

Online Supplement to

Sustainability of Healthcare Information Exchanges: A Game-Theoretic Approach

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Proofs

Proofs of Lemmas 1, 2, and 3

Because r follows the uniform distribution with parameters \underline{r} and \bar{r} , we can replace $f(r_{i^*})$ and $F(r_{i^*})$ with $\frac{1}{\bar{r}-\underline{r}}$ and $\frac{r_{i^*}-\underline{r}}{\bar{r}-\underline{r}}$ respectively in Equation (12). Hence, we get a closed-form expression for r_{i^*} that we substitute into $\mathbb{E}[N_b^* | p_b^*] = N(1 - F(r_{i^*}))$. This procedure reveals that when there is a partial network, its size is $\frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r}-\underline{r})(c_b-g))}}{3(\bar{r}-\underline{r})}$ as discussed in part (a) of Lemma 1.

The fact that the HIE provider would not leave any value to the HP that is indifferent between joining and not joining implies that the optimal price is set such that the following satisfies: $p_b^* = r_{i^*} \mathbb{E}[N_b^* | p_b^*] = N r_{i^*} (1 - F(r_{i^*}))$. Hence, after substituting r_{i^*} and $F(r_{i^*})$ into this equality, we get $p_b^* = \frac{N\bar{r}^2 + 3(\bar{r}-\underline{r})(c_b-g) + \bar{r}\sqrt{N(N\bar{r}^2 - 3(\bar{r}-\underline{r})(c_b-g))}}{9(\bar{r}-\underline{r})}$ as presented in Lemma 2.

Equilibrium values of the HPs and the HIE provider depicted in Lemma 3 are derived by substituting the solution into the objective functions of the parties presented in Equations (6) and (7).

Next, we would like to discuss parts (b) and (c) of Lemma 1. The derivation of the conditions presented in these parts of the Lemma is due to three constraints we have in our model. The first two constraints are related to the bounds on the network size. It is bounded below by 0 (i.e.,

no HP joins the network), and above by N (i.e., every HP joins the HIE). Hence, we have: $0 \leq \frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}{3(\bar{r} - \underline{r})} \leq N$. Equation (9) constitutes the basis for the third constraint. It implies that the HIE provider would not establish the HIE, if its expected value provided in Lemma 3 is below zero. Hence, consideration of these three constraints simultaneously, results in the conditions outlined in Lemma 1. Please note that second order conditions confirm our results.

In addition, note that $\frac{dp_b^*}{d\bar{r}} = \frac{N(\bar{r} - 2\underline{r}) \left(2\bar{r} \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}$. Given that it is a partial network, part (a) of Lemma 1 states that $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ and $\underline{r} < \frac{\bar{r}}{2}$. Hence, $2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \geq \frac{5N\bar{r}^2}{4} > 0$. Therefore, $\frac{dp_b^*}{d\bar{r}} > 0$, or p_b^* increases in \bar{r} .

Finally, $\frac{dp_b^*}{d\underline{r}} = \frac{\bar{r}N \left(2\bar{r} \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}} > 0$. To see this, observe that we have a partial network. Hence, part (a) of Lemma 1 implies that $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ and $\underline{r} < \frac{\bar{r}}{2}$. Hence, $2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \geq \frac{5N\bar{r}^2}{4} > 0$. Therefore, $\frac{dp_b^*}{d\underline{r}} > 0$, or p_b^* increases in \underline{r} . ■

Proof of Proposition 1

The presented condition is derived by examining when the partial network size can be made equal to N . Part (b) of Lemma 1 reveals that it is possible when $c_b \leq g + N(3\underline{r} - \bar{r})$ and $\frac{\bar{r}}{3} - \frac{g}{3N} \leq \underline{r} \leq \frac{\bar{r}}{2}$, or $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ and $\underline{r} \geq \frac{\bar{r}}{2}$. Hence, when $\underline{r} < \frac{\bar{r}}{3} - \frac{g}{3N}$, no matter what the values of cost (i.e., c_b) and reimbursement (i.e., g) parameters are (as long as the government does not cover all the costs, i.e., $g < c_b$), network size cannot be equal to N . ■

Proof of Proposition 2

Observe that partial network size $N_b = \frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}{3(\bar{r} - \underline{r})}$ strictly decreases in cost parameter c_b . Since this is a partial network, part (a) of Lemma 1 states that $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$. Hence, if we substitute $c_b = g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ into $\frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}{3(\bar{r} - \underline{r})}$, we get $N_b = \frac{\bar{r}N}{2(\bar{r} - \underline{r})}$. Setting \underline{r} to 0 results in the smallest possible network size that is $\frac{N}{2}$. ■

Proof of Proposition 3

Derivative of the equilibrium network size with respect to the lower bound parameter equals to:

$$\frac{d(N_b)}{d\underline{r}} = \frac{N \left(2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) + 2\bar{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} \right)}{6(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}}. \text{ We know from Lemma 1 that } c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}.$$

Hence, $2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \geq \frac{5}{4}\bar{r}^2 N > 0$, which implies that $\frac{d(N_b)}{d\underline{r}} > 0$. Hence, part (a) of the proposition.

Derivative of the equilibrium network size with respect to the upper bound parameter equals to:

$$\frac{d(N_b)}{d\bar{r}} = -\frac{N \left(2\underline{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}\underline{r}N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{6(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}}. \text{ We know from}$$

Lemma 1 that $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$. Hence, the derivative is positive if and only if

$$N \left(2\underline{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}\underline{r}N - 3(\bar{r} - \underline{r})(c_b - g) \right) < 0. \text{ This condition equals to}$$

$$2\underline{r} \sqrt{N(-3(\bar{r} - \underline{r})(c_b - g) + \bar{r}^2 N)} < 3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N. \text{ Hence, if } 3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N > 0,$$

the condition can be rewritten as $4\underline{r}^2 N(-3(\bar{r} - \underline{r})(c_b - g) + \bar{r}^2 N) < (3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N)^2$ that

further reduces to $(\bar{r} - \underline{r})^2 c_b(-3(c_b - g) + 4\underline{r}N) < 0$. In order for this to hold, we need to have

$$-3(c_b - g) + 4\underline{r}N < 0 \text{ and } 3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N > 0. \text{ In addition, we need to have } c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$$

as discussed in Lemma 1. In summary, we need to have the following satisfied in order for the

derivative to be positive.

$$-3c_b - 3g + 4\underline{r}N < 0, \tag{24}$$

$$3(c_b - g)(\bar{r} - \underline{r}) - 2\bar{r}\underline{r}N > 0, \tag{25}$$

$$c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}. \tag{26}$$

Conditions (24) and (25) are satisfied at the same time if $(\bar{r} - 4\underline{r})(3\bar{r} - 4\underline{r}) > 0$. On the other hand,

conditions (25) and (26) are both satisfied when $3\bar{r} > 8\underline{r}$. However given $3\bar{r} > 8\underline{r}$, the condition

$(\bar{r} - 4\underline{r})(3\bar{r} - 4\underline{r}) > 0$ reduces to $\bar{r} > 4\underline{r}$. In addition, condition (25) becomes redundant since we

have $\bar{r} > 4\underline{r}$ and $c_b < g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ at the same time. In summary, in order for $\frac{d(N_b)}{d\bar{r}}$ to be positive, we

need to have $\bar{r} > 4\underline{r}$ and $g + \frac{4\underline{r}N}{3} < c_b < g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$. Since $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ is already required for the establishment of the HIE, we do not present it in part (b) of the proposition. ■

Proof of Proposition 4

Derivative of the equilibrium value of HP i with respect to the upper bound parameter is given by:

$$-\frac{N(\bar{r} - 2\underline{r}) \left(2\bar{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}} - \frac{3N\underline{r}_i \left(2\underline{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}\underline{r}_i N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}}.$$

Hence, the derivative is negative if and only if

$$\begin{aligned} \omega &= (\bar{r} - 2\underline{r}) \left(2\bar{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right) \\ &+ 3\underline{r}_i \left(2\underline{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}\underline{r}_i N - 3(\bar{r} - \underline{r})(c_b - g) \right) > 0. \end{aligned}$$

Observe that ω strictly decreases in c_b . Hence, if we evaluate ω at the upper bound on c_b (i.e., $g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ due to part (a) of Lemma 1) we have:

$$\omega \geq \omega|_{c_b = g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}} = \frac{9}{4} \bar{r} N (\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i)).$$

Hence, the derivative of the equilibrium value of HP i with respect to \bar{r} is negative if $\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i) > 0$. Since we have a partial network, part (a) of Lemma 1 states that $\underline{r} < \frac{\bar{r}}{2}$. So, observe that $\frac{d(\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i))}{d\bar{r}} = 2\bar{r} - 2\underline{r} - \underline{r}_i > 0$. Hence, $\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i)$ strictly increases in \bar{r} . Therefore, $\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i) \geq \lim_{\bar{r} \rightarrow \underline{r}_i} [\bar{r}^2 + 4\underline{r}\underline{r}_i - \bar{r}(2\underline{r} - \underline{r}_i)] = 2\underline{r}\underline{r}_i \geq 0$. Hence, part (a) of the proposition.

On the other hand, the derivative of the equilibrium value of HP i with respect to the lower bound parameter equals to $\frac{N(3\underline{r}_i - \bar{r}) \left(2\bar{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}}$. Because we have $c_b \leq g + \frac{\bar{r}^2 N}{4(\bar{r} - \underline{r})}$ from Lemma 1, the sign of the derivative is positive if and only if $\underline{r}_i > \frac{\bar{r}}{3}$. Hence, part (b) of the proposition.

Next, we analyze the derivative of the equilibrium value of the HIE provider with respect to the upper bound parameter. The net value of the HIE provider is given by:

$$\frac{\left(N\bar{r}^2 - 6(\bar{r} - \underline{r})(c_b - g) + \bar{r} \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))} \right) \left(N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))} \right)}{27(\bar{r} - \underline{r})^2}.$$

The derivative with respect to \bar{r} is positive if and only if the following expression is positive: $\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))}(3(\bar{r} - \underline{r})(c_b - g) + 2\bar{r}N(\bar{r} - 3\underline{r})) + N(2\bar{r}^2N(\bar{r} - 3\underline{r}) + 9\underline{r}(\bar{r} - \underline{r})(c_b - g))$.

The derivative of the above expression with respect to c_b shows that it strictly increases in c_b . However, part (a) of Lemma 1 (since we have a partial network) states that $c_b \leq g + \frac{\bar{r}^2N}{4(\bar{r} - \underline{r})}$. Hence, $\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))}(3(\bar{r} - \underline{r})(c_b - g) + 2\bar{r}N(\bar{r} - 3\underline{r})) + N(2\bar{r}^2N(\bar{r} - 3\underline{r}) + 9\underline{r}(\bar{r} - \underline{r})(c_b - g))$ evaluated at $c_b = g + \frac{\bar{r}^2N}{4(\bar{r} - \underline{r})}$ becomes $\frac{27}{8}\bar{r}^2(\bar{r} - 2\underline{r})N^2$. Since part (a) of Lemma 1 also states that $\underline{r} < \frac{\bar{r}}{2}$, we have $\frac{27}{8}\bar{r}^2(\bar{r} - 2\underline{r})N^2 > 0$. Therefore, the derivative of the equilibrium value of the HIE provider with respect to \bar{r} is always positive. ■

Proofs of Lemmas 4 and 5

Let us emphasize some properties of the solution:

- (a) There are two possibilities regarding the relationship between N_b and N_v at the equilibrium.

In particular, we need to have either $N_b = N_v$, or $N_b > N_v$, because in order for an HP to request the value-added service, it has to be a member of the HIE.

- (b) In the solution of the problem, without loss of generality, we order r_i such that $r_i < r_{i+1}$ as discussed in the main paper. Note that this order is preserved under linear transformations of r_i . Hence, the same ordering is preserved with respect to the (i) net benefit of the value-added service (i.e., $r_i s \mathbb{E}[N_v | p_v] d - p_v$), (ii) net benefit of the base service (i.e., $r_i \mathbb{E}[N_b | p_b] - p_b$), and (iii) total net benefit (i.e., $r_i \mathbb{E}[N_b | p_b] - p_b + r_i s \mathbb{E}[N_v | p_v] d - p_v$).

Note that for HP i , we can have $r_i \mathbb{E}[N_b | p_b] - p_b < 0$ and $r_i \mathbb{E}[N_b | p_b] - p_b + r_i s \mathbb{E}[N_v | p_v] d - p_v \geq 0$ at the equilibrium and that implies that this HP requests the value-added service. In this case, if HP x is one such HP that is indifferent between requesting the value-added service (by joining the HIE) or not, then we have $r_x \mathbb{E}[N_b | p_b] - p_b < 0$ and $r_x \mathbb{E}[N_b | p_b] - p_b + r_x s \mathbb{E}[N_v | p_v] d - p_v = 0$ at

the equilibrium. Observe that HP k , where $k < x$, would not join the HIE because of the order preserving property discussed above. In addition, HP l , where $l > x$, would join the HIE. Hence, we have $N_b = N_v$ in this case (because there will not be any HIE for which $r_i \mathbb{E}[N_b|p_b] - p_b \geq 0$ and $r_i \mathbb{E}[N_b|p_b] - p_b + r_i s \mathbb{E}[N_v|p_v]d - p_v < 0$).

Similarly, observe that if $r_k s \mathbb{E}[N_v|p_v]d - p_v \geq 0$ and $r_k \mathbb{E}[N_b|p_b] - p_b + r_k s \mathbb{E}[N_v|p_v]d - p_v < 0$, then HP k does not join the HIE at the equilibrium solution. Since $r_i s \mathbb{E}[N_v|p_v]d - p_v$ and $r_i \mathbb{E}[N_b|p_b] - p_b + r_i s \mathbb{E}[N_v|p_v]d - p_v$ both increase in r_i , there might be some HP x for which $r_x s \mathbb{E}[N_v|p_v]d - p_v \geq 0$ and $r_x \mathbb{E}[N_b|p_b] - p_b + r_x s \mathbb{E}[N_v|p_v]d - p_v = 0$. This implies that for HP x , $r_x \mathbb{E}[N_b|p_b] - p_b < 0$ and $r_x \mathbb{E}[N_b|p_b] - p_b + r_x s \mathbb{E}[N_v|p_v]d - p_v = 0$. Hence, this case is equivalent to the previous case discussed above. Therefore, we again have $N_b = N_v$ at the equilibrium. In other words, if one solves for Equations (21) and (22) and observes that $N_b < N_v$, it means that there would be HPs that join the HIE in order to benefit from the value-added service and that $N_b = N_v$ (but the solutions of Equations (21) and (22) would not hold). Hence, in such a case, we need to solve for Equation (23) to determine the equilibrium values of N_b and N_v and we cannot solve for Equations (21) and (22).

On the other hand, if one solves for Equations (21) and (22) and observes that $N_b > N_v$ at the equilibrium, we obviously cannot have $r_x \mathbb{E}[N_b|p_b] - p_b < 0$ where HP x is the HP that is indifferent between joining the HIE or not (based on the total benefit). In fact, all of the HPs that request the value-added service already find the base service valuable because of the ordering property and the fact that $N_b > N_v$. Hence, the solution of Equations (21) and (22) and the following arguments help us determine the equilibrium values of N_b and N_v . Note that the HIE provider does not consider the decisions of each of the HPs separately. However, it considers the total number of HPs interested in the base HIE service, and the total number of HPs interested in the value-added services.

Let us recall here the objective functions of the HIE provider and the HPs:

$$\begin{aligned} \max_{s, p_b, p_v} \quad & p_v \mathbb{E}[N_v] + p_b \mathbb{E}[N_b] + g \mathbb{E}[N_b] - c_b \mathbb{E}[N_b] - c_v \mathbb{E}[N_v] s^2 \\ \max_{q_{bi}, q_{vi}} \quad & q_{bi} (r_i \mathbb{E}[N_b] - p_b) + q_{vi} (r_i s \mathbb{E}[N_v] d - p_v) \quad \forall i \end{aligned}$$

After substituting the values of p_b , p_v , $\mathbb{E}[N_b]$, and $\mathbb{E}[N_v]$ discussed in the main paper into the objective functions of the parties, we can solve the game among the parties by utilizing Equations (21) and (22). Rather than the payment terms p_b and p_v , we can consider r_b and r_v as decision variables because of the one-to-one correspondences between these variables and the payment terms as discussed in the main paper. In addition, the HIE provider needs to also decide the level of value-added service level s . Hence, first order conditions with respect to s , r_b , and r_v reveal the equilibrium values of the variables below. Hence, Lemma 4.

$$\begin{aligned} s &= \frac{Nr_v^* (\bar{r} - r_v^*) d}{2c_v (\bar{r} - \underline{r})}, \\ r_b^* &= \frac{2\bar{r}N - \sqrt{N(\bar{r}^2 N + 3c_b(\underline{r} - \bar{r}))}}{3N}, \\ r_v^* &= \frac{2\bar{r}Nd - \sqrt{Nd(\bar{r}^2 Nd + 3c_v s(\underline{r} - \bar{r}))}}{3Nd}. \end{aligned}$$

After solving the above equations simultaneously, second order conditions imply that the following service level is optimal and the equilibrium number of HPs in the HIE, and those who request value-added services are:

$$\begin{aligned} s^* &= \frac{3\bar{r}^2 Nd}{25c_v (\bar{r} - \underline{r})}, \\ N_b &= \text{Min} \left\{ \frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}{3(\bar{r} - \underline{r})}, N \right\}, \\ N_v &= \text{Min} \left\{ \frac{3\bar{r}N}{5(\bar{r} - \underline{r})}, N \right\}. \end{aligned}$$

As discussed earlier, this solution is valid only if the number of HPs in the network is more than the number of HPs that request value-added services, i.e., if $\frac{N\bar{r} + \sqrt{N(N\bar{r}^2 - 3(\bar{r} - \underline{r})(c_b - g))}}{3(\bar{r} - \underline{r})} \geq \frac{3\bar{r}N}{5(\bar{r} - \underline{r})}$. It is easy to show that this condition is true if and only if $c_b < g + \frac{3\bar{r}^2 N}{25(\bar{r} - \underline{r})}$. Hence, if $c_b > g + \frac{3\bar{r}^2 N}{25(\bar{r} - \underline{r})}$, there are HPs interested in value-added services but do not find base HIE services beneficial. In such a case, some HPs (or all of them) decide to join the HIE. Getting a closed-form expression of the resultant network size is not possible due to the complexity of the equations in the solution. We do not provide these equations for brevity and clarity. However, we provide the conditions when value-added services increase the network size in parts (a) and (b) of Lemma 5, together with the bounds on the network size and the number of HPs that request value-added services. ■

Proof of Lemma 6

Optimal fees are derived by substituting the solution discussed in the proofs of Lemmas 4, and 5 into equations $p_b^* = r_b^* \mathbb{E}[N_b | p_b^*]$, and $p_v^* = r_v^* s^* \mathbb{E}[N_v | p_v^*] d$. Optimal level of value-added services is derived in the proofs of Lemmas 4, and 5.

Now let us analyze the sensitivity of these terms with respect to \bar{r} and \underline{r} . First, $\frac{ds^*}{d\bar{r}} = \frac{3\bar{r}(\bar{r} - 2\underline{r})dN}{25(\bar{r} - \underline{r})^2 c_v} > 0$. Since $\bar{r} - 2\underline{r} > 0$, given that $\underline{r} \leq \frac{2}{5}\bar{r}$ (due to part (d) of Lemma 5), we conclude that s^* increases with \bar{r} . Similarly, $\frac{ds^*}{d\underline{r}} = \frac{3\bar{r}^2 dN}{25(\bar{r} - \underline{r})^2 c_v} > 0$. Hence, s^* increases with \underline{r} .

Moreover, $\frac{dp_v^*}{d\bar{r}} = \frac{36\bar{r}^3 d^2 N^2 (\bar{r} - 2\underline{r})}{625(\bar{r} - \underline{r})^3 c_v}$. Since $\bar{r} - 2\underline{r} > 0$, given that $\underline{r} \leq \frac{2}{5}\bar{r}$ (due to part (d) of Lemma 5), we conclude that $\frac{36\bar{r}^3 d^2 N^2 (\bar{r} - 2\underline{r})}{625(\bar{r} - \underline{r})^3 c_v} > 0$. Therefore, p_v^* increases with \bar{r} . Furthermore, $\frac{dp_v^*}{d\underline{r}} = \frac{36\bar{r}^4 d^2 N^2}{625(\bar{r} - \underline{r})^3 c_v} > 0$. Hence, p_v^* increases with \underline{r} .

In addition, note that $\frac{dp_b^*}{d\bar{r}} = \frac{N(\bar{r} - 2\underline{r}) \left(2\bar{r} \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))}}$. Note that given $\underline{r} \leq \frac{2}{5}\bar{r}$, we have $\bar{r} - 2\underline{r} > 0$. In addition, observe that $2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g)$ strictly decreases in c_b . Given the upper bound of c_b (in part (d) of Lemma 5), we have $2\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g) \geq \frac{41}{25}\bar{r}^2 N > 0$.

Hence, $\frac{dp_b^*}{d\bar{r}} > 0$. Finally, $\frac{dp_b^*}{d\underline{r}} = \frac{\bar{r}N \left(2\bar{r}\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))} + 2\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g) \right)}{18(\bar{r} - \underline{r})^2\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))}} > 0$, so that p_b^* increases with \underline{r} . ■

Proof of Proposition 5

Derivative of the equilibrium network size with respect to the lower bound parameter equals to:

$\frac{d(N_b)}{d\underline{r}} = \frac{N \left(2\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g) + 2\bar{r}\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))} \right)}{6(\bar{r} - \underline{r})^2\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))}}$. Note that part (d) of Lemma 5 states that $c_b < g + \frac{3\bar{r}^2N}{25(\bar{r} - \underline{r})}$. Hence, $2\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g) \geq \frac{41}{25}\bar{r}^2N > 0$, which implies that $\frac{d(N_b)}{d\underline{r}} > 0$. Hence, part (a) of the proposition.

The derivative of N_b with respect to \bar{r} is given by: $\frac{N \left(3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N - 2\underline{r}\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))} \right)}{6(\bar{r} - \underline{r})^2\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))}}$.

Hence, this derivative is positive if and only if

$$\omega = 3(\bar{r} - \underline{r})(c_b - g) - 2\bar{r}\underline{r}N - 2\underline{r}\sqrt{N(\bar{r}^2N - 3(\bar{r} - \underline{r})(c_b - g))} > 0.$$

Note that ω strictly increases in c_b . By solving $\omega = 0$ for c_b in the above equation, we get $\omega > 0$ iff $c_b > g + \frac{4\underline{r}N}{3}$. Furthermore, note that part (d) of Lemma 5 states that $c_b < g + \frac{3\bar{r}^2N}{25(\bar{r} - \underline{r})}$. Therefore, these two conditions satisfy only if $\frac{4\underline{r}N}{3} < \frac{3\bar{r}^2N}{25(\bar{r} - \underline{r})}$. This condition can be rewritten as $\{\bar{r} < \frac{10}{9}\underline{r} \mid \bar{r} > 10\underline{r}\}$. Since part (d) of Lemma 5 also implies that $\underline{r} \leq \frac{2}{5}\bar{r}$, we can have $\omega > 0$ only if $\bar{r} > 10\underline{r}$, whereas the sufficient condition is $c_b > g + \frac{4\underline{r}N}{3}$. Hence, part (b) of the proposition.

Our extensive numerical experiments reveal that the HIE settings outlined in parts (a) and (b) of Lemma 5 depict the same behavior discussed in part (a) of Proposition 5. On the other hand, in these settings, we observe that if \bar{r} increases, a fewer number of HPs would join the HIE and fewer number of HPs would request the value-added service. For each setting, we provide a set of parameter values below. A small increase in \bar{r} reveals that N_b and N_v decrease (in each corresponding setting) in \bar{r} . On the other hand, a small increase in \underline{r} results in an increase in both N_b and N_v .

- For part (a)(1): $\{N = 100, \bar{r} = 7, \underline{r} = 0, d = 1.1, c_b = 180, c_v = 99, g = 0\}$

- For part (b)(1): $\{N = 100, \bar{r} = 10, \underline{r} = 0, d = 0.3, c_b = 180, c_v = 200, g = 0\}$
- For part (b)(2): $\{N = 100, \bar{r} = 10, \underline{r} = 4.5, d = 0.3, c_b = 400, c_v = 200, g = 0\}$ ■

Proof of Lemma 7

Equilibrium values of the parties are derived by substituting the solution discussed in the proofs of Lemmas 4, and 5 into the objective functions of the parties. ■

Proof of Proposition 6

At the equilibrium, derivative of the value of HP i with respect to \bar{r} equals to:

$$-\frac{162\bar{r}^2 N^2 d^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} (4\bar{r}(\bar{r} - 2\underline{r}) - 5(\bar{r} - 3\underline{r})r_i)}{11250(\bar{r} - \underline{r})^3 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} c_v} - \frac{625(\bar{r} - \underline{r}) N c_v \left(-3(\bar{r} - \underline{r})(c_b - g)(\bar{r} - 2\underline{r} + 3r_i) + 2 \left(\bar{r} N + \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} \right) (\bar{r}^2 - 2\bar{r}\underline{r} + 3r_i) \right)}{11250(\bar{r} - \underline{r})^3 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} c_v}.$$

This value is positive if and only if γ that is defined below is positive:

$$\gamma = -162\bar{r}^2 N d^2 \sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} (4\bar{r}(\bar{r} - 2\underline{r}) - 5(\bar{r} - 3\underline{r})r_i) - 625c_v(\bar{r} - \underline{r}) \left(2 \left(\sqrt{N(\bar{r}^2 N - 3(\bar{r} - \underline{r})(c_b - g))} + \bar{r} N \right) (\bar{r}^2 - 2\bar{r}\underline{r} + 3r_i) - 3(c_b - g)(\bar{r} - \underline{r})(\bar{r} - 2\underline{r} + 3r_i) \right).$$

If $r_i \leq \frac{4}{5}\bar{r}$, γ is always negative. This is because in order to have a partial network where some but not all HPs request value-added services, we need to also have $c_b < g + \frac{3\bar{r}^2 N}{25(\bar{r} - \underline{r})}$. Hence, part (a1) of the proposition.

In order to prove part (a2) of the proposition, let us give an example when the value of HP i increases with \bar{r} . First set $\underline{r} = 0$ in order to have high heterogeneity, and rewrite γ as:

$$\bar{r}^2 \left(-162\bar{r} N d^2 \sqrt{\bar{r} N (\bar{r} N - 3(c_b - g))} (4\bar{r} - 5r_i) - 625c_v \left(2\bar{r} \left(\sqrt{\bar{r} N (\bar{r} N - 3(c_b - g))} + \bar{r} N \right) - 3(c_b - g)(\bar{r} + 3r_i) \right) \right).$$

Observe that, if $r_i > \frac{4}{5}\bar{r}$ and if $d > \frac{25\sqrt{c_v(3(c_b - g)(\bar{r} + 3r_i) - 2\bar{r}(\sqrt{\bar{r} N (\bar{r} N - 3(c_b - g))} + \bar{r} N))}}{9\sqrt{2}\sqrt{\bar{r} N} \sqrt{\bar{r} N (\bar{r} N - 3(c_b - g))} (5r_i - 4\bar{r})}$, value of HP i

increases with \bar{r} . Hence, part (a2) of the proposition. The proof for part (b) is trivial, hence omitted.

Our extensive numerical experiments reveal that the HIE settings outlined in parts (a) and (b) of Lemma 5 depict the same behavior discussed in part (b) of Proposition 6. On the other hand,

in these settings, we observe that if \bar{r} increases, the values of all HPs decrease regardless of the values of r_i and d . For parts (a) and (b) of Lemma 5, we provide a set of parameter values below.

A small increase in \bar{r} reveals that values of each HP decrease (in each corresponding setting) in \bar{r} .

On the other hand, a small increase in \underline{r} results in an increase in the values of all HPs.

- For part (a)(1): $\{N = 100, \bar{r} = 7, \underline{r} = 0, d = 1.1, c_b = 180, c_v = 99, g = 0, r_i = 6\}$
- For part (b)(1): $\{N = 100, \bar{r} = 10, \underline{r} = 0, d = 0.3, c_b = 180, c_v = 200, g = 0, r_i = 7\}$
- For part (b)(2): $\{N = 100, \bar{r} = 10, \underline{r} = 4.5, d = 0.3, c_b = 400, c_v = 200, g = 0, r_i = 7\}$ ■