{F}(\sigma(t))$ we have $p(t) = h(\sigma(t))$. Since (RC.i) holds, h is strictly increasing and, thus, $\sigma(t) = \theta(t)$. Consequently $N_F(t) = N(t) = mG(t)$ for any $t \geq 0$, and F yields the observed adoption path as G .

Uniqueness. Assume that F_1 and F_2 satisfy the requirements and lead to equivalence. We have $1 - F_1(\theta_1(t)) = 1 - F_2(\theta_2(t)) = G(t)$. Since $f_1(\theta) > 0$ and $f_2(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, $\theta_1(t)$ and $\theta_2(t)$ are uniquely defined. Given that F_1 and F_2 are continuously increasing, $G(0) = 0$ and $G(\infty) = 1$, it follows that $\theta_1(t)$ and $\theta_2(t)$ are bijections between $[0, \infty)$ and $(\underline{\theta}, \bar{\theta}]$. Moreover, since full adoption occurs at infinity, $\theta_1(t) = p(t) - \nu m \bar{F}_1(\theta_1(t)) = p(t) - \nu m G(t) = p(t) - \nu m \bar{F}_2(\theta_2(t)) = \theta_2(t)$. Also, $\bar{F}(\theta_1(t)) = \bar{F}(\theta_2(t))$. Thus, $F_1 \equiv F_2$.

(b) Note that the conditions are necessary in order for adoption to start smoothly at 0, continue in a strictly increasing manner, and end with full adoption. The condition $p(0) = \bar{\theta}$ indicates that adoption starts at 0. Moreover, since h is increasing, there is no jump at 0. In addition, since price is continuously decreasing, adoption never stalls. Also from Lemma A2, the lowest type must adopt at infinity. If any of the conditions are violated, no distribution function satisfying (RC) can generate the observed adoption path G . \square

Lemma A3. *If $0 < \alpha < 1$, $p \in \mathcal{P}$, and condition (RC) is satisfied, the following two relationships hold for $t \geq 0$:*

$$(a) \quad Q(t) - N_F(t) = (1 - \alpha)e^{-\alpha t} \int_0^t e^{\alpha z} \dot{Q}(z) dz, \quad (\text{A.6})$$

$$(b) \quad Q(t) = \frac{e^{-t}}{\alpha} \int_0^t e^z [\dot{N}_F(z) + \alpha N_F(z)] dz. \quad (\text{A.7})$$

Proof. (a) We can rewrite equation (6) as

$$[\dot{Q}(t) - \dot{N}_F(t)] + \alpha[Q(t) - N_F(t)] = (1 - \alpha)\dot{Q}(t).$$

Multiplying both sides by the integration factor $e^{\alpha t}$, we have

$$\frac{\partial}{\partial t} \{e^{\alpha t} [Q(t) - N_F(t)]\} = e^{\alpha t} (1 - \alpha) \dot{Q}(t).$$

Since $\dot{Q}(t)$ can only be discontinuous at $t_M \triangleq \inf\{t | Q(t) = m\}$, both sides are integrable. Integrating with respect to t , we obtain $e^{\alpha t} [Q(t) - N_F(t)] = (1 - \alpha) \int_0^t e^{\alpha z} \dot{Q}(z) dz + K_1$, where K_1 is a constant.

Using the boundary condition at $t = 0$, $Q(0) = N_F(0) = 0$ (based on Lemma A1), we have $K_1 = 0$ and the result (A.6) follows immediately.

(b) We can rewrite equation (6) as $\frac{1}{\alpha}[\dot{N}_F(t) + \alpha N_F(t)] = \dot{Q}(t) + Q(t)$. Multiplying both sides by the integration factor e^t , we have $\frac{e^t}{\alpha}[\dot{N}_F(t) + \alpha N_F(t)] = \frac{\partial}{\partial t}[e^t Q(t)]$. By integration and use of the boundary conditions, we obtain the desired equation (A.7).

For both parts (a) and (b), note that the equalities, (A.6) and (A.7), hold even when $t_M < \infty$. First, the above approach guarantees that (A.6) and (A.7) hold for $t < t_M$. When $t \geq t_M$, $Q(t) = m$ and $\dot{N}_F(t) = \alpha[m - N_F(t)]$. Solving this differential equation, we obtain $N_F(t) = m - K_2 e^{-\alpha t}$ for all $t \geq t_M$, where K_2 is a constant. Then

$$(1 - \alpha)e^{-\alpha t} \int_0^t e^{\alpha z} \dot{Q}(z) dz = e^{-\alpha(t-t_M)} [m - N_F(t_M)] = K_2 e^{-\alpha t} = m - N_F(t).$$

Thus, (a) holds. Part (b) follows immediately by writing $\dot{N}(t) + \alpha N(t) = \alpha m$ for all $t > t_M$, as specified by equation (6). \square

Lemma A4. Define $N_F(\infty) \triangleq \lim_{t \rightarrow \infty} N_F(t)$ and $Q(\infty) \triangleq \lim_{t \rightarrow \infty} Q(t)$. If condition (RC) is satisfied, $p \in \mathcal{P}$, and $0 < \alpha < 1$, then,

(a) $N_F(t) < m$, and $N_F(t)$ is strictly increasing, for all $0 \leq t < \infty$; i.e. full adoption occurs only at infinity, if ever;

(b) $\theta(t)$ is strictly decreasing and $Q(t)$ is strictly increasing for $\theta(t) > \underline{\theta}$;

(c) $Q(\infty) = N_F(\infty)$; i.e., all qualified customers ultimately adopt;

(d) for full asymptotic adoption ($N_F(\infty) = Q(\infty) = m$) to occur, it is necessary that $h(\underline{\theta}) \geq p(\infty)$;

(e) if $h(\theta) > h(\underline{\theta}) \geq p(\infty)$ for all $\theta > \underline{\theta}$, then F yields full asymptotic adoption.

Proof. (a,b). We prove (a) and (b) simultaneously. We first look at time $t = 0$. Since $\theta(0) = p(0) = \bar{\theta}$ (Lemma A1), due to right continuity at $t = 0$, we have:

$$\begin{aligned} \dot{\theta}(0) &= \dot{p}(0) - \nu \dot{N}_F(0), \\ \dot{N}_F(0) &= \alpha \dot{Q}(0), \\ \dot{Q}(0) &= -mf(\bar{\theta})\dot{\theta}(0). \end{aligned}$$

Therefore, we have $\dot{N}_F(0) = -\frac{\alpha mf(\bar{\theta})\dot{p}(0)}{1 - \nu m \alpha f(\bar{\theta})}$. Using conditions (RC.i) and the fact that $\dot{p}(0) < 0$, it follows that $\dot{N}_F(0) > 0$. Then $\dot{\theta}(0) < 0$ and $\dot{Q}(0) > \dot{N}(0) > 0$.

For the monotonicity properties at $t > 0$, note that $\dot{N}_F(t)$ depends on $Q(t) - N_F(t)$ and $\dot{Q}(t)$ from (6). Condition (RC.ii) ensures that not all customers are qualified at the origin, and $Q(\cdot)$ and $\dot{Q}(\cdot)$ are continuous in the immediate vicinity of 0. Given that $\dot{Q}(0) > 0$, there exists $\epsilon > 0$ such that $\dot{Q}(t) > 0$ for $t \in [0, \epsilon)$. Moreover, the installed base does not shrink, since the price is weakly decreasing and existing customers continue to subscribe to the service. Consequently, $\theta(t)$ is weakly decreasing, $Q(t)$ is weakly increasing, and thus $\dot{Q}(t) \geq 0$ for $t > 0$.

For the strict monotonicity, first, note that $Q(t) - N_F(t) > 0$ for $0 < t < \infty$, since $\dot{Q}(t) > 0$ in the vicinity of the origin, and from equation (A.6). Then, it follows that N_F is strictly increasing from (6). Next, from $\theta(t) = \max\{\underline{\theta}, p(t) - \nu N_F(t)\}$, $\theta(t)$ is strictly decreasing for $\theta(t) > \underline{\theta}$. Consequently, from equation (5), $Q(t)$ is strictly increasing for $\theta(t) > \underline{\theta}$.

(c) Since both $Q(t)$ and $N_F(t)$ are continuous, increasing, and bounded from above by m , it follows that $\lim_{t \rightarrow \infty} \dot{Q}(t) = \lim_{t \rightarrow \infty} \dot{N}_F(t) = 0$. Therefore, from equation (6), note that $\lim_{t \rightarrow \infty} [Q(t) - N_F(t)] = 0$. Thus $Q(\infty) = N_F(\infty)$.

(d) It follows immediately since the lowest type must adopt as well.

(e) Suppose that we only have partial adoption. From part (c), note that $\theta(\infty) > \underline{\theta}$ (otherwise $Q(\infty) = m = N_F(\infty)$). Then, at any time t , we have $Q(t) < m$, $N_F(t) < m$, and $\theta(t) = p(t) - \nu N_F(t)$. Taking the limit $t \rightarrow \infty$ and using part (c), we obtain

$$\theta(\infty) = p(\infty) - \nu N_F(\infty) = p(\infty) - \nu Q(\infty) = p(\infty) - \nu m \bar{F}(\theta(\infty)).$$

Therefore, $h(\theta(\infty)) = p(\infty) \leq h(\underline{\theta})$, which contradicts the fact that $h(\theta) > h(\underline{\theta})$ for all $\theta > \underline{\theta}$. Thus, it follows that $\theta(\infty) = \underline{\theta}$ and $Q(\infty) = m$. Using part (c), we obtain $N_F(\infty) = m$. \square

Lemma A5. *If condition (RC) is satisfied, $p \in \mathcal{P}$, and $0 < \alpha < 1$, the following properties hold:*

- (a) *in order for all customers to become qualified in finite time (i.e. there exists $t_M < \infty$ such that $\theta(t_M) = \underline{\theta}$ and $Q(t_M) = m$), it is necessary that $h(\underline{\theta}) > p(\infty)$;*
- (b) *if $h(\theta) > h(\underline{\theta}) > p(\infty)$ for all $\theta > \underline{\theta}$, then all customers become qualified in finite time;*
- (c) *if $Q(\infty) = m$, and $h(\underline{\theta}) > p(\infty)$, then all customers become qualified in finite time.*

Proof. (a) Since $Q(t_M) = m$, we have $N_F(\infty) = m > N_F(t_M)$ from parts (a) and (c) of Lemma A4. At time $t_M < \infty$, we have $\underline{\theta} \geq p(t_M) - \nu N_F(t_M) > p(\infty) - \nu m$. Thus, $h(\underline{\theta}) > p(\infty)$.

(b) From part (e) of Lemma A4, it follows that $Q(\infty) = N_F(\infty) = m$ and $\theta(\infty) = \underline{\theta}$. Suppose that it takes infinite time for all customers to become qualified. That means $Q(t) < m$ and $\theta(t) > \underline{\theta}$ for all $t < \infty$. We then have $\theta(t) = p(t) - \nu N_F(t)$ for all $t < \infty$. Taking limit $t \rightarrow \infty$, we obtain $\underline{\theta} = \theta(\infty) = p(\infty) - \nu m$, or $h(\underline{\theta}) = p(\infty)$, which is a contradiction. Therefore, there exists $t_M < \infty$ such that $Q(t_M) = m$ and $\theta(t_M) = \underline{\theta}$.

(c) The proof is similar to part (b). \square

Lemma A6. *When $p \in \mathcal{P}$ and $0 < \alpha < 1$, for a distribution F satisfying regularity conditions (RC) to generate a service adoption path G , the following conditions are necessary:*

- (a) $\underline{\theta} + \nu m \geq p(\infty)$;
- (b) $\int_0^t \frac{e^{z-t} g(z) dz}{g(t)} < \frac{1}{1-\alpha}$, for $t \in [0, t_G]$;
- (c) $\int_0^{t_G} e^{z-t_G} [g(z) + \alpha G(z)] dz = \alpha$; and
- (d) *if $h(\underline{\theta}) > p(\infty)$, there exists a finite time t_c such that $0 \leq t_c \leq t_G$ and $e^{\alpha t} [1 - G(t)]$ is constant over the interval $[t_c, \infty)$.*

Proof. (a) This condition is necessary for full asymptotic adoption since the lowest type customer must adopt (albeit at infinity).

(b) Suppose that there exists F that satisfies (RC) and generates the adoption path G . Then $N_F(t) = N(t) = mG(t)$, and consequently, $\theta(t) = \max\{\underline{\theta}, p(t) - \nu mG(t)\}$. Subsequently, it follows that $\theta(t_G) = \underline{\theta}$ and $\theta(t) > \underline{\theta}$ for all $t \in [0, t_G)$. Since $p(t)$ is weakly decreasing and $G(t)$ is strictly increasing, $\theta(t)$ is strictly decreasing for $t \in [0, t_G]$. Using equations (5) and (A.7), we have

$$F(\theta(t)) = 1 - \frac{e^{-t}}{\alpha} \int_0^t e^z [g(z) + \alpha G(z)] dz, \quad \forall t \in [0, t_G]. \quad (\text{A.8})$$

Even though \dot{Q} may not exist at t_G , $Q(t)$ remains continuous. As a result, F is continuous, and therefore, equation (A.8) is valid for t_G . Differentiating equation (A.8) with respect to time, using integration by parts, and simplifying the expression, we get

$$\frac{\partial[F(\theta(t))]}{\partial t} = \left(\frac{1}{\alpha} - 1\right) e^{-t} \int_0^t e^z g(z) dz - \frac{g(t)}{\alpha}. \quad (\text{A.9})$$

Since F is strictly increasing in θ due to (RC), and $\theta(t)$ is strictly decreasing for $t \in [0, t_G]$, $F(\theta(t))$ should be strictly decreasing in t over $[0, t_G)$: $\frac{\partial[F(\theta(t))]}{\partial t} < 0$ for all $t \in [0, t_G)$. This is equivalent to $\frac{\int_0^t e^{z-t} g(z) dz}{g(t)} < \frac{1}{1-\alpha}$ for $t \in [0, t_G)$.

(c) Since $\theta(t_G) = \underline{\theta}$ from part (b) and $F(\underline{\theta}) = 0$, using equation (A.8), the condition follows.

(d) Note that $\lim_{t \rightarrow \infty} G(t) = 1$ and a distribution F generates an adoption path G , all customers are eventually qualified and adopt, i.e., $Q(\infty) = N_F(\infty) = m$. Subsequently, according to part (c) of Lemma A5, $Q(t)$ converges to m in finite time. From the definition of t_G , it follows that $t_G = \inf\{t | Q(t) = m\} < \infty$. Consequently, from part (a) of Lemma A4, it follows that $N_F(t_G) < m$. Hence, for $t \geq t_G$, $\dot{Q}(t) = 0$, and from equation (6), $N_F(t)$ evolves according to equation:

$$\dot{N}_F(t) = \alpha[m - N_F(t)]. \quad (\text{A.10})$$

Solving this differential equation, we obtain $N_F(t) = m - ke^{-\alpha t}$, for all $t \geq t_G$, where $k = \frac{m - N_F(t_G)}{e^{-\alpha t_G}}$. Since $N_F(t) = mG(t)$, this completes the proof. \square

Lemma A7. *When $0 < \alpha < 1$, the following holds*

$$\lim_{t \rightarrow \infty} e^{-t} \int_0^t e^z [g(z) + \alpha G(z)] dz = \alpha.$$

Proof. Via integration by parts and using $G(0) = 0$, it follows that

$$\int_0^t e^z g(z) dz = e^t G(t) - \int_0^t e^z G(z) dz.$$

Therefore, we obtain

$$e^{-t} \int_0^t e^z [g(z) + \alpha G(z)] dz = G(t) - e^{-t}(1 - \alpha) \int_0^t e^z G(z) dz. \quad (\text{A.11})$$

Since $G(t) \leq 1$, it follows that

$$\lim_{t \rightarrow \infty} e^{-t} \int_0^t e^z G(z) dz \leq 1. \quad (\text{A.12})$$

Since $G(t)$ is strictly increasing and smooth, for any $0 < \epsilon < 1$, there exists t_ϵ such that for all $z > t_\epsilon$ we have $G(z) > 1 - \epsilon$. Therefore, for $t > t_\epsilon$, we obtain

$$e^{-t} \int_0^t e^z G(z) dz \geq e^{-t} \int_{t_\epsilon}^t e^z G(z) dz > e^{-t} \int_{t_\epsilon}^t e^z (1 - \epsilon) dz = (1 - \epsilon) (1 - e^{t_\epsilon - t}). \quad (\text{A.13})$$

Taking limit over $t \rightarrow \infty$ in (A.13), and combining with upper bound (A.12) we obtain

$$1 - \epsilon \leq \lim_{t \rightarrow \infty} e^{-t} \int_0^t e^z G(z) dz \leq 1. \quad (\text{A.14})$$

Further, taking limit over $\epsilon \downarrow 0$ in (A.14), it follows that

$$\lim_{t \rightarrow \infty} e^{-t} \int_0^t e^z G(z) dz = 1. \quad (\text{A.15})$$

Therefore, taking limit over $t \rightarrow \infty$ in (A.11), we obtain

$$\lim_{t \rightarrow \infty} e^{-t} \int_0^t e^z [g(z) + \alpha G(z)] dz = 1 - (1 - \alpha) = \alpha. \quad \square$$

Proof of Theorem 2. First, from Lemma A6, note that condition (i) is necessary for the existence of a distribution F_α that would generate the observed subscription path G . Lemma A7 ensures that condition (c) in Lemma A6 is satisfied since in this case, $t_G = \infty$. Condition (d) in Lemma A6 is not necessary in this case since $p(\infty) = h(\underline{\theta})$.

Existence. Note that since $\sigma(t) = p(t) - \nu m G(t)$, $\sigma(t)$ is a strictly decreasing bijection between $[0, \infty)$ and $[\underline{\theta}, \bar{\theta}]$. Therefore, $\sigma^{-1}(\theta)$ is well-defined. We propose the following construction

$$F(\sigma(t)) = 1 - \frac{1}{\alpha} \int_0^t e^{z-t} [g(z) + \alpha G(z)] dz. \quad (\text{A.16})$$

First, we verify that F is a properly defined distribution function satisfying regularity condition (RC). From $p(\infty) = \underline{\theta} + \nu m$, we have $t_G = \infty$. Since we assume that G is twice continuously differentiable up to t_G , we have $F \in \mathcal{C}^2$. Using $\dot{\sigma}(t) = \dot{p}(t) - \nu m g(t) < 0$ and integration by parts to derive a similar result to equation (A.9) together with condition (i) in Theorem 2 (the same as part (b) of Lemma A6), we obtain

$$0 < f(\sigma(t)) = \frac{\left(\frac{1}{\alpha} - 1\right) e^{-t} \int_0^t e^z g(z) dz - \frac{g(t)}{\alpha}}{\dot{p}(t) - \nu m g(t)}. \quad (\text{A.17})$$

Therefore, $F(\theta)$ is strictly increasing in θ over $[\underline{\theta}, \bar{\theta}]$. Further, $F(\bar{\theta}) = F(\sigma(0)) = 1$. From Lemma A7, it also follows that $F(\underline{\theta}) = F(\sigma(\infty)) = \lim_{t \rightarrow \infty} F(\sigma(t)) = 0$. Thus F is an appropriate distribution. From (A.17), we have $f(\sigma(t)) < \frac{g(t)}{\alpha[\nu m g(t) - \dot{p}(t)]} \leq \frac{1}{\alpha \nu m}$ for $t > 0$. Further, if $t = 0$, then we have

$f(\sigma(0)) = \frac{g(0)}{\alpha[\nu m g(0) - \dot{p}(0)]} < \frac{1}{\alpha \nu m}$ from $\dot{p}(0) < 0$. Thus, condition (RC.i) is satisfied. Using integration by parts, equation (A.16) can be rewritten as

$$\bar{F}(\sigma(t)) = \frac{e^{-t}(1-\alpha)}{\alpha} \int_0^t e^z g(z) dz + G(t).$$

Using condition (ii) in the Theorem and the fact that $\sigma(t) = p(t) - \nu m G(t)$, it follows that $h(\sigma(t)) = \sigma(t) + \nu m \bar{F}(\sigma(t)) < p(0) = \bar{\theta} = h(\bar{\theta})$ for all $t > 0$. Since $\sigma(t)$ is a strictly decreasing bijection between $[0, \infty)$ and $(\underline{\theta}, \bar{\theta}]$, condition (RC.ii) is also satisfied. Thus, condition (RC) is completely satisfied.

Second, we show that $N_F(t) = N(t) = mG(t)$ where F is given by (A.16). Define $\Phi(t) = m(1 - F(\sigma(t)))$. Replacing $F(\sigma(t))$ with (A.16) and differentiating $\Phi(t)$ with respect to time t , we obtain $\dot{\Phi}(t) = \frac{mg(t) + \alpha m G(t)}{\alpha} - \Phi(t)$, which can be rewritten as $\dot{N}(t) = \alpha[\Phi(t) - N(t) + \dot{\Phi}(t)]$. Since $t_G = \infty$, it follows that the lowest type can only adopt at infinity. Consequently, $\theta(t)$ can be rewritten as $\theta(t) = p(t) - \nu N_F(t)$. Therefore, $\{\theta(t), Q(t), N_F(t)\}$ and $\{\sigma(t), \Phi(t), N(t)\}$ satisfy the same differential system, comprised of equations (4), (5), and (6). By substituting (4) into (5), and (5) into (6), N_F satisfies the following ODE:

$$\dot{N}_F(t) = \frac{\alpha[m\bar{F}(p(t) - \nu N_F(t)) - N_F(t) - mf(p(t) - \nu N_F(t))\dot{p}(t)]}{1 - \alpha m \nu f(p(t) - \nu N_F(t))}, \quad (\text{A.18})$$

with the boundary conditions $N_F(0) = 0$ (via Lemma A1). Thus we can write $\dot{N}_F(t) = \Psi(t, N_F(t))$. Since F satisfies condition (RC), we see that $\Psi(t, N_F) \in C^{0,1}$. Thus, Ψ is locally Lipschitz in N_F . This is a sufficient condition for the existence and *uniqueness* of a solution for equation (A.18), for any given boundary condition (e.g. see Amann 1990). Since $\{\theta(t), Q(t), N_F(t)\}$ and $\{\sigma(t), \Phi(t), N(t)\}$ satisfy the same differential system, it follows that $N_F(t) = N(t) = mG(t)$ and $\theta(t) = \sigma(t)$ for all t .

Uniqueness. Note that $\theta(t)$ is unique and it is a bijection. Further, from (A.7), $Q(t)$ is unique. Since $Q(t) = m\bar{F}(t)$, the uniqueness of F follows. \square

Proof of Proposition 1. (a) First, note that t_G is independent of α . Second, since both $1 - \alpha$ and $\frac{1-\alpha}{\alpha}$ are decreasing in α over the interval $(0, 1)$, if conditions (i) and (ii) in Theorem 2 hold for a value $\alpha < 1$, then they hold for all values $\tilde{\alpha} \in (\alpha, 1)$.

(b) From Theorem 2, if $\alpha < 1$ satisfies conditions (i) and (ii), then there exists a unique distribution F_α satisfying (RC) that generates observed adoption path $G(t)$. Moreover, from part (a.1) of this proposition, if conditions (i) and (ii) are satisfied for one value $\alpha < 1$, then they are satisfied for infinitely many values of α . Therefore, we either have an infinite number of pairs $\{\alpha, F_\alpha\}$ generating $G(t)$ and satisfying (RC), or none. Note that Example 1 following Theorem 2 in the main text provides an existence scenario. For no existence scenario, consider $G(t) = 1 - e^{-t}$. In this case, $g(t) = e^{-t}$ and $\frac{\int_0^t e^{z-t} g(z) dz}{g(t)} = t$ which cannot be bounded by a constant (in this case, $\frac{1}{1-\alpha}$) for $t \in (0, \infty)$. Therefore, condition (i) in Theorem 2, which is necessary, is never satisfied.

(c) Note that given N and implied distributions F_α s, according to the proof of Theorem 2, $\theta(t) = p(t) - \nu m G(t) = \sigma(t)$ does not depend on the parameter α . Therefore, $\theta(t)$ is a bijection between $[0, \infty)$ and $(\underline{\theta}, \bar{\theta}]$. For any $\alpha \in (0, 1)$, using integration by parts, we can rewrite (A.16) as

$$F_\alpha(\theta(t)) = 1 - G(t) + \left(1 - \frac{1}{\alpha}\right) e^{-t} \int_0^t e^z g(z) dz. \quad (\text{A.19})$$

It then follows that $\frac{\partial}{\partial \alpha} F_\alpha(\theta(t)) > 0$, which immediately yields the stochastic dominance property. \square

Proof of Theorem 3. From Lemma A6, note that conditions (i), (iii), and (iv) are necessary for the existence of distribution F_α .

Existence. Since $p(\infty) < \underline{\theta} + \nu m$, we have $t_G < \infty$. Further, $\sigma(t) = \max\{\underline{\theta}, p(t) - \nu m G(t)\}$, and it is a weakly decreasing surjective (onto) function from $[0, \infty)$ to $(\underline{\theta}, \bar{\theta}]$; specifically, $\sigma(t)$ is strictly decreasing on $[0, t_G]$ and constant (equal to $\underline{\theta}$) afterwards. Since Q should satisfy equation (A.7), we define the type distribution F in a similar way as in (A.16):

$$F(\sigma(t)) = 1 - \frac{1}{\alpha} \int_0^{\min\{t, t_G\}} e^{z - \min\{t, t_G\}} [g(z) + \alpha G(z)] dz.$$

From the proof of Theorem 2, $F(\sigma(t))$ is strictly decreasing in t over $[0, t_G]$, with $F(\bar{\theta}) = F(\sigma(0)) = 1$ and $F(\underline{\theta}) = F(\sigma(t_G)) = 0$ (because of condition (iii)). Moreover, $F(\sigma(t)) = F(\sigma(t_G)) = 0$ for all $t \geq t_G$. Thus, F is a proper distribution function. Similar to the proof of Theorem 2, one can show that F satisfies (RC).

Next, we show that $N(t) = N_F(t)$. Denote $t_M \triangleq \inf\{t | \theta(t) = \underline{\theta}\} = \inf\{t | Q(t) = m\}$. Since $t_G < \infty$, we have two cases: $t_M \geq t_G$ and $0 < t_M < t_G$ (since we know that not all customers are qualified initially due to condition (RC.ii), $t_G > 0$ holds). First, if $t_M \geq t_G$, then $\theta(t) > \underline{\theta}$ and $\sigma(t) > \underline{\theta}$ for all $t \in [0, t_G)$. Similar to the proof of Theorem 2, it follows that $mG(t) = N(t) = N_F(t)$ and that $\sigma(t) = \theta(t)$ over the interval $[0, t_G]$. Moreover, for all $t > t_G$, we have $\dot{N}_F(t) = \alpha[m - N_F(t)]$. Furthermore, since $t_G \geq t_c$ from condition (iv), we have that $e^{\alpha t} [1 - G(t)]$ is a constant for all $t > t_G$, which implies that $g(t) = \alpha[1 - G(t)]$, or $\dot{N}(t) = \alpha[m - N(t)]$ for all $t > t_G$. Since we have identical differential equations with identical boundary condition at t_G ($N_F(t_G) = N(t_G)$), it immediately follows that $N_F(t) = N(t)$ for all $t > t_G$. Lastly, it follows that $t_G = t_M$, the moment when all consumers become qualified. Second, if $0 < t_M < t_G$, then $\theta(t) > \underline{\theta}$ and $\sigma(t) > \underline{\theta}$ for all $t \in [0, t_M)$. Similar to part (a), it follows that $mG(t) = N(t) = N_F(t)$ and that $\sigma(t) = \theta(t)$ over the interval $[0, t_M]$. However, in this case, $\theta(t_M) = \underline{\theta}$. Since $t_M < t_G$, $\sigma(t_M) > \underline{\theta}$. Therefore $p(t_M) - \nu N_G(t_M) > \underline{\theta} \geq p(t_M) - \nu N_F(t_M)$, which is a contradiction. Consequently, this second case cannot exist.

Uniqueness. Similar to the proof of Theorem 2, uniqueness over the interval $[0, t_G]$ follows. Further, as shown in the *existence* proof above, N_F obeys a unique ODE $\dot{N}_F(t) = \alpha[m - N_F(t)]$ over the interval $[t_G, \infty)$ with boundary condition $N_F(t_G) = N(t_G)$. This guarantees uniqueness of the solution. \square

Proof of Proposition 2: If $t_G < \infty$, for F_α to generate the observed adoption path G , there must exist a finite time $t_c < t_G$ such that $e^{\alpha t}[1 - G(t)]$ is constant for all $t > t_c$ from condition (iii) of Theorem 3. This cannot happen for two different values $\alpha_1 \neq \alpha_2$. Therefore, there exists at most one parameter $\alpha \in (0, 1)$ that can generate the adoption path G . \square

Proof of Proposition 3: (a) Note that from the condition (iv) in Theorem 3 in §3.2.2 for the case in which full qualification occurs in finite time, i.e., $t_G < \infty$, α is uniquely defined using the adoption G_α over the time interval $[t_G, t_1]$. Consequently, from (12), the distribution F_α is uniquely defined.

(b) First, consider $G_\alpha(t)$ as in (13) and the corresponding $p(t)$ in Example 4, from which we can derive the corresponding distribution function as given in the end of Example 4. For any given $t_0 < t_{G_\alpha} = 1/\beta_1$, where β_1 is the unique solution to (14), define a different adoption function $G_0(t)$ for any $a_2 > 0$ as follows:

$$G_0(t) = \begin{cases} a_1 t & \text{if } t \leq t_0; \\ a_1 t + a_2 (t - t_0)^3 & \text{if } t_0 < t < t_{G_0}; \\ 1 - a_3 e^{-\alpha_0 t} & \text{if } t_{G_0} \leq t, \end{cases} \quad (\text{A.20})$$

where $a_1 = \beta_1(1 - 1/e)$, and t_{G_0} is the unique solution greater than t_0 of the following equation:

$$a_2(t_{G_0} - t_0)^3 + (a + \delta)t - \bar{\theta} = 0. \quad (\text{A.21})$$

Furthermore,

$$\alpha_0 = \frac{\int_0^{t_{G_0}} e^{z-t_{G_0}} g_0(z) dz}{1 - \int_0^{t_{G_0}} e^{z-t_{G_0}} G_0(z) dz}, \quad (\text{A.22})$$

$$a_3 = e^{\alpha_0 t_{G_0}} (1 - a_1 t_{G_0} - a_2 (t_{G_0} - t_0)^3), \quad (\text{A.23})$$

and $p(t)$ is given in Example 4. Lastly, consider

$$\bar{\theta} > \max \left(\frac{e-1}{\alpha_0 e}, 1 - \frac{1}{e} + \frac{(1-\alpha_0)(a_1 + a_2)}{\beta_1 \alpha_0} \right). \quad (\text{A.24})$$

It then follows that the corresponding G_0 in (A.20) satisfies conditions (i) – (iv) in Theorem 3, and hence there exists the corresponding F_{α_0} . Note that we can pick any $a_2 > 0$, which guarantees the existence of multiple distinct adoption paths G_0 , each supported by a pair $\{\alpha_0, F_{\alpha_0}\}$ that satisfies (RC) such that $G_0 \neq G_\alpha$ but $G_0(t) = G_\alpha(t)$ for $t \in [0, t_0]$. \square

B. Discussion on the Regularity Conditions

In this section, we discuss the implications of (RC) being violated in the simple case when $\alpha = 1$. Note that when $\alpha = 1$, (RC) is equivalent to (RC.i) because $h(\theta) = \theta + \nu m \bar{F}(\theta)$ is strictly increasing in θ , and hence, (RC.i) implies (RC.ii). First, if (RC.i) is violated, there might be multiple adoption paths in equilibrium. For example, if $\underline{\theta} + \nu m > p(0) > \bar{\theta}$, then we have at least two equilibria at time

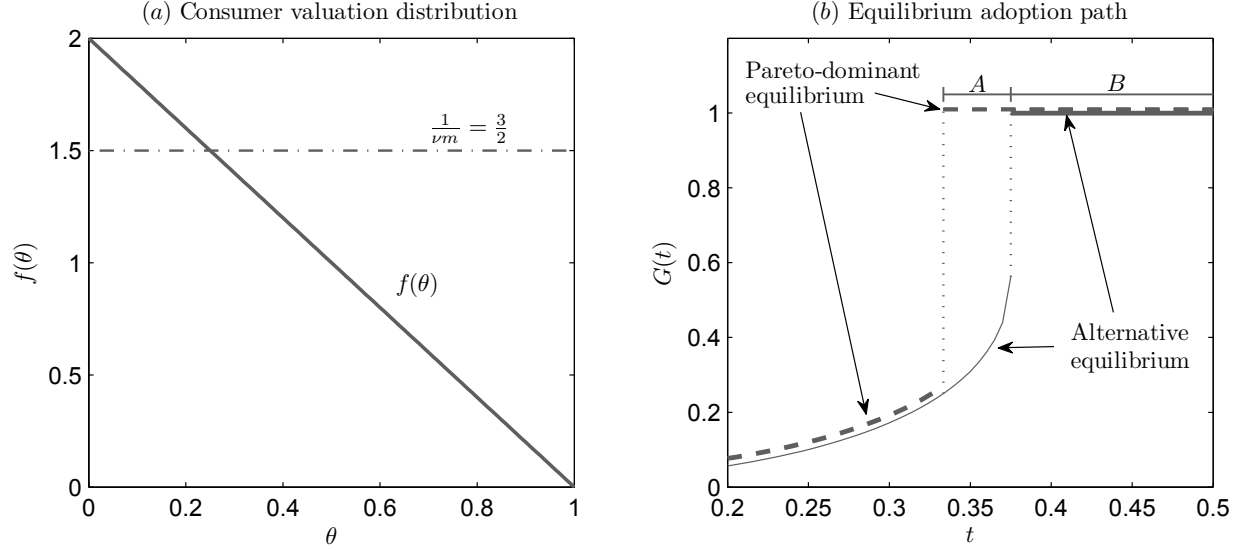


Figure 7 Illustration of a subscription path when $\alpha = 1$ and (RC) is violated. Panel (a) plots a consumer valuation distribution function when RC is violated. Panel (b) depicts the corresponding adoption path. The subscription rate considered is $p(t) = 1 - t$ for $t \in [0, 1]$ and 0 for $t > 1$. The consumer type density function is $f(\theta) = 2(1 - \theta)$ for $\theta \in [0, 1]$, $\bar{\theta} = 1$, and $\underline{\theta} = 0$. The other parameters are $\alpha = 1$, and $\nu m = 2/3$.

$t = 0$: (1) all customers adopt, or (2) all customers wait. We may also obtain multiple equilibria if h given in (A.1) is not one-to-one (injective). Alternatively, if $f(\theta) = 0$ over some interval (θ_1, θ_2) with $\theta_1 > \underline{\theta}$ and $F(\theta_1) > 0$, adoption stalls before reaching full saturation once θ_2 -type customers have adopted, and it may never re-start if the price does not drop enough in the future for the lower type customers to jump in. If adoption restarts, then $\theta(t)$ would be discontinuous.

On the other hand, taking the derivative of $\theta(t)$ with respect to t for $\theta(t) > \underline{\theta}$ under differentiability assumptions, we obtain:

$$\dot{\theta}(t)(1 - \nu m f(\theta(t))) = \dot{p}(t).$$

If for some $\theta_1 \in (\underline{\theta}, \theta(0))$ we have $1 - \nu m f(\theta_1) = 0$, then the path of $\theta(t)$ cannot pass through or cannot be differentiable at that point if the price is strictly decreasing. This leads to jumps or stalling in adoption.

Last, if for some $\theta_2 \in (\underline{\theta}, \theta(0))$ we have $1 - \nu m f(\theta_2) < 0$, one of the previously described situations occurs. If $f(\theta) > \frac{1}{\nu m}$ for all θ , since $\int_{\underline{\theta}}^{\bar{\theta}} f(\theta) d\theta = 1$, it must be the case that $\bar{\theta} < \nu m + \underline{\theta}$, which, depending on the initial price values might lead to multiple equilibria. Alternatively, if there exists another $\theta_3 \in [\underline{\theta}, \bar{\theta}]$ such that $f(\theta_3) \leq \frac{1}{\nu m}$, then under the continuity assumption of f , there must exist $\theta_4 \in [\theta_3, \theta_2]$ such that $f(\theta_4) = \frac{1}{\nu m}$, which leads to jumps or stalling.

Figure 7 also illustrates visually an example of multiple equilibria when (RC) is violated and the distribution is heavily skewed towards lower types. As depicted in Panel (a), if $\theta < 0.25$, then $1 - \nu m f(\theta) < 0$. In this case, as revealed in Panel (b), there exist multiple equilibria in period A ($0.33 <$

$t < 0.375$) as well as jumps in that period. In the Pareto-dominant equilibrium, the jump occurs at the beginning of period A (at $t = 0.33$). There also exist a continuum of alternative equilibria since, in equilibrium, the jump can occur at any time within period A . Technically, from (4), (5), and $N_F(t) = Q(t)$, we obtain $\theta(t) = \frac{1+\sqrt{9-24t}}{4}$, which is well-defined and monotone decreasing when $t < 0.375$. Correspondingly, the resulting adoption path becomes $G(t) = 1 - (2\theta(t) - \theta(t)^2)$ when $t < 0.375$. As t increases further, full adoption becomes a feasible choice for customers, generating the alternative equilibrium in period A . Furthermore, beyond $t = 0.375$, full adoption is the only equilibrium, as illustrated in period B in the figure.

C. Estimation of Unique Subscriber Installed Base for Mobile Voice Services in Japan

As discussed in §6.1, we want to adjust the total number of active voice services accounts in order to obtain the total number of unique voice subscribers. According to a GSM Asia Pacific report by Garner (2006), as of Q1 of FY2006, the multiple-connections phenomenon (whereby a user has several active wireless voice accounts) is very small in Japan, due partly to the low prevalence of prepaid services.⁹ A Forrester Research report by Browne et al. (2009) documents a recent surge in the multiple-connection phenomenon in Japan, with 6% of female Internet users and 10% of male Internet users reporting having at least two active accounts at the end of Q2 of FY2008. Same report states that 96% of Internet users have a mobile phone. Also, we note that NTT DoCoMo, the leading telecommunications firm in the Japanese market, is also offering 2in1 plans whereby users can keep two different phone numbers. Prior to March 2008, each 2in1 plan (and, implicitly, the two numbers in use) corresponded to a single subscription. Starting in March 2008, two subscriptions were necessary for the same user for voice services but data consumption was still quantified in a consolidated way in association with only one of these two subscriptions.

Taking all this information into consideration we derived an approximation for the number of adopters with multiple voice accounts based on the number of mobile Internet subscriptions at the industry level and 2in1 NTT DoCoMo plans as well. First, in order to connect Internet user numbers with mobile Internet user numbers, we are assuming that as of Q2 of FY2008, the vast majority of Japanese Internet users who have a cell phone are also *mobile* Internet users. In the absence of additional historical data, we further assume that the multiple-connection phenomenon is negligible among voice subscribers who do not use mobile Internet. In other words, we assume users who need to communicate a lot and/or, due to specific reasons, need multiple accounts in order to separate communication streams, are also consumers of mobile Internet since mobile data services provide additional communication channels (MMS, social networking, etc). Moreover,

⁹ As of December 2007, less than 2.5% of the Japanese mobile telecom market consisted of prepaid customers (GSM Asia Pacific 2008).

in the absence of specific market information, we consider negligible the number of subscribers with three or more active voice accounts and consider that the multiple-connection phenomenon is predominantly induced by users with two active accounts. Prior to the introduction of mobile Internet services in 1999, we assume that the multiple connection phenomenon was negligible. We consider that at the beginning most of the adoption is still first-time adoption while later, as the market approaches saturation, adoption exhibits an increasing percentage of multiple-connections users. In that sense, we assumed that the percentage of mobile Internet subscribers that have multiple voice accounts grew linearly from February 1999 until March 2006 from 0% to 5% and, after that, it increased faster, at a rate of an additional 1% per year. Then, number of unique subscribers is derived then by subtracting the number of users with multiple accounts from the total number of active mobile voice accounts.

D. NTT DoCoMo Voice ARPU Details

NTT DoCoMo reports yearly ARPU as the ratio between the fiscal year revenue from monthly related charges (net of any activation fees or other unrelated charges) and the sum of active subscriptions during each month of the current fiscal year. For any given month, the active subscriptions are computed as the average between the subscriptions at the end of the previous month and the subscriptions at the end of the current month.