Mathematical Online Appendix to "Should Online Content Providers be allowed to Subsidize Content? – an Economic Analysis"

Appendix A: Proofs

Proof of Lemma 1

Proof: In the benchmark equilibrium, the ISP does not have an option to allow data sponsorship. The timeline of the game is given as follows: in stage 1, the ISP announces *p*, which is the usage-based price for consumers; in stage 2, the end consumers choose content from either *L* or *H*. We solve this game-theoretical model using the backward induction.

In stage 2, the price charged by the ISP, *p*, is given. To find out the market share of both CPs, we identify the location of marginal consumers who are indifferent between choosing content from L and from *H*: $U_L(x^*) = U_H(x^*)$, where $U_L(x) = V(\lambda) - p\lambda - tx$ and $U_H(x) = V(\lambda) - p\lambda - t(1-x)$. This leads to $x^* = \frac{1}{2}$ and this result implies equal market share for the two CPs. In stage 1, the ISP chooses p to maximize its profit given the participation constraints for consumers and content providers:

$$\max_p \pi_{ISP} = p\lambda,$$

$$s.t.$$

$$U_L\left(\frac{1}{2}\right) \geq 0, U_H\left(\frac{1}{2}\right) \geq 0.$$

From the constraints, $U_H\left(\frac{1}{2}\right) \geq 0$ and $U_L\left(\frac{1}{2}\right) \geq 0$, we can obtain the upper bound of p, $\frac{2V(\lambda) - t}{2\lambda} \geq p$. Therefore, when the constraint is binding, the highest value of the objective function is $\pi_{ISP} = \frac{2V(\lambda) - t}{2}$ where $2V(\lambda) - t \geq 0$. ■

Proof of Lemma 2

Proof: Case 1. The ISP allows no CPs to subsidize their consumers' data usage. Case 1 is equivalent to the benchmark case described in Proposition 1.

Case 2. The ISP allows CP L to subsidize its consumers' data usage. In this case, the timeline of the game is given as follows: in stage 1, the ISP announces p, which is the usage-based price for both consumers and CP L; in stage 2, L decides the amount of its subsidization, $\lambda_{s,L} \in [0, \lambda]$; finally, in stage 3, the end consumers choose either L or H. We solve this game-theoretical model using backward induction.

In stage 3, the marginal consumer is given by: $U_L(x^*) = U_H(x^*)$ where $U_L(x) = V(\lambda) - p(\lambda - \lambda_{s,L}) - tx \text{ and } U_H(x) = V(\lambda) - p\lambda - t(1-x)$. Since L subsidizes consumers' data usage by $\lambda_{s,L}$, the ISP does not charge p for the quantity of the data-packet $\lambda_{s,L}$. In contrast with Case 1, we find that the utility of a consumer increases by $p\lambda_{s,L}$ in Case 2. The marginal consumer is given by $x^* = \frac{1}{2} + \frac{p\lambda_{s,L}}{2t}$; Because the marginal consumer is located at $x^* \in [0,1]$, the following inequality conditions should be satisfied: $0 \le x^* = \frac{1}{2} + \frac{p\lambda_{s,L}}{2t} \le 1$. From the constraint, two results can be shown as follows: (1) the upper bound of p is $p \le \frac{t}{\lambda_{s,L}}$; (2) the market share of CP L is $x^* \ge \frac{1}{2}$. If we compare x^* with the market share of CP L in Case 1, the market share of CP L increases by $\frac{p\lambda_{s,L}}{2t}$ with the subsidization.

In stage 2, given p, CP L decides the subsidized amount $\lambda_{s,L}$ to maximize its profit: $\pi_L = r_L\lambda x^* - p\lambda_{s,L}x^*$. From the first order condition of the profit function, $\frac{\partial \pi_L}{\partial {\lambda_{s,L}}^*} = 0$, we can obtain the optimal the subsidized amount of CP L, ${\lambda_{s,L}}^* = \frac{r_L\lambda - t}{2p}$.

From this result, we can derive the optimal value of ${\lambda_{s,L}}^*$ based on $\lambda_{s,L} \in [0, \lambda]$:

$$\lambda_{s,L}{}^{*} = \begin{cases} 0, \text{if } r_L < \frac{t}{\lambda} \\ \frac{r_L\lambda - t}{2p}, \text{if } \frac{t}{\lambda} \le r_L < \frac{2p+t}{\lambda} \\ \lambda, \text{if } r_L \ge \frac{2p+t}{\lambda} \end{cases}.$$

In stage 1, the ISP decides p with the participation constraints and the formulation is as follows:

$$\max_{p} \pi_{ISP} = p\lambda$$

$$s.t.$$

$$U_L(x^*) \ge 0, x^* \in [0,1]$$

$$U_H(x^*) \ge 0, x^* \in [0,1]$$

$$\pi_L(\lambda_{s,L}{}^{*}) \ge 0$$

From the marginal consumer's participation constraint $U_L(x^*) \ge 0$ and $U_H(x^*) \ge 0$, which is demonstrated as $V(\lambda) - p\left(\lambda - \lambda_{s,L}{}^{*}\right) - tx^* \ge 0$, we can obtain the upper bound of p as follows: if $r_L < \frac{t}{\lambda}$, $\frac{2V(\lambda) - t}{2\lambda} \ge p$; if $\frac{t}{\lambda} \le r_L < \frac{2p+t}{\lambda}$, $V(\lambda) + \frac{r_L\lambda - 3t}{4\lambda} \ge p$; if $r_L \ge \frac{2p+t}{\lambda}$, $\frac{2V(\lambda) - t}{\lambda} \ge p$. The constraint $\pi_L(\lambda_{s,L}{}^{*}) \ge 0$ is always satisfied.

Therefore, the equilibrium is given by the following three cases: (1) If $r_L < \frac{t}{\lambda}$, $x^* = \frac{1}{2}$, $p^* = \frac{2V(\lambda) - t}{2\lambda}$, and $\lambda_{s,L}{}^{*} = 0$ with a constraint $2V(\lambda) - t \ge 0$;

(2) If $\frac{t}{\lambda} \le r_L < \frac{2p+t}{\lambda}$, $x^* = \frac{1}{2} + \frac{r_L\lambda - t}{4t}$, $p^* = \frac{V(\lambda)}{\lambda} + \frac{r_L\lambda - 3t}{4\lambda}$, and $\lambda_{s,L}{}^{*} = \frac{r_L\lambda - t}{2p}$ with constraints $V(\lambda) + \frac{r_L\lambda - 3t}{4\lambda} \ge 0$ and $r_L\lambda < 3t$;

(3) If $r_L \ge \frac{2p+t}{\lambda}$, $x^* = \frac{V(\lambda)}{t}$, $p^* = \frac{2V(\lambda) - t}{\lambda}$, and $\lambda_{s,L}{}^{*} = \lambda$ with constraints $2V(\lambda) - t \ge 0$ and $V(\lambda) < t$.

Case 3. The ISP allows CP H to subsidize its consumers' data usage by $\lambda_{s,H}$. The results of Case 2 and Case 3 are symmetric. Therefore, we can obtain the following three possible results that yield the highest objective values:

(1) If $r_H < \frac{t}{\lambda}$, $x^* = \frac{1}{2}$, $p^* = \frac{2V(\lambda) - t}{2\lambda}$, and ${\lambda_{s,H}}^* = 0$ with a constraint $2V(\lambda) - t \geq 0$;

(2) If $\frac{t}{\lambda} \leq r_H < \frac{2p+t}{\lambda}$, $x^* = \frac{1}{2} - \frac{r_H\lambda - t}{4t}$, $p^* = \frac{V(\lambda)}{\lambda} + \frac{r_H\lambda - 3t}{4\lambda}$, and ${\lambda_{s,H}}^* = \frac{r_H\lambda - t}{2p}$ with constraints $V(\lambda) + \frac{r_H\lambda - 3t}{4\lambda} \geq 0$ and $r_H\lambda < 3t$;

(3) If $r_H \geq \frac{2p+t}{\lambda}$, $x^* = \frac{1}{2} - \frac{p\lambda}{2t} = 1 - \frac{V(\lambda)}{t}$, $p^* = \frac{2V(\lambda) - t}{\lambda}$, and ${\lambda_{s,H}}^* = \lambda$ with constraints $2V(\lambda) - t \geq 0$ and $V(\lambda) < t$.

Case 4. The ISP allows both CP L and CP H to subsidize their consumers' data usage by $\lambda_{s,L}$ and $\lambda_{s,H}$, respectively. In this case, the timeline of the game is given as follows: in stage 1, the ISP announces p, which is the usage-based price for consumers and two CPs; in stage 2, L and H decide the amount of their subsidization, $\lambda_{s,L} \in [0, \lambda]$ and $\lambda_{s,H} \in [0, \lambda]$ respectively; finally, in stage 3, the end consumers choose either L or H. We solve this game-theoretical model using backward induction.

In stage 3, the marginal consumer is given by $U_L(x^*) = U_H(x^*)$ where $U_L(x) = V(\lambda) - p(\lambda - \lambda_{s,L}) - tx$ and $U_H(x) = V(\lambda) - p(\lambda - \lambda_{s,H}) - t(1-x)$. Since L and H subsidize consumers' data usage by $\lambda_{s,L}$ and $\lambda_{s,H}$, the ISP does not charge p for the quantity of the data-packet $\lambda_{s,L}$ and $\lambda_{s,H}$. Accordingly, we find that a consumer's utility increases by $p\lambda_{s,L}$ and $p\lambda_{s,H}$ comparing Case 4 with Case 1. From the above equation, we can derive the market share of content providers: the market share for content provider L is given by $x^* = \frac{1}{2} + \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t}$; the market share for content provider H is given by $(1 - x^*) = \frac{1}{2} - \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t}$. Because the marginal consumer is located at $x^* \in [0,1]$, the following inequality condition should be satisfied: $0 \leq x^* = \frac{1}{2} + \frac{p(\lambda_{s,L}-\lambda_{s,H})}{2t} \leq 1$. From the constraint, we obtain the following results: the upper bound of p, $p \leq \frac{t}{(\lambda_{s,L}-\lambda_{s,H})}$ if $\lambda_{s,L} \geq \lambda_{s,H}$ and $p \leq \frac{t}{(\lambda_{s,H}-\lambda_{s,L})}$ if $\lambda_{s,H} \geq \lambda_{s,L}$.

In stage 2, given p, CP L and CP H decide the amount of subsidization $\lambda_{s,L}$ and $\lambda_{s,H}$to maximize their profit: $\pi_L = r_L\lambda x^* - p\lambda_{s,L}x^*$ and $\pi_H = r_H\lambda x^* - p\lambda_{s,H}x^*$. From the first order conditions of the profit functions, $\frac{\partial \pi_L}{\partial \lambda_{s,L}^*} = 0$ and $\frac{\partial \pi_H}{\partial \lambda_{s,H}^*} = 0$, we obtain the following reaction functions, $\lambda_{s,L}^*(\lambda_{s,H}) = \frac{-t+p\lambda_{s,H}+r_L\lambda}{2p}$ and $\lambda_{s,H}^*(\lambda_{s,L}) = \frac{-t+p\lambda_{s,L}+r_H\lambda}{2p}$.

Therefore, the reaction functions $\lambda_{s,L}^*(\lambda_{s,H})$ and $\lambda_{s,H}^*(\lambda_{s,L})$ are given by:

$$\lambda_{s,L}^*(\lambda_{s,H}) = \begin{cases} 0, \text{ if } \lambda_{s,H} \leq \frac{t-r_L\lambda}{p} \\ \frac{-t+p\lambda_{s,H}+r_L\lambda}{2p}, \text{ if } \frac{t-r_L\lambda}{p} < \lambda_{s,H} \leq \frac{t-r_L\lambda}{p} + 2\lambda, \\ \lambda, \text{ if } \lambda_{s,H} > \frac{t-r_L\lambda}{p} + 2\lambda \end{cases}$$

$$\lambda_{s,H}^*(\lambda_{s,L}) = \begin{cases} 0, \text{ if } \lambda_{s,L} \leq \frac{t-r_H\lambda}{p} \\ \frac{-t+p\lambda_{s,L}+r_H\lambda}{2p}, \text{ if } \frac{t-r_H\lambda}{p} < \lambda_{s,L} \leq \frac{t-r_H\lambda}{p} + 2\lambda. \\ \lambda, \text{ if } \lambda_{s,L} > \frac{t-r_H\lambda}{p} + 2\lambda \end{cases}$$

In stage 1, the ISP chooses p given the participation constraints:

$$\max_p \pi_{ISP} = p\lambda$$

$$s.t.$$

$$U_L(x^*) \geq 0, U_H(x^*) \geq 0$$

$$0 \leq x^* = \frac{1}{2} + \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t} \leq 1$$

$$\pi_L(\lambda_{s,L}^*) \geq 0, \pi_H(\lambda_{s,H}^*) \geq 0.$$

The optimal p is derived as $p^* = \frac{2V(\lambda)-t}{2\lambda-(\lambda_{s,L}+\lambda_{s,H})}$.

Combining $\lambda_{s,L}{}^{*}(\lambda_{s,H})$ and $\lambda_{s,H}{}^{*}(\lambda_{s,L})$ with $p^{*} = \frac{2V(\lambda)-t}{2\lambda-(\lambda_{s,L}{}^{*}+\lambda_{s,H}{}^{*})}$, we can obtain five equilibria as follows:

(1) If $\lambda_{s,L} \leq \frac{t-r_H\lambda}{p}$ and $\lambda_{s,H} \leq \frac{t-r_L\lambda}{p}$, $\lambda_{s,L}{}^{*} = 0$, $\lambda_{s,H}{}^{*} = 0$, and $p^{*} = \frac{2V(\lambda)-t}{2\lambda}$.

(2) If $\frac{t-r_H\lambda}{p} < \lambda_{s,L} \leq \frac{t-r_H\lambda}{p} + 2\lambda$ and $\lambda_{s,H} \leq \frac{t-r_L\lambda}{p}$, $\lambda_{s,L}{}^{*} = 0$, $\lambda_{s,H}{}^{*} = \frac{2(t\lambda-r_H\lambda^2)}{3t-4V(\lambda)-r_H\lambda}$, and $p^{*} = \frac{4V(\lambda)+r_H\lambda-3t}{4\lambda}$.

(3) If $\lambda_{s,L} > \frac{t-r_H\lambda}{p} + 2\lambda$ and $\lambda_{s,H} \leq \frac{t-r_L\lambda}{p}$, $\lambda_{s,L}{}^{*} = 0$, $\lambda_{s,H}{}^{*} = \lambda$, and $p^{*} = \frac{2V(\lambda)-t}{\lambda}$.

(4) If $\frac{t-r_H\lambda}{p} < \lambda_{s,L} \leq \frac{t-r_H\lambda}{p} + 2\lambda$ and $\frac{t-r_L\lambda}{p} < \lambda_{s,H} \leq \frac{t-r_L\lambda}{p} + 2\lambda$, $\lambda_{s,L}{}^{*} = \frac{2\lambda(-3t+(r_H+2r_L)\lambda)}{3(-3t+2V(\lambda)+(r_H+r_L)\lambda)}$, $\lambda_{s,H}{}^{*} = \frac{2\lambda(-3t+(2r_H+r_L)\lambda)}{3(-3t+2V(\lambda)+(r_H+r_L)\lambda)}$, and $p^{*} = \frac{-3t+2V(\lambda)+(r_H+r_L)\lambda}{2\lambda}$.

(5) If $\lambda_{s,L} > \frac{t-r_H\lambda}{p} + 2\lambda$ and $\frac{t-r_L\lambda}{p} < \lambda_{s,H} \leq \frac{t-r_L\lambda}{p} + 2\lambda$, $\lambda_{s,L}{}^{*} = \frac{\lambda(-2t+2V(\lambda)+r_L\lambda)}{-3t+4V(\lambda)+r_L\lambda}$, $\lambda_{s,H}{}^{*} = \lambda$, and $p^{*} = \frac{4V(\lambda)-3t+r_L\lambda}{\lambda}$. ■

Proof of Proposition 1

Proof: From the results of Proposition 2 (Tables 3, 4, and 5), we see that $\lambda_{s,L}{}^{*} = \lambda$ when r_L is high, and $\lambda_{s,H}{}^{*} = \lambda$ when r_H is high. ■

Proof of Proposition 2

Proof: From Table 5, we can find that under all market conditions, $\lambda_{s,H}{}^{*} \geq \lambda_{s,L}{}^{*}$, in Case 4. ■

Proof of Proposition 3

Proof: From the proofs of Propositions 1 and 2, the result follows. ■

Proof of Proposition 4

Proof: From the proof of Proposition 2, we can see that in Cases 2, 3, and 4, when the fit cost $t \geq \lambda r_H$, ${\lambda_{s,H}}^* = {\lambda_{s,L}}^* = 0$. ■

Proof of Proposition 5

Proof: From the proof of Proposition 2, we know that in Case 3, if $\frac{t}{\lambda} \leq r_H < \frac{2p+t}{\lambda}$, $x^* = \frac{1}{2} - \frac{r_H\lambda - t}{4t}$, and if $r_H \geq \frac{2p+t}{\lambda}$, $x^* = \max\left\{1 - \frac{V(\lambda)}{t}, 0\right\}$. Therefore, if $r_H \geq \frac{3t}{\lambda}$ and $V(\lambda) \geq t$, then $x^* = 0$. ■

Appendix B: Vertical Product Differentiation

Our model can be extended to the vertical differentiation case with $V_L(\lambda)$ and $V_H(\lambda)$ for each CP, and the analytical insights remain the same. We solve Cases 1 and 3 in this appendix, and Cases 2 and 4 are similar: the analysis carries through with one parameter replacing another in the relevant expressions.

In Case 1, we identify the location of marginal consumers who are indifferent between choosing content from L and from H: $U_L(x^*) = U_H(x^*)$, where $U_L(x) = V_L(\lambda) - p\lambda - tx$ and $U_H(x) = V_H(\lambda) - p\lambda - t(1-x)$.

The equilibrium marginal consumer and per-packet fee is given by:

$$x^* = \frac{1}{2t}[V_L(\lambda) - V_H(\lambda)] + \frac{1}{2},$$

and

$$p^* = \frac{1}{2\lambda}[V_L(\lambda) + V_H(\lambda)] - \frac{t}{2\lambda}.$$

In Case 3, the equilibrium marginal consumer is given as follows:

(1) If $r_H < \frac{V_H(\lambda)-V_L(\lambda)+t}{\lambda}$, $x^* = \frac{1}{2t}[V_L(\lambda) - V_H(\lambda)] + \frac{1}{2}$;

(2) If $\frac{t}{\lambda} \leq r_H < \frac{2p+t}{\lambda}$, $x^* = \frac{3}{4} - \frac{r_L\lambda+V_H(\lambda)-V_L(\lambda)}{4t}$;

(3) If $r_H \geq \frac{2p+t}{\lambda}$, $x^* = 1 - \frac{V_H(\lambda)}{t}$.

Therefore, if $r_H \geq \frac{3t+V_L(\lambda)-V_H(\lambda)}{\lambda}$, $x^* = 0$, which means that CP L might be driven out of the market even if $V_L(\lambda) > V_H(\lambda)$.

Appendix C: A Fixed Subscription Fee

In our main model, we adopt the assumption of a linear per-packet fee. In this appendix, we consider a fixed subscription fee F. We solve Case 1 as follows and other cases are similar: the analysis carries through with one parameter replacing another in the relevant expressions. To find out the market share of both CPs, we identify the location of marginal consumers who are indifferent between choosing content from L and from H: $U_L(x^*) = U_H(x^*)$, where $U_L(x) = V(\lambda) - F - tx$ and $U_H(x) = V(\lambda) - F - t(1-x)$. This leads to $x^* = \frac{1}{2}$ and this result implies equal market share for the two CPs. In stage 1, the ISP chooses a fixed fee F to maximize its profit given the participation constraints for consumers and content providers:

$$\max_p \pi_{ISP} = F,$$

s.t.

$$U_L\left(\frac{1}{2}\right) \geq 0, U_H\left(\frac{1}{2}\right) \geq 0.$$

From the constraints, $U_H\left(\frac{1}{2}\right) \geq 0$ and $U_L\left(\frac{1}{2}\right) \geq 0$, we can obtain the upper bound of F, $\frac{2V(\lambda)-t}{2} \geq F$. Therefore, when the constraint is binding, the highest value of the objective function is $\pi_{ISP} = \frac{2V(\lambda)-t}{2}$ where $2V(\lambda)- t \geq 0$.

Appendix D: Extended Model

There are two key features of the extended model: (1) in the previous model, the ISP charges the CPs the same per-packet price when the CPs are allowed to sponsor data traffic. However, in order to maximize its profit, the ISP can implement price discrimination by charging consumers *p* per-packet and CPs *c* per-packet. (2) The per-packet price faced by consumers is fixed at the benchmark case in which neither CPs are allowed to subsidize data traffic. Our justification is that some consumers have long-term contracts with their ISPs, so that a dynamic pricing policy (adjusting the price when CPs are allowed to sponsor data) might not always be feasible.

Case 1 (Benchmark Case). The ISP allows no CPs to subsidize their consumers' data usage. The results of Case 1 in the extended problem are same as the results of Case 1 in the baseline problem.

Case 2. The ISP allows CP L to subsidize the data usage of CP L's consumers. In this case, the timeline of the game is given as follows: in stage 1, the ISP announces *c*, which is the usage-based price for CP *L*; in stage 2, *L* decides the amount of its subsidization, $\lambda_{s,L} \in [0, \lambda]$; finally, in stage 3, the end consumers choose either *L* or *H*. We solve this game-theoretical model using backward induction.

The stage 3 in the extended problem is same as the stage 3 in our baseline problem. The market share for content provider L is $x^* = \frac{1}{2} + \frac{p\lambda_{s,L}}{2t}$. The market share for content provider H is $(1 - x^*) = \frac{1}{2} - \frac{p\lambda_{s,L}}{2t}$. In stage 2, with the assumption, c is given, CP L decides the subsidized amount $\lambda_{s,L}$ to maximize its profit, which is demonstrated with the following equation: $\pi_L = r_L\lambda x^* - c\lambda_{s,L}x^*$. From the first order condition of the profit function, $\frac{\partial \pi_L}{\partial \lambda_{s,L}{}^*} = 0$, we can find the optimal the subsidized amount of CP L, $\lambda_{s,L}{}^* = \frac{-ct + pr_L\lambda}{2cp}$.

We can derive the optimal value of $\lambda_{s,L}{}^*$ based on $\lambda_{s,L} \in [0, \lambda]$

$$\lambda_{s,L}{}^* = \begin{cases} 0, \text{if } \lambda_{s,L}{}^* < 0 \\ \frac{-ct + pr_L\lambda}{2cp}, \text{if } 0 \le \lambda_{s,L}{}^* < \lambda. \\ \lambda, \text{if } \lambda_{s,L}{}^* \ge \lambda \end{cases}$$

In stage 1, the ISP chooses c given the participation constraints:

$$\max_{c} \pi_{ISP} = p\lambda - p\lambda_{s,L}x^* + c\lambda_{s,L}x^*$$

$$s.t.$$

$$U_L(x^*) \ge 0, x^* \in [0, 1]$$

$$U_H(x^*) \ge 0, x^* \in [0, 1]$$

$$\pi_L(\lambda_{s,L}{}^*) \ge 0.$$

We can obtain the optimal c^* for CP L as follows:

$c^* = \frac{3^{1/3}p^2r_L{}^2\lambda^2 - \frac{\left[-9p^3r_L{}^2t^4\lambda^2 + \sqrt{3}\sqrt{p^6r_L{}^4t^6\lambda^4(27t^2 + r_L{}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3r_L{}^2t^4\lambda^2 + \sqrt{3}\sqrt{p^6r_L{}^4t^6\lambda^4(27t^2 + r_L{}^2\lambda^2)}\right]^{1/3}}$. Substituting the optimal value c^*, we find the equilibrium subsidized amount $\lambda_{s,L}{}^* = \frac{-c^*t + pr_L\lambda}{2c^*p}$, and equilibrium market share $x^* = \frac{1}{2} + \frac{p\lambda_{s,L}{}^*}{2t}$ and the constraint $p \le \frac{t}{\lambda_{s,L}{}^*}$ from $0 \le x^* \le 1$.

We summarize these results as follows:

If CP L makes an exclusive contract with ISP to subsidize the data usage of L's consumers,

1) ISP will choose the usage based price c for content provider L:

$$c^* = \frac{3^{1/3}p^2{r_L}^2\lambda^2 - \frac{\left[-9p^3{r_L}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_L}^4t^6\lambda^4(27t^2+{r_L}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3{r_L}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_L}^4t^6\lambda^4(27t^2+{r_L}^2\lambda^2)}\right]^{1/3}}.$$

2) CP L will subsidize its consumers' usage by ${\lambda_{s,L}}^*$.

- If ${\lambda_{s,L}}^* < 0$, ${\lambda_{s,L}}^* = 0$.
- If $0 < {\lambda_{s,L}}^* \leq \lambda$, ${\lambda_{s,L}}^*$ will be subsidized.
- If ${\lambda_{s,L}}^* > \lambda$, ${\lambda_{s,L}}^* = \lambda$.

3) The market share of L is $x^* = \frac{1}{2} + \frac{p{\lambda_{s,L}}^*}{2t}$.

Case 3. The ISP allows CP H to subsidize the data usage of CP H's consumers. In this case, the timeline of the game is given as follows: in stage 1, the ISP announces c, which is the usage-based price for CP H; in stage 2, H decides the amount of its subsidization, $\lambda_{s,H} \in [0, \lambda]$; finally, in stage 3, the end consumers choose either L or H. We solve this game-theoretical model using backward induction.

The stage 3 in the extended problem is the same as the stage 3 in the baseline problem. The market share for content provider L is $x^* = \frac{1}{2} - \frac{p\lambda_{s,H}}{2t}$. The market share for content provider H is $(1 - x^*) = \frac{1}{2} + \frac{p\lambda_{s,H}}{2t}$.

In stage 2, given c, CP H chooses the subsidized amount $\lambda_{s,H}$ to maximize its profit: $\pi_H = r_H\lambda x^* - c\lambda_{s,H}x^*$. From the first order condition of the profit function, $\frac{\partial \pi_H}{\partial {\lambda_{s,H}}^*} = 0$, we can obtain the optimal subsidized amount of CP H:

$$\lambda_{s,H}{}^{*} = \begin{cases} 0, \text{if } \lambda_{s,H}{}^{*} < 0 \\ \frac{-ct+pr_H\lambda}{2cp}, \text{if } 0 \le \lambda_{s,H}{}^{*} < \lambda. \\ \lambda, \text{if } \lambda_{s,H}{}^{*} \ge \lambda \end{cases}$$

In stage 1, the ISP chooses c given the participation constraints:

$$\max_{c} \pi_{ISP} = p\lambda - p\lambda_{s,H}(1 - x^*) + c\lambda_{s,H}(1 - x^*)$$

$$s.t.$$

$$U_L(x^*) \ge 0, x^* \in [0, 1]$$

$$U_H(x^*) \ge 0, x^* \in [0, 1]$$

$$\pi_H(\lambda_{s,H}{}^{*}) \ge 0.$$

We can obtain the optimal c^* for CP H as follows:

$c^* = \frac{3^{1/3}p^2{r_H}^2\lambda^2 - \frac{\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{1/3}}$. Substituting the optimal value c^*, we find the equilibrium subsidized amount $\lambda_{s,H}{}^{*} = \frac{-c^*t+pr_H\lambda}{2c^*p}$, and the equilibrium market share $x^* = \frac{1}{2} - \frac{p\lambda_{s,H}{}^{*}}{2t}$.

We summarize these results as follows:

If the CP H makes an exclusive contract with ISP to subsidize the data usage of H's consumers,

1) ISP will choose the usage based price c for content provider H as

$$c^* = \frac{3^{1/3}p^2{r_H}^2\lambda^2 - \frac{\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{1/3}}.$$

2) Content provider H will subsidize its consumers' usage by $\lambda_{s,H}{}^{*}$.

- If $\lambda_{s,H}{}^{*} < 0$, $\lambda_{s,H}{}^{*} = 0$.

- If $0 < \lambda_{s,H}^* \leq \lambda$, $\lambda_{s,H}^*$ will be subsidized.
- If $\lambda_{s,H}^* > \lambda$, $\lambda_{s,H}^* = \lambda$.

3) The market share of L is $x^* = \frac{1}{2} - \frac{p\lambda_{s,H}^*}{2t}$.

Case 4. ISP allows both CP L and CL H to subsidize the data usage. In this case, the timeline of the game is given as follows: In stage 1, the ISP announces c, which is the usage-based price for CP L; in stage 2, L and H decides the amount of its subsidization, $\lambda_{s,L} \in [0, \lambda]$ and $\lambda_{s,H} \in [0, \lambda]$, respectively; finally, in stage 3, the end consumers choose either L or H. We solve this game-theoretical model using backward induction.

The stage 3 in the extended problem is the same as the stage 3 in the baseline problem. The market share for content provider L is $x^* = \frac{1}{2} + \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t}$, and the market share for content provider H is $(1 - x^*) = \frac{1}{2} - \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t}$.

In stage 2, with the assumption, c is given, CP L and CP H decide the amount of subsidization $\lambda_{s,L}$ and $\lambda_{s,H}$ to maximize their profit, which is demonstrated with the following equations: $\pi_L = r_L \lambda x^* - c\lambda_{s,L} x^*$ and $\pi_H = r_H \lambda x^* - c\lambda_{s,H} x^*$. From the first order conditions of the profit functions, $\frac{\partial \pi_L}{\partial \lambda_{s,L}^*} = 0$ and $\frac{\partial \pi_H}{\partial \lambda_{s,H}^*} = 0$, we can obtain the following reaction functions:

$$\lambda_{s,L}^*(\lambda_{s,H}) = \begin{cases} 0, \text{if } \dfrac{-ct + cp\lambda_{s,H}^*(\lambda_{s,L}) + pr_L\lambda}{2cp} \leq 0 \\ \dfrac{-ct + cp\lambda_{s,H}^*(\lambda_{s,L}) + pr_L\lambda}{2cp}, \quad \text{if } 0 < \dfrac{-ct + cp\lambda_{s,H}^*(\lambda_{s,L}) + pr_L\lambda}{2cp} \leq \lambda \\ \lambda, \text{if } \dfrac{-ct + cp\lambda_{s,H}^*(\lambda_{s,L}) + pr_L\lambda}{2cp} > \lambda \end{cases}$$

$$\lambda_{s,H}{}^{*}(\lambda_{s,L}) = \begin{cases} 0, \text{if } \dfrac{-ct + cp\lambda_{s,L}{}^{*}(\lambda_{s,H}) + p\,r_H\lambda}{2cp} \leq 0 \\ \dfrac{-ct + cp\lambda_{s,L}{}^{*}(\lambda_{s,H}) + p\,r_H\lambda}{2cp}, \text{ if } 0 < \dfrac{-ct + cp\lambda_{s,L}{}^{*}(\lambda_{s,H}) + p\,r_H\lambda}{2cp} \leq \lambda \\ \lambda, \text{if } \dfrac{-ct + cp\lambda_{s,L}{}^{*}(\lambda_{s,H}) + p\,r_H\lambda}{2cp} > \lambda \end{cases}.$$

In stage 1, the ISP chooses *c* given the participation constraints:

$$\max_{c} \pi_{ISP} = p\lambda - p\lambda_{s,L}x^* + c\lambda_{s,L}x^* - p\lambda_{s,H}(1 - x^*) + c\lambda_{s,H}(1 - x^*)$$

$$s.t.$$

$$U_L(x^*) \geq 0,\ U_H(x^*) \geq 0$$

$$0 \leq x^* = \frac{1}{2} + \frac{p(\lambda_{s,L} - \lambda_{s,H})}{2t} \leq 1$$

$$\pi_L(\lambda_{s,L}{}^{*}) \geq 0,\ \pi_H(\lambda_{s,H}{}^{*}) \geq 0.$$

From the participation constraints, we obtain the following three inequalities:

(1) $-t + V(\lambda) + \frac{1}{2}p(-\lambda_{s,L} + 2\lambda_{s,H} - 2\lambda + \frac{r_L\lambda}{c}) \geq 0,$

(2) $\frac{\{t+p[\lambda_{s,L}{}^{*}(\lambda_{s,H})-\lambda_{s,H}{}^{*}(\lambda_{s,L})]\}\{c[t-p\lambda_{s,H}{}^{*}(\lambda_{s,L})]+pr_L\lambda\}}{4pt} \geq 0,$

(3) $\frac{\{t+p[-\lambda_{s,L}{}^{*}(\lambda_{s,H})+\lambda_{s,H}{}^{*}(\lambda_{s,L})]\}\{c[t-p\lambda_{s,L}{}^{*}(\lambda_{s,H})]+pr_H\lambda\}}{4pt} \geq 0.$

We solve the equations of reaction function simultaneously and substitute the equilibrium subsidized amount $\lambda_{s,L}{}^{*}$ and $\lambda_{s,H}{}^{*}$. From the first order condition of the objective function, we find nine equilibria as follow:

(1) If $\frac{-ct+pr_L\lambda}{2cp} \leq 0$ and $\frac{-ct+p\,r_H\lambda}{2cp} \leq 0$, $\lambda_{s,L}{}^{*} = 0$, $\lambda_{s,H}{}^{*} = 0$ (there is no subsidization).

(2) If $\frac{-3ct+p(r_H+2r_L)\lambda}{4cp} \leq 0$ and $0 < \frac{-ct+p\,r_H\lambda}{2cp} \leq \lambda$, $\lambda_{s,L}{}^* = 0$, $\lambda_{s,H}{}^* = \frac{-ct+p\,r_H\lambda}{2cp}$, and

$$c^* = \frac{3^{1/3}p^2{r_H}^2\lambda^2 - \frac{\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3{r_H}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_H}^4t^6\lambda^4(27t^2+{r_H}^2\lambda^2)}\right]^{1/3}}$$

(3) If $\frac{-ct+cp\lambda+pr_L\lambda}{2cp} \leq 0$ and $\frac{-ct+p\,r_H\lambda}{2cp} > \lambda$, $\lambda_{s,L}{}^* = 0$, $\lambda_{s,H}{}^* = \lambda$, and $c^* = \frac{pr_H\lambda}{2\lambda p+t}$

(4) If $0 < \frac{-ct+pr_L\lambda}{2cp} \leq \lambda$ and $\frac{-3ct+p(2r_H+r_L)\lambda}{4cp} \leq 0$, $\lambda_{s,L}{}^* = \frac{-ct+pr_L\lambda}{2cp}$, $\lambda_{s,H}{}^* = 0$, and

$$c^* = \frac{3^{1/3}p^2{r_L}^2\lambda^2 - \frac{\left[-9p^3{r_L}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_L}^4t^6\lambda^4(27t^2+{r_L}^2\lambda^2)}\right]^{2/3}}{t^2}}{3^{2/3}\left[-9p^3{r_L}^2t^4\lambda^2+\sqrt{3}\sqrt{p^6{r_L}^4t^6\lambda^4(27t^2+{r_L}^2\lambda^2)}\right]^{1/3}}.$$

(5) If $0 < \frac{-ct+pr_L\lambda+\frac{1}{3}(-3ct-2pr_H\lambda+pr_L\lambda)}{2cp} \leq \lambda$ and $0 < \frac{-ct+pr_H\lambda+\frac{1}{3}(-3ct-pr_H\lambda+2pr_L\lambda)}{2cp} \leq \lambda$,

$\lambda_{s,L}{}^* = -\frac{3ct+pr_H\lambda-2pr_L\lambda}{3cp}$, $\lambda_{s,H}{}^* = -\frac{3ct+2pr_H\lambda-pr_L\lambda}{3cp}$, and

$c^* = \text{(a)} - \frac{1}{t^2}\left[(b)+\sqrt{6}\sqrt{(c)}\right]^{2/3} / \left\{36^{2/3}\left[(d)+\sqrt{6}\sqrt{(c)}\right]\right\}^{1/3}$ where

(a) $= 6^{1/3}p^2\lambda(9r_Ht - 9r_Lt + {r_H}^2\lambda + 2r_Hr_L\lambda + {r_L}^2\lambda)$,

(b) $= -54p^3{r_H}^2t^4\lambda^2 - 108p^3r_Hr_Lt^4\lambda^2 - 54p^3{r_L}^2t^4\lambda^2$,

(c) $= p^6t^6\lambda^3[486(r_H+r_L)^4t^2\lambda + (9r_Ht - 9r_Lt + {r_H}^2\lambda + 2r_Hr_L\lambda + {r_L}^2\lambda)^3]$,

(d) $= -54p^3{r_H}^2t^4\lambda^2 - 108p^3r_Hr_Lt^4\lambda^2 - 54p^3{r_L}^2t^4\lambda^2$.

(6) If $0 < \frac{-ct+cp\lambda+pr_L\lambda}{2cp} \leq \lambda$ and $\frac{p(2r_H+r_L)\lambda+c(-3t+p\lambda)}{4cp} > \lambda$, $\lambda_{s,L}{}^* = \frac{-ct+cp\lambda+pr_L\lambda}{2cp}$,

$\lambda_{s,H}{}^* = \lambda$, and $c^* = \left\{(a)+(b)+3^{1/3}\left[\text{(c)}+\sqrt{3}\sqrt{(d)(e)}\right]^{2/3}\right\} / \left\{(f)\left[\text{(c)}+\sqrt{3}\sqrt{-(d)(e)}\right]^{1/3}\right\}$ where

(a) $= -3^{2/3}p^2{r_L}^2t^2\lambda^2 + 23^{2/3}p^5r_L\lambda^4$,

(b) $= 3^{2/3}p^4r_L\lambda^3(16t + r_L\lambda) + 23^{2/3}p^3r_Lt\lambda^2(-t + 4r_L\lambda)$,

(c) $= -9p^3{r_L}^2t^4\lambda^2 + 144p^4{r_L}^2t^3\lambda^3 - 558p^5{r_L}^2t^2\lambda^4 - 144p^6{r_L}^2t\lambda^5 - 9p^7{r_L}^2\lambda^6$,

(d) $= -p^6{r_L}^3\lambda^4(-t^2 + 8pt\lambda + p^2\lambda^2)^3$,

(e) $= 8p^3\lambda^2 + 6p{r_L}^2\lambda^2 + {r_L}^3\lambda^2 + 3r_L(9t^2 - 72pt\lambda - 5p^2\lambda^2)$,

(f) $= 3(-t^2 + 8pt\lambda + p^2\lambda^2)$.

(7) If $\frac{-ct+pr_L\lambda}{2cp} > \lambda$ and $\frac{-ct+cp\lambda+p\,r_H\lambda}{2cp} \leq 0$, ${\lambda_{s,L}}^* = \lambda$, ${\lambda_{s,H}}^* = 0$, and $c^* = \frac{p\,r_H\lambda}{(t-p\lambda)}$.

(8) If $\frac{p(r_H+2r_L)\lambda+c(-3t+p\lambda)}{4cp} > \lambda$ and $0 < \frac{-ct+cp\lambda+p\,r_H\lambda}{2cp} \leq \lambda$, ${\lambda_{s,L}}^* = \lambda$, ${\lambda_{s,H}}^* = \frac{-ct+cp\lambda+p\,r_H\lambda}{2cp}$, and $c^* = \left\{(a) + (b) + 3^{1/3}\left[(c) + \sqrt{3}\sqrt{(d)(e)}\right]^{\frac{2}{3}}\right\} / \left\{(f)\left[(c) + \sqrt{3}\sqrt{-(d)(e)}\right]^{\frac{1}{3}}\right\}$ where

(a) $= -3^{2/3}p^2{r_H}^2t^2\lambda^2 + 23^{2/3}p^5r_H\lambda^4$,

(b) $= 3^{2/3}p^4r_H\lambda^3(16t + r_H\lambda) + 23^{2/3}p^3r_Ht\lambda^2(-t + 4r_H\lambda)$,

(c) $= -9p^3{r_H}^2t^4\lambda^2 + 144p^4{r_H}^2t^3\lambda^3 - 558p^5{r_H}^2t^2\lambda^4 - 144p^6{r_H}^2t\lambda^5 - 9p^7{r_H}^2\lambda^6$,

(d) $= -p^6{r_H}^3\lambda^4(-t^2 + 8pt\lambda + p^2\lambda^2)^3$,

(e) $= 8p^3\lambda^2 + 6p{r_H}^2\lambda^2 + {r_H}^3\lambda^2 + 3r_H(9t^2 - 72pt\lambda - 5p^2\lambda^2)$,

(f) $= 3(-t^2 + 8pt\lambda + p^2\lambda^2)$.

(9) If $\frac{-ct+cp\lambda+pr_L\lambda}{2cp} > \lambda$ and $\frac{-ct+cp\lambda+p\,r_H\lambda}{2cp} > \lambda$, ${\lambda_{s,L}}^* = \lambda$, ${\lambda_{s,H}}^* = \lambda$, and $c^* \to \infty$.

From the following results, we can find the differences between our main model and this extended model.

Result D.1 (ISP's Network Management Choices). In the extended model, if the fit cost is low, Case 3 maximizes ISP profit. However, if the fit cost is relatively large, the ISP will choose Case 4. The result is shown in Figure D.1 graphically.

In contrast with our main model, we find that in the extended model where price discrimination can be implemented, the ISP will allow data subsidization in a larger range of market conditions. The intuition is that the ISP can more effectively extract CPs' profits using data subsidization when price discrimination can be implemented.

Result D.2 The usage-based cost *c* that the ISP charges to CPs is greater than the usage-based price *p* for its customers under most of the market conditions. The results are shown in Figures D.2, D.3 and D.4.

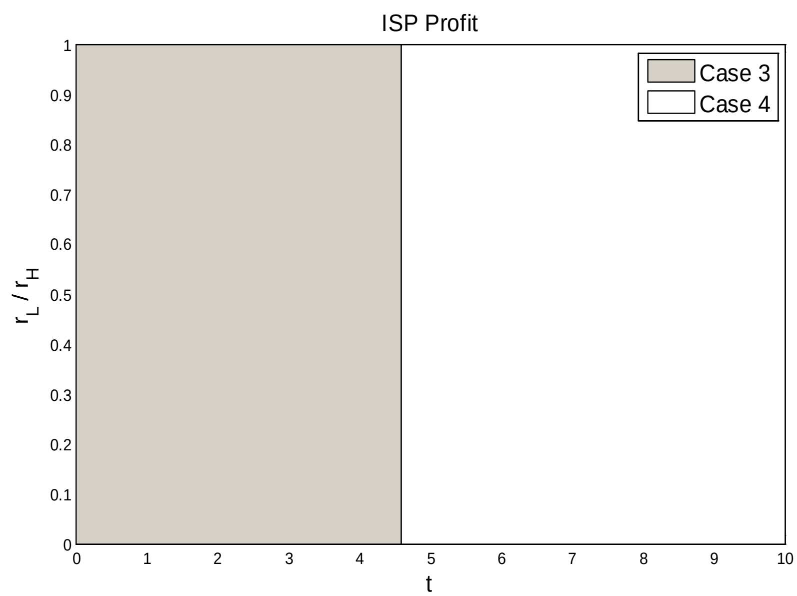

Figure D.1 Internet Service Provider's network management choices in the extended model

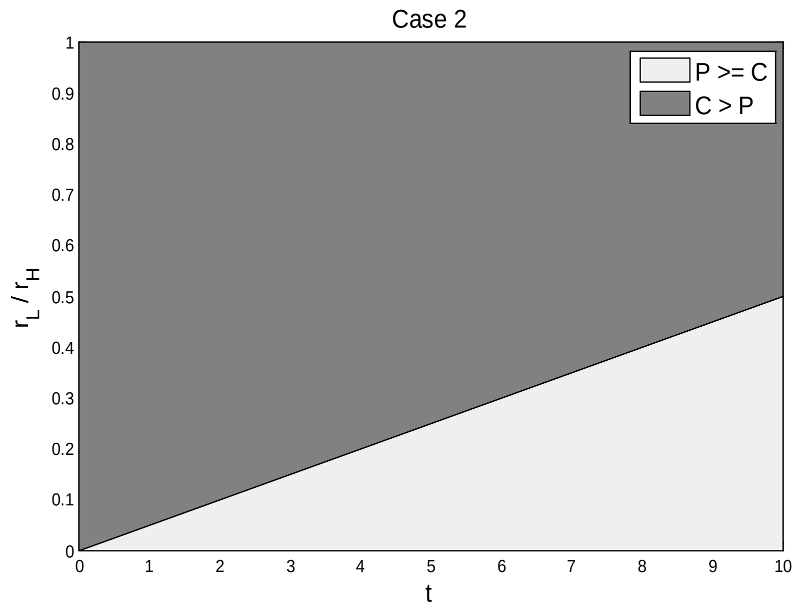

Figure D.2 The usage-based cost *c* that the ISP charges to CPs vs. the usage-based price *p* for its customers in Case 2

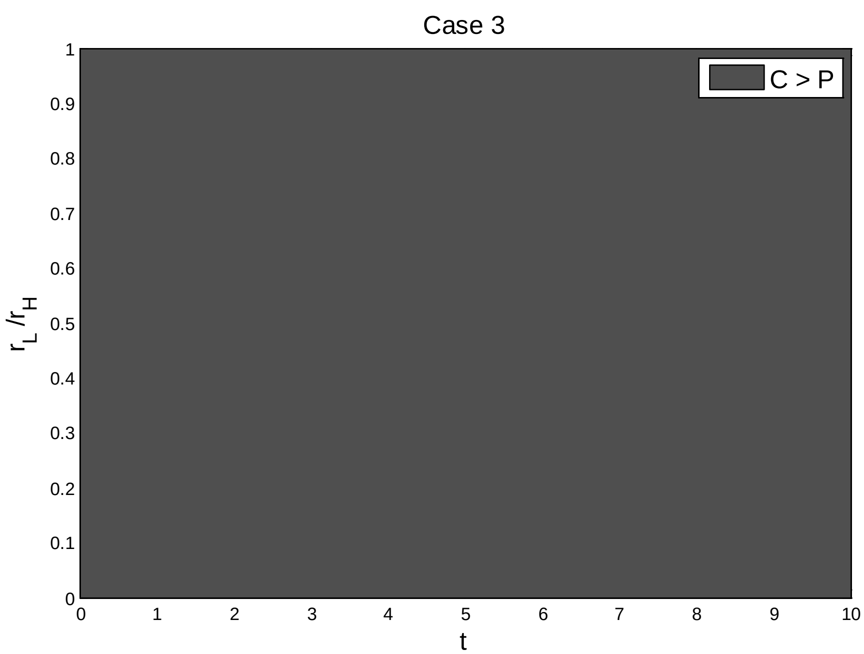

Figure D.3 The usage-based cost *c* that the ISP charges to CPs vs. the usage-based price *p* for its customers in Case 3

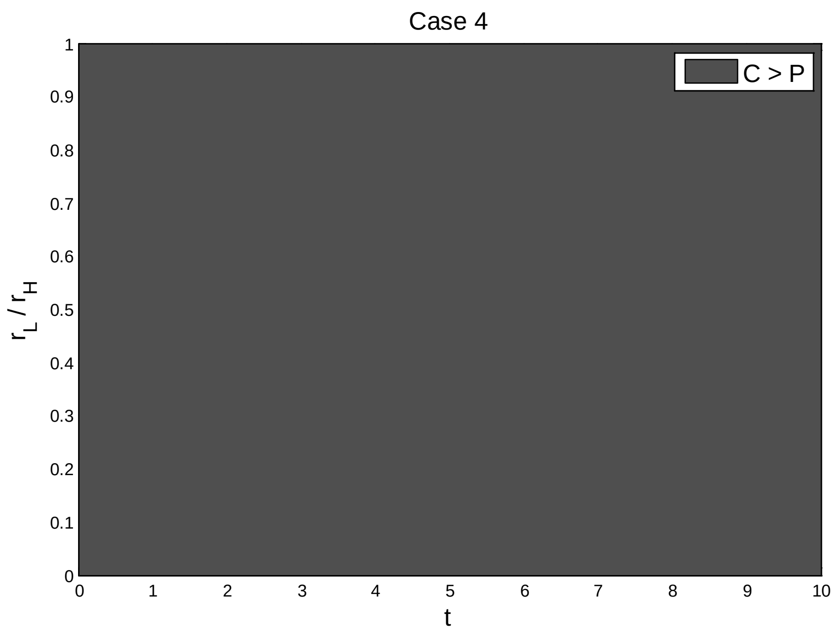

Figure D.4 The usage-based cost *c* that the ISP charges to CPs vs. the usage-based price *p* for its customers in Case 4

From Figures D.2, D.3 and D.4, we can see that in most of the cases, the usage-based cost *c* that the ISP charges to CPs is greater than the usage-based price *p* for its customers.

Appendix E: Institutional Details of Data Subsidization Contracts

In AT&T's sponsored data service, content providers (not the ISPs) sponsor data traffic: AT&T enables content providers to "sponsor the data usage for specific content on behalf of eligible AT&T wireless customers."[1] Under data sponsorship, customers can browse and stream online content without impacting their monthly data plan allowance. AT&T launched data sponsorship service in January, 2014, and this sponsorship scheme won Ovum's Innovative Service of the Award.[2]

[1] http://www.att.com/att/sponsoreddata/en/index.html#tab2.

[2] http://developer.att.com/static-assets/documents/sponsored-data/ovum-att-sponsored-data.pdf.

For content providers who sponsor data traffic, they will have access to billing and analytics information through the AT&T Developer portal atdeveloper.att.com.[3] According to AT&T's description, the contract between ISP (AT&T) and content providers (such as how much content a content provider wants to sponsor) is on a case-by-case basis. A data subsidization contract between ISP and CPs consists of the amount of data content the CPs want to sponsor and data charges billed to the CPs. With the new Sponsored Data service, data charges resulting from eligible uses will be billed directly to the sponsoring company. In reality, different content providers may want to sponsor different amount of data content in their data subsidization contracts. AT&T allows content providers to put their different limits on the amount of sponsored data content they are responsible for. AT&T provides a set of (what they call) Sponsored Data API for content providers to manage their data sponsorship. The developer portal website includes intuitive features which allow content providers to manage their offers, check billing and measure impact of offers using a robust analytics engine. Similarly, in our analytical model, we allow content providers to choose the amount of data content they want to sponsor, which is endogenously determined by market conditions.

From consumers' perspective, when content, such as a video, is sponsored by content providers, they will see an AT&T sponsor icon identifying that the video is sponsored. When the customer clicks the icon to play the video, the data usage incurred while watching the video is not applied to the customer's data allowance.

---

[3] http://developer.att.com/support/faqs/sponsored-data-api-faqs.