Appendix. Proofs of Lemmas and Theorems

Proof of Theorem 1 We consider a special case of our original problem by assuming that the cloud user's probabilistic guarantee is one, the switching cost is zero, e.g., $\beta = 1$ and $S = 0$.

Problem $\mathcal{P}^A$

$$Z^{w_a} = \min_{\mathbf{x}} \quad \sum_{t=1}^{T}\sum_{i=1}^{m} c_i x_{it}$$

subject to

$$\sum_{i=1}^{m} x_{it} \leq 1, \qquad \forall t, \tag{1}$$

$$\mathbb{P}\left[\sum_{t=1}^{T}\sum_{i=1}^{m} \tilde{q}_{it} x_{it} \geq 1\right] \geq 1, \tag{2}$$

$$x_{it} \in \{0,1\} \qquad \forall i, t. \tag{3}$$

Let us introduce problem $\overline{\mathcal{P}}^A$ and show that problem $\mathcal{P}^A$ is equivalent to problem $\overline{\mathcal{P}}^A$. Given any instance of problem $\mathcal{P}^A$, an equivalent instance of problem $\overline{\mathcal{P}}^A$ will have an additional computing resource K_0 with $c_0 = 0$ and $\tilde{q}_{0t} = 0, \forall t$, while hourly rental cost for other computation resources is $c_i, \forall i$ (computing resource K_0 gives no extra cost). Each feasible solution to problem $\mathcal{P}^A$ is feasible solution to problem $\overline{\mathcal{P}}^A$ and vice versa, with the same objective function value, $Z^{w_a} = \overline{Z}^{w_a}$. Thus problem $\mathcal{P}^A$ is equivalent to problem $\overline{\mathcal{P}}^A$.

Problem $\overline{\mathcal{P}}^A$

$$\overline{Z}^{w_a} = \min_{\mathbf{x}} \quad \sum_{t=1}^{T}\sum_{i=0}^{m} c_i x_{it}$$

subject to

$$\sum_{i=0}^{m} x_{it} = 1, \qquad \forall t, \tag{4}$$

$$\mathbb{P}\left[\sum_{t=1}^{T}\sum_{i=0}^{m} \tilde{q}_{it} x_{it} \geq 1\right] \geq 1, \tag{5}$$

$$x_{it} \in \{0,1\} \qquad \forall i, t. \tag{6}$$

Let us assume there are ζ scenarios (very large number) to represent constraint (5). Note that each scenario $\varepsilon = 1, 2, \cdots, \zeta$, should satisfy $\sum_{t=1}^{T}\sum_{i=0}^{m} q_{it}^{\varepsilon} x_{it} \geq 1$ in an optimal solution as $\beta = 1$, where q_{it}^{ε} is fraction of computing task done in the optimal solution x_{it}. The worst case scenario $\hat{\varepsilon}$ occurs with $q_{it}^{\hat{\varepsilon}} = \underline{q}_{it}$ (lower bound) for all i and t. Now it is easy to see that solving problem $\overline{\mathcal{P}}^A$ is equivalent to solving problem $\mathcal{P}^B$ as described below:

Problem $\mathcal{P}^B$

$$Z^{w_b} = \min_{\mathbf{x}} \quad \sum_{t=1}^{T}\sum_{i=0}^{m} c_i x_{it}$$

$$\text{subject to} \qquad (4) \quad \& \quad (6) \tag{7}$$

$$\sum_{t=1}^{T}\sum_{i=0}^{m} \underline{q}_{it} x_{it} \geq 1. \tag{8}$$

Notice that by scaling up every $\underline{q}_{it}$ to $\min\{\psi \underline{q}_{it}, 1\}$, we can replace (8) by a more general form $\sum_{t=1}^{T}\sum_{i=0}^{m} \underline{q}_{it} x_{it} \geq \psi$ for any given ψ. We next prove that the recognition version of problem $\mathcal{P}^B$ is NP-complete.

Du and Leung (1990) has proved the following restricted version of EVEN-ODD PARTITION as NP-complete problem.

RESTRICTED EVEN-ODD PARTITION: Given an even integer number F, a set of $2n$ positive integers $\Lambda = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ such that $a_j < a_{j+1}$ for each $1 \le j < 2n$, $a_{2i} + \rho < a_{2i+1}$, for each $1 \le i < n$, $a_j > n(4n+1)\rho + 5n(a_{2n} - a_1)$ for each $1 \le j \le 2n$, and $\sum_{a_i \in \Lambda} a_i = F$, where $\rho = \sum_{i=1}^{n}(a_{2i} - a_{2i-1})$, is there a partition of Λ into two disjoint subsets Λ_1 and Λ_2 such that $\sum_{a_k \in \Lambda_1} a_k = \sum_{a_j \in \Lambda_2} a_j = \frac{F}{2}$, $\Lambda = \Lambda_1 \cup \Lambda_2$ and that each of Λ_1, Λ_2 contains exactly one of a_{2i-1} and a_{2i} for $i = 1, 2, \ldots, n$?

Given an instance of RESTRICTED EVEN-ODD PARTITION, we construct a specific instance of the decision version of problem $\mathcal{P}^B$ as follows for given computing task J:

- A computing task J must be processed in within period T, $T = n$. There are m computing resources are available, $m = 2n$. The computing resource set Ω, where: $\Omega = \{K_1, K_2, \ldots, K_{2n}\}$.
- $q_{2k-1} = \frac{(L+2a_{2k-1})}{\psi}$, $k = 1, 2, \ldots, n$.
- $q_{2k} = \frac{(L+2a_{2k})}{\psi}$, $k = 1, 2, \ldots, n$, where $\psi = nL + F$ and $L = 3F$.
- $c_{2k-1} = L + 2a_{2k-1}$, $k = 1, 2, \ldots, n$.
- $c_{2k} = L + 2a_{2k}$, $k = 1, 2, \ldots, n$.
- $S = 0$, where S is a switching cost.

Since the complexity theory deals with integer numbers we normalize the ratios to be integers by multiply by an integer number ψ and assume that the computing task content is equal to ψ.

- $\psi q_{2k-1} = (L + 2a_{2k-1})$, $k = 1, 2, \ldots, n$.
- $\psi q_{2k} = (L + 2a_{2k})$, $k = 1, 2, \ldots, n$.

A threshold value, $D = nL + F$. We let $\Gamma_{\phi(J)}$ denote the total computing task content processed by the computing resource sequence $\phi(J)$.

Decision Problem (DQ): Does there exist a schedule $\phi(J)$ such that the total rental cost of $\phi(J)$, $Z_{\phi(J)}$ satisfies $Z_{\phi(J)} \le D$ with $\Gamma_{\phi(J)} \ge \psi$?

It is obvious that the construction of the instance of the Decision Problem, DQ from the arbitrary instance of EVEN-ODD PARTITION can be done in polynomial time in $m = 2n$. The decision problem is clearly in class NP. Also, it is easy to verify that the construction of the decision problem can be done in polynomial time. We now show that there exists a schedule $\phi(J)$ such that $Z_{\phi(J)} \le D$ with $\Gamma_{\phi(J)} \ge \psi$ if and only if there exists a solution to the EVEN-ODD PARTITION problem.

If Part: Suppose there exists a Even-Odd Partition. Without loss of generality, we may assume $a_1 + a_4 + a_5 + \cdots + a_{2n} = a_2 + a_3 + a_6 + \cdots + a_{2n-1} = \frac{F}{2}$. Consider schedule $\phi(J) = (K_{2n}, \ldots, K_5, K_4, K_1)$ corresponding to partition $a_1 + a_4 + a_5 + \cdots + a_{2n} = \frac{F}{2}$. The total rental cost $Z_{\phi(J)}$ can be calculated as below:

$$
\begin{aligned}
Z_{\phi(J)} &= c_{2n} + \cdots + S + c_5 + S + c_4 + S + c_1 \\
&= c_{2n} + \cdots + c_5 + c_4 + c_1 \\
&= L + 2a_{2n} + \ldots L + 2a_5 + L + 2a_4 + L + 2a_1
\end{aligned}
$$

$= nL + F = D$

i.e. $Z_{\phi(J)} \leq D$ holds. In addition, the process of computing task J can be completed in T by using feasible schedule $\phi(J)$. i.e.,

$$\Gamma_{\phi(J)} = \psi q_{2n} + \cdots + \psi q_5 + \psi q_4 + \psi q_1 = (L + a_{2n}) + \cdots + (L + a_5) + (L + a_4) + (L + a_1),$$

$$\Gamma_{\phi(J)} = \psi q_{2n} + \cdots + \psi q_5 + \psi q_4 + \psi q_1 = (nL + F) = \psi.$$

Only If Part: Now we show that if Problem $\mathcal{P}^B$ has a schedule $\phi_0(J)$ with $Z_{\phi_0(J)} \leq D$ with $\Gamma_{\phi_0(J)} \geq \psi$, then Even-Odd Partition has a solution.

We now assume that there exists a schedule $\phi_0(J)$ with $Z_{\phi_0(J)} \leq D$ with $\Gamma_{\phi_0(J)} \geq \psi$. First we show that $\phi_0(J)$ is similar to the structure of the schedule $\phi(J)$ obtained in *If Part*. We characterize the structure of the schedule $\phi_0(J)$ in a series of claims below. Let $\hat{\Omega}$ be the set of computing resources processed computing task J in $\phi_0(J)$. Since $S = 0$, we arrange computing resource in nonincreasing order of computing resource indices in $\phi_0(J)$.

Claim 1 *In $\phi_0(J)$, each period of n periods must be processed by a computing resource and $|\hat{\Omega}| = n$.*

Proof: Suppose computing task J is processed by computing resource sequence $\phi_0(J)$ with less than n period. Then $\Gamma_{\phi_0} \leq (n-1)L + \sum_{k \in \hat{\Omega}} 2\hat{a}_k < nL - L + 2F = nL + F = \psi$. This contradicts with the fact that $\Gamma_{\phi_0(J)} \geq \psi$. Thus, the claim holds true and $|\hat{\Omega}| = n$.

Note that $\Omega = \{K_1, K_2, \ldots, K_{2n}\}$, i.e., $\Omega = \{K_{2k-1}, K_{2k} | k = 1, 2, \ldots, n\}$. Next we prove in Claim 2 that $\hat{\Omega}$ contains exactly one of computing resource pair (K_{2k-1}, K_{2k}) for $k = 1, 2 \ldots, n$, belonging to Ω. The computing resource selected in pair (K_{2k-1}, K_{2k}) for $k = 1, 2 \ldots, n$, for $\hat{\Omega}$ is used only once in ϕ_0.

Claim 2 *$\hat{\Omega}$ contains exactly one of computing resource pair (K_{2k-1}, K_{2k}) for $k = 1, 2 \ldots, n$, belonging to Ω. The computing resource selected in pair (K_{2k-1}, K_{2k}) for $k = 1, 2 \ldots, n$, for $\hat{\Omega}$ is used only once in ϕ_0.*

Proof: From Claim 1, $|\hat{\Omega}| = n$. We assume that r is the first largest index such that either K_{2r+1} or K_{2r+2} belongs to both $\hat{\Omega}$, where $1 \leq r \leq n$. Since $\rho = \sum_{k=1}^{n} (a_{2k} - a_{2k-1})$, $\hat{\Omega}$ cannot contain both K_{2r+1} or K_{2r+2}, for any r, $r = 1, 2 \ldots, n$. Otherwise $\hat{\Omega}$ will satisfy the following relation: Say $\hat{\Omega}$ will have $\sum_{a_k \in \hat{\Omega}} a_k \geq \sum_{k=1}^{n} a_{2k-1} + \rho = \sum_{k=1}^{n} a_{2k} > \frac{F}{2}$. This in turn implies that $Z_{\phi_0(J)} > nL + F = D$. More over, we can show that computing resource K_{2r+2} cannot be used more than once in $\phi_0(J)$. Otherwise $Z_{\phi_0(J)} > nL + F = D$. Similarly we can show that computing resource K_{2r+1} cannot be used more than once in $\phi_0(J)$. Thus $\hat{\Omega}$ cannot contain both (K_{2k+1}, K_{2k+2}) for any k, $k = 1, 2 \ldots, n$, in a feasible schedule ϕ_0.

As a consequence of above claims, $\hat{\Omega}$ contains exactly one computing resource of the pair (K_{2k+1}, K_{2k+2}) for $k = 1, 2 \ldots, n$, and $Z_{\phi_0(J)} \leq nL + \sum_{a_k \in \hat{\Omega}} 2a_k \leq F$. Since $\Gamma_{\phi_0(J)} = nL + \sum_{a_k \in \hat{\Omega}} 2a_k \geq \psi = nL + F$, we have $\sum_{a_k \in \hat{\Omega}} 2a_k = F$. Hence, there is a solution to EVEN-ODD PARTITION with $Z_{\phi_0(J)} = nL + F = D$ and $\Gamma_{\phi_0(J)} = \psi = nL + F$. This completes the proof of Theorem 1. $\square$

Proof of Lemma 1 As we explained in Section 3, we assume a concave relationship between q_i and c_i. Hence, for three computing resources K_{j_1}, K_{j_2}, and K_{j_3}, which $c_{j_1} < c_{j_2} < c_{j_3}$ and $q_{j_1} < q_{j_2} < q_{j_3}$, we have $\frac{q_{j_2} - q_{j_1}}{c_{j_2} - c_{j_1}} \geq \frac{q_{j_3} - q_{j_2}}{c_{j_3} - c_{j_2}}$ and $\frac{q_{j_3} - q_{j_1}}{c_{j_3} - c_{j_1}} \leq \frac{q_{j_2} - q_{j_1}}{c_{j_2} - c_{j_1}}$. Because of the concavity assumption, the intersection points of μ_i in spot of two functions $f_i(\mu)$ and $f_{i+1}(\mu)$ have the following relationship $\mu_0 \leq \mu_1 \leq ... \leq \mu_{m-1}$.

Let us consider interval $[\mu_{i-1}, \mu_i] \quad \forall i \in \{0, ..., m\}$(we consider $\mu_{-1} = 0$ and $\mu_m = \bar{\mu}$), and show that $Z_R^{wo-LR}(\mu) = f_i(\mu)$ for any μ belongs to this interval.

Since point μ_i is the intersection of two functions $f_i(\mu)$ and $f_{i+1}(\mu)$, then $f_i(\mu_i) = f_{i+1}(\mu_i)$ and for any $\mu < \mu_i$, we have

$$f_i(\mu) < f_{i+1}(\mu), \tag{9}$$

and for any $\mu > \mu_i$, we have

$$f_{i+1}(\mu) < f_i(\mu) \tag{10}$$

Therefore, for any $\mu < \mu_l \quad \forall l \in \{i+1, i+2, ..., m\}$, we have

$$f_l(\mu) < f_{l+1}(\mu) \tag{11}$$

and for any $\mu > \mu_j \quad \forall j \in \{i-1, i-2, ..., -1\}$, we have

$$f_{j+1}(\mu) < f_j(\mu) \tag{12}$$

From (9)-(12), we can say

$$\begin{aligned} &f_i(\mu) < f_{i-1}(\mu) < f_{i-2}(\mu) < ... < f_0(\mu) \\ &f_i(\mu) < f_{i+1}(\mu) < f_{i+2}(\mu) < ... < f_m(\mu) \end{aligned} \tag{13}$$

Therefore, we have

$$Z_R^{wo-LR}(\beta, \mu) = f_i(\mu) \quad \forall \quad \mu \in [\mu_{i-1}, \mu_i]$$

□

Proof of Theorem 3 Let us explain the proof of part (iii) where $j = \gamma$. Two first parts can be proven by using the same procedure. Since computing resource K_j is the slowest feasible computing resource or the first linear function with negative slope, and we know $\beta - Tq_1 > \beta - Tq_2 > ... > \beta - Tq_m$, then

$$\beta - Tq_i > 0 \quad \forall i \in \{0, ..., j-1\}$$

$$\beta - Tq_i \leq 0 \quad \forall i \in \{j, ..., m\}$$

Hence, from above findings and Lemma 1, function $Z_R^{wo-LR}(\beta, \mu)$ is increasing in μ for $0 \leq \mu \leq \mu^*$, and it is decreasing in μ for $\mu^* \leq \mu \leq \bar{\mu}$. Therefore, function $Z_R^{wo-LR}(\beta, \mu)$ is maximized at point $\mu^* = \frac{c_j - c_{j-1}}{q_j - q_{j-1}}$ which is the intersection of two functions f_{j-1} and f_j. Therefore,

$$Z_R^{wo-D}(\beta) = Z_R^{wo-LR}(\beta, \mu^*) = \left(\frac{Tq_j - \beta}{q_j - q_{j-1}}\right) c_{j-1} + \left(\frac{\beta - Tq_{j-1}}{q_j - q_{j-1}}\right) c_j$$

Hence, the optimal solution of problem $\mathcal{P}_R^{wo-D}(\beta)$ is to assign computing task to both computing resource K_{j-1} and K_j for T units of time. Function $Z_R^{wo-D}(\beta) = Z_R^{wo-LR}(\beta, \mu^*)$ would be the best lower bound for problem $\mathcal{P}_R^{wo-LP}(\beta)$. For showing the optimum objective value of problem $\mathcal{P}_R^{wo-LP}(\beta)$ is equal to $Z_R^{wo-D}(\beta)$, we need to build a feasible solution for problem $\mathcal{P}_R^{wo-LP}(\beta)$ whose objective function value is equal to $Z_R^{wo-D}(\beta)$. This feasible solution is to assign computing task to computing resource K_{j-1} and K_j for $\left(\dfrac{Tq_j - \beta}{q_j - q_{j-1}}\right)$ and $\left(\dfrac{\beta - Tq_{j-1}}{q_j - q_{j-1}}\right)$ units of time respectively. Therefore, $Z_R^{wo-LP}(\beta) = Z_R^{wo-D}(\beta)$. □

Proof of Lemma 2 Let us explain the proof of part (iii) where $g = \hat{\gamma}$. Two first parts can be proven by using the same procedure. We assume that $a = \dfrac{Tq_g - \beta'}{q_g - q_{g-1}}$ and $b = \dfrac{\beta' - Tq_{g-1}}{q_g - q_{g-1}}$. Then, we can say

$$a + b = T \tag{14}$$

$$aq_{g-1} + bq_g = \beta' \tag{15}$$

Solution explained in this theorem is feasible for problem $\mathcal{P}_R^{wo}(\beta')$ if it satisfies constraint (9) with Right Hand Side (R.H.S) value $\beta = \beta'$. Then, we need to show that

$$\lfloor a \rfloor q_{g-1} + \lceil b \rceil q_g \geq \beta'$$

or from (15), we need to show

$$\lfloor a \rfloor q_{g-1} + \lceil b \rceil q_g \geq aq_{g-1} + bq_g \tag{16}$$

We substitute (14) into (16), then we have

$$\lfloor a \rfloor q_{g-1} + \lceil T - a \rceil q_g \geq aq_{g-1} + (T - a)q_g$$
$$\left(\lceil T - a \rceil - T + a\right)q_g \geq \left(a - \lfloor a \rfloor\right)q_{g-1}$$

Since T is an integer number and from (14), we can say that

$$\lceil T - a \rceil - T = -\lfloor a \rfloor$$

Since $q_g > q_{g-1}$, then we can say

$$\left(\lceil T - a \rceil - T + a\right)q_g \geq \left(a - \lfloor a \rfloor\right)q_{g-1}$$

and, (16) holds. Therefore, the presented solution is a feasible solution for problem $\mathcal{P}_R^{wo}(\beta')$. □

Proof of Lemma 3 Because the size of U is at most $\lfloor 1/\delta \rfloor$, thus it takes at most $O(m^{1/\delta})$ time to enumerate all possible U. For a given U, since $r \leq m$, we can enumerate all possible r in time m. Thus the total time complexity is $O(m^{1/\delta+1})$. We next prove that our solution achieves $(1+\delta)$ approximation ratio. Assume at some stage of our algorithm, we are given the true U and K_r, we have $c_r < \delta Z_R^{wo}(\beta')$ and $O \setminus U$ is a feasible solution to problem $P_R^{wo}(\beta', U, r)$. First of all, we can show that $Z_R^{wo}(\beta', U, r) \leq Z_R^{wo}(\beta')$, this is because $Z_R^{wo}(\beta', U, r) - c(U) \leq c(O \setminus U)$. In addition, because $c_g < \delta Z_R^{wo}(\beta')$, we have $c(H) < Z_R^{wo}(\beta', U, r) - c(U) + \delta Z_R^{wo}(\beta')$. Therefore, $Z_R^{wo}(\beta', U^*, r^*) \leq c(U) + c(H) < Z_R^{wo}(\beta') + \delta Z_R^{wo}(\beta') = (1+\delta) Z_R^{wo}(\beta')$. This finishes the proof. □

Proof of Theorem 4 Let us assume η is the integrality gap of problem $\mathcal{P}_R^{wo-LP}(\beta')$ and δ is an error bound. We first prove that

$$\frac{Z_R^{wo}(\beta',U^*,r^*)}{Z^{wo}(\beta)} \le (1+\delta)\left(\frac{q_j-q_{j-1}}{q_g-q_{g-1}}\right)\left(\frac{\left(Tq_g - z_\beta\sigma\sqrt{T}-1\right)c_{g-1} + \left(z_\beta\sigma\sqrt{T}+1-Tq_{g-1}\right)c_g + \eta\left(q_g-q_{g-1}\right)}{\left(Tq_j-\beta\right)c_{j-1}+\left(\beta-Tq_{j-1}\right)c_j}\right).$$

The objective function value of problem $\mathcal{P}_R^{wo-LP}(\beta)$ plays role a lower-bound for problem $\mathcal{P}^{wo}(\beta)$. Therefore, we can say

$$Z^{wo}(\beta) \ge Z_R^{wo-LP}(\beta)$$

Since $Z_R^{wo-LP}(\beta') = \left(\frac{Tq_g-\beta'}{q_g-q_{g-1}}\right)c_{g-1} + \left(\frac{\beta'-Tq_{g-1}}{q_g-q_{g-1}}\right)c_g$, we have $Z_R^{wo}(\beta') = \left(\frac{Tq_g-\beta'}{q_g-q_{g-1}}\right)c_{g-1} + \left(\frac{\beta'-Tq_{g-1}}{q_g-q_{g-1}}\right)c_g + \eta$. According to Lemma 3, we have $Z_R^{wo}(\beta',U^*,r^*) \le (1+\delta)[\left(\frac{Tq_g-\beta'}{q_g-q_{g-1}}\right)c_{g-1} + \left(\frac{\beta'-Tq_{g-1}}{q_g-q_{g-1}}\right)c_g + \eta]$. Therefore,

$$\frac{Z_R^{wo}(\beta',U^*,r^*)}{Z^{wo}(\beta)} \le \frac{Z_R^{wo}(\beta',U^*,r^*)}{Z_R^{wo-LP}(\beta)} < \frac{(1+\delta)[\left(\frac{Tq_g-\beta'}{q_g-q_{g-1}}\right)c_{g-1} + \left(\frac{\beta'-Tq_{g-1}}{q_g-q_{g-1}}\right)c_g + \eta]}{\left(\frac{Tq_j-\beta}{q_j-q_{j-1}}\right)c_{j-1} + \left(\frac{\beta-Tq_{j-1}}{q_j-q_{j-1}}\right)c_j}$$

$$= (1+\delta)\left(\frac{q_j-q_{j-1}}{q_g-q_{g-1}}\right)\left(\frac{\left(Tq_g - z_\beta\sigma\sqrt{T}-1\right)c_{g-1} + \left(z_\beta\sigma\sqrt{T}+1-Tq_{g-1}\right)c_g + \eta\left(q_g-q_{g-1}\right)}{\left(Tq_j-\beta\right)c_{j-1}+\left(\beta-Tq_{j-1}\right)c_j}\right).$$

□

Proof of Theorem 5 The proof of two first cases is trivial. Let us explain proof of case (iii) and (iv). We assume $j=\gamma$.The optimal solution of problem $\mathcal{P}_R^{w-LP}(\beta)$ can include either a combination of different computing resources (switching between different computing resources happens in the solution) or only one computing resource (switching doesn't happen in the solution).

Let us assume that ϖ is a feasible solution for problem $\mathcal{P}_R^{wo-LP}(\beta)$ which includes more than two different computing resources and its objective function value denotes by $Z_R^{wo-LP^{UB}}(\beta)$. Then, we can say

$$Z_R^{wo-LP}(\beta) \le Z_R^{wo-LP^{UB}}(\beta) \tag{17}$$

Now, we assume there exists constant cost S for switching the computing task between two different computing resources. Hence, the total rental cost of solution ϖ would be $Z_R^{wo-LP^{UB}}(\beta)+n_sS$, where $n_s>1$ is the numbers of time that the computing task has been switched between two different computing resources in solution ϖ. Since only one switching happens in optimal solution of problem $\mathcal{P}_R^{wo-LP}(\beta)$ and from (17), then we have

$$Z_R^{wo-LP}(\beta)+S \le Z_R^{wo-LP^{UB}}(\beta)+n_sS \tag{18}$$

Therefore, we conclude when optimal solution of problem $\mathcal{P}_R^{w-LP}(\beta)$ is a switching solution, it only includes two different resources or one time switching with objective value $Z_R^{w-LP}(\beta) = Z_R^{wo-LP}(\beta)+S$.

Now, we need to show when the optimal solution of problem $\mathcal{P}_R^{w-LP}(\beta)$ is a switching solution.

$$Z_R^{wo-LP}(\beta)+S \le \frac{\beta}{q_j}c_j \tag{19}$$

$$\left(\frac{Tq_j-\beta}{q_j-q_{j-1}}\right)c_{j-1}+\left(\frac{1-Tq_{j-1}}{q_j-q_{j-1}}\right)c_j+S\le\frac{\beta}{q_j}c_j \tag{20}$$

$$S\le\left(\frac{Tq_j-\beta}{q_j-q_{j-1}}\right)\left(\frac{q_{j-1}c_j-q_jc_{j-1}}{q_j}\right) \tag{21}$$

□

Proof of Lemma 5 Similar to the proof of Lemma 2, the total time complexity is $O(m^{1/\delta+1})$. We next prove that our solution achieves $(1+\delta)$ approximation ratio. Assume at some stage of our algorithm, we are given the true U and K_r: if $O\setminus U$ contains only one type of computing resources, e.g., computing resource K_r, then we achieve the optimal solution since we also compute the cost when using K_r only, e.g., $c_r\lceil\frac{\beta'-q(U)}{q_r}\rceil$; if $O\setminus U$ contains at least two computing resources, the switching cost of $O\setminus U$ is at least S, then based on similar proof to Lemma 2, we can prove that our solution achieves $(1+\delta)$ approximation ratio. Therefore, our solution achieves at least $(1+\delta)$ approximation ratio. □

Proof of Theorem 6 The proof is similar to the proof of Theorem 4 except that the $Z_R^{w-LP}(\beta')=\left(\frac{Tq_g-\beta'}{q_g-q_{g-1}}\right)c_{g-1}+\left(\frac{\beta'-Tq_{g-1}}{q_g-q_{g-1}}\right)c_g+I_1S$ and $Z_R^{w-LP}(\beta)=\left(\frac{Tq_j-\beta}{q_j-q_{j-1}}\right)c_{j-1}+\left(\frac{\beta-Tq_{j-1}}{q_j-q_{j-1}}\right)c_j+I_2S.$ □

Proof of Theorem 7 Here we prove that the solution obtained from $\mathrm{EnhancedRounding}^{OC}$ is a feasible solution for the problem under OC dependency structure.

We use the updated expected computing performance of resources calculated in (19) and implement $\mathrm{EnhancedRounding}$ approach to find solution $\mathbf{x}^{\mathrm{ub}}$ for problem $\mathcal{P}_R^w(\beta')$. In solution $\mathbf{x}^{\mathrm{ub}}$, for computing resource K_i scheduled at time unit t, we define two sets A_i and B_i. Set A_i includes all resources which are scheduled in the first $(t-1)$ time units and are weaker than resource K_i. Set B_i is a set of resources which are scheduled in the first $(t-1)$ time units and are stronger than resource K_i. According to OC definition, the updated expected computing performance of resource K_i in solution $\mathbf{x}^{\mathrm{ub}}$ denoted by q_i^{oc} is calculated as

$$q_i^{\mathrm{oc}}=\mathbb{E}\Big[\tilde q_{it}\Big|\max_{j\in A_i}\{\theta_j\}\le\tilde q_{it}\le\min_{j\in B_i}\{\theta_j\}\Big], \tag{22}$$

where θ_j is realization of resource K_j's computing performance. We can rewrite q_i^{oc} as below

$$q_i^{\mathrm{oc}}=\frac{1}{G_i\big(\min\{\overline{q}_i,\min_{j\in B_i}\{\theta_j\}\}\big)-G_i\big(\max\{\underline{q}_i,\max_{j\in A_i}\{\theta_j\}\}\big)}\int_{\max\{\underline{q}_i,\max_{j\in A_i}\{\theta_j\}\}}^{\min\{\overline{q}_i,\min_{j\in B_i}\{\theta_j\}\}}q_{it}g_i(q_{it})dq_{it}.$$

Since $\underline{q}_j\le\theta_j$ and $\underline{q}_j\le\underline{q}_{j+1},\forall j\in B_i$, then we have

$$\min\{\overline{q}_i,\underline{q}_{i+1}\}\le\min\{\overline{q}_i,\min_{j\in B_i}\{\theta_j\}\},\qquad\forall i. \tag{23}$$

Since $qg_i(q_{it})\ge q_{it}g_i(q_{it})$ for any $q_{it}\le q$, then we can say

$$\frac{\partial}{\partial q}\left[\frac{1}{G_i(q)-G_i\big(\max\{\underline{q}_i,\max_{j\in A_i}\{\theta_j\}\}\big)}\int_{\max\{\underline{q}_i,\max_{j\in A_i}\{\theta_j\}\}}^{q}q_{it}g_i(q_{it})dq_{it}\right]\ge0 \tag{24}$$

For any $q_{it}\ge q$, we know that $qg_i(q_{it})\le q_{it}g_i(q_{it})$, hence

$$\frac{\partial}{\partial q}\left[\frac{1}{G_i\big(\min\{\overline{q}_i,\min_{j\in B_i}\{\theta_j\}\}\big)-G_i(q)}\int_{q}^{\min\{\overline{q}_i,\min_{j\in B_i}\{\theta_j\}\}}q_{it}g_i(q_{it})dq_{it}\right]\ge0 \tag{25}$$

From (23)-(25) and the fact that $\underline{q}_i \leq \max\{\underline{q}_i, \max_{j\in A_i}\{\theta_j\}\}$, we conclude

$$q_i^{\text{ub}} \leq q_i^{\text{oc}}, \qquad \forall i.$$

Therefore, for solution $\mathbf{x}^{\text{ub}}$ with components x_{it}^{ub}, we have

$$\sum_{t=1}^{T}\sum_{i=0}^{m} q_i^{\text{oc}} x_{it}^{\text{ub}} \geq \sum_{t=1}^{T}\sum_{i=0}^{m} q_i^{\text{ub}} x_{it}^{\text{ub}} \geq \beta'. \tag{26}$$

Therefore, based on above analysis, the solution produced by EnhancedRounding approach where the expected value of the computing resources' performance are calculated using equation (19) is a feasible solution for problem $\mathcal{P}^w(\beta)$ under OC dependency structure. After pruning solution $\mathbf{x}^{\text{ub}}$, the generated solution is still feasible since the remaining resources are able to complete the process of the computing task before or by deadline.

Proof of Theorem 8 We use the expected computing performance of resources from (20) and find the optimal solution of problem $\mathcal{P}_R^{w-LP}(\beta)$ denoted by $\mathbf{x}^{\text{lb}}$ with components x_{it}^{lb}. Since $\theta_j \leq \overline{q}_j$ and $\overline{q}_{j-1} \leq \overline{q}_j, \forall j \in A_i$, then we have

$$\max\{\underline{q}_i, \max_{j\in A_i}\{\theta_j\}\} \leq \max\{\underline{q}_i, \overline{q}_{i-1}\}, \qquad \forall i. \tag{27}$$

From (24), (25), and (27) and the fact that $\min\{\overline{q}_i, \min_{j\in B_i}\{\theta_j\}\} \leq \overline{q}_i$, we can say

$$q_i^{\text{oc}} \leq q_i^{\text{lb}} \qquad \forall i.$$

Hence, for solution $\mathbf{x}^{\text{lb}}$, we say

$$\sum_{t=1}^{T}\sum_{i=0}^{m} q_i^{\text{lb}} x_{it}^{\text{lb}} \geq \sum_{t=1}^{T}\sum_{i=0}^{m} q_i^{\text{oc}} x_{it}^{\text{lb}} \geq \beta. \tag{28}$$

Therefore, based on above analysis, the solution $\mathbf{x}^{\text{lb}}$ is an infeasible solution for problem $\mathcal{P}_R^{w-LP}(\beta)$ under OC case. Therefore, its optimal objective function value would be smaller than or equal to that of the problem under OC case. Therefore, it can be considered as a lower bound for problem $\mathcal{P}^w(\beta)$ under OC dependency structure.

Appendix. Switching of Resources

All VM instances are launched from a Machine Image (MI) which is a virtual hard disk file containing an operating system, data files, and applications. A MI is saved on a storage volume called a root device volume. A root device volume can be either an *instance* store volume or a *remote* store volume.

An instance store volume provides ephemeral storage and cannot be detached from one VM instance and attached to a new one. Thus, all data saved on instance store volumes persists only during the lifetime of the associated VM instance. If an instance store-backed VM instance stops or terminates, all data in the instance store is lost. Instance store volume costs are included in the hourly rental cost of the VM instance and hence, do not incur any extra rental cost. On the other hand, a remote store volume provides persistent storage and can be detached from one VM instance and attached to a new one. However, cloud users need to pay separately for remote store volumes.

When the cloud users want to switch their computing tasks from one instance store-backed VM instance to another (without any loss of processing), they must create a machine image of their existing VM instance and keep the image in remote store volumes. This step can planned ahead and completed before the scheduled switching time. The new VM instance can be launched from the image and the existing instance (together with the remote store volumes) can be terminated. The above process can be repeated for each switch. Therefore, the switching cost is the cost of renting the remote store volumes and the switching time (the time needed to launch a new instance) is negligible.

For switching the application between two remote store-backed VM instances, the cloud users first need to stop the existing remote store-backed VM instance and change the size of the instance (which can be done quickly by changing its instance type) and then restart the stopped VM instance. Hence, the switching cost is zero and the switching time (the time needed to restart the stopped instance) is negligible. However, in this case, the user must pay for remote storage for the entire duration of processing the job. Thus, while each switch is free, a (single) fixed cost of switching is incurred.

Figure 1 **Instance Store-backed VM Instance vs Remote Store-backed VM Instance**