

ONLINE APPENDIX: Welfare Implications in Intermediary Networks

Thành Nguyen*

Karthik Kannan†

Abstract

We study the welfare implications of competing middlemen in a two-sided market, where goods are intermediated between providers and purchasers. In our model, each intermediary sets the quantities it intermediates, and the prices are a consequence of a Cournot competition. Our analysis shows that, unlike traditional markets, increasing competition is not always beneficial for market efficiency and that mergers can have an ambiguous effect on efficiency. We also analyze how the underlying network influences social welfare. We define parameter w_G as the intermediary capacity of the network G and show that the price of anarchy is at least $1 - \frac{1}{2w_G+1}$. These results suggest an intuitive and simple measure for the level of competitiveness in a networked market involving intermediaries.

A Proof of Theorem 4.1

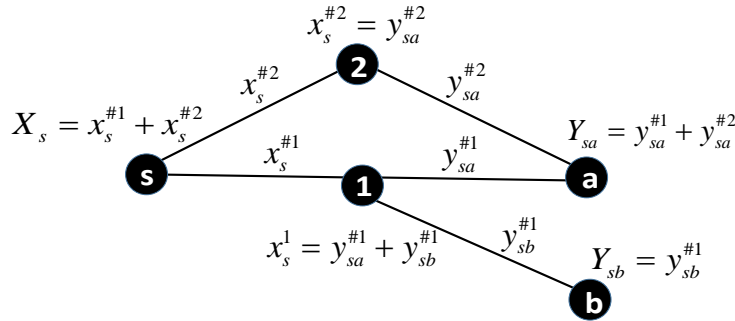


Figure 1: Illustrating example for equilibrium

Proof. We first illustrate the main idea of the proof with an example of the network in Figure 1. The example will also subsequently be used to demonstrate some seemingly counterintuitive results when we later study the sensitivity of the results. As for the example, assume a networked structured as in Figure 1, with two intermediaries ($\#1$ and $\#2$), two purchasers (a and b), and a provider (s).

Readers not interested in the characterization of the proof may skip to the end of the section.

*Krannert School of Management, Purdue University, email: nguy161@purdue.edu

†Krannert School of Management, Purdue University, email: kkarthik@purdue.edu

We will use the optimality conditions of individual players' payoffs to characterize the equilibrium. We already know that, at equilibrium, the inventory constraints imply $x_s^{\#1} = y_{sa}^{\#1} + y_{sb}^{\#1}$ and $x_s^{\#2} = y_{sa}^{\#2}$. Further, Intermediary #2 connects seller s and purchaser a and so $x_s^{\#2} = y_{sa}^{\#2}$. Therefore, we can write the payoff of Intermediary #2 as functions of y :

$$\begin{aligned}
\Pi_2(y) &= (\mu_{sa} - \beta_{sa}Y_{sa})y_{sa}^{\#2} - (\theta_s + \alpha_s X_s)x_s^{\#2} \\
&= (\mu_{sa} - \beta_{sa}Y_{sa})y_{sa}^{\#2} - (\theta_s + \alpha_s X_s)y_{sa}^{\#2} \\
&= (\mu_{sa} - \theta_s)y_{sa}^{\#2} - \beta_{sa}Y_{sa}y_{sa}^{\#2} - \alpha_s X_s y_{sa}^{\#2}.
\end{aligned}$$

Notice that $Y_{sa} = y_{sa}^{\#2} + y_{sa}^{\#1}$ and $X_s = x_s^{\#2} + x_s^{\#1} = y_{sa}^{\#2} + y_{sa}^{\#1} + y_{sb}^{\#1}$, which are functions of $y_{sa}^{\#2}$.

Thus, the derivative of $\Pi_2(y)$ according to $y_{sa}^{\#2}$ is

$$\begin{aligned}
\frac{\partial \Pi_2(y)}{\partial y_{sa}^{\#2}} &= (\mu_{sa} - \theta_s) - \beta_{sa}Y_{sa} - \beta_{sa}y_{sa}^{\#2} \frac{\partial Y_{sa}}{\partial y_{sa}^{\#2}} - \alpha_s X_s - \alpha_s y_{sa}^{\#2} \frac{\partial X_s}{\partial y_{sa}^{\#2}} \\
&= (\mu_{sa} - \theta_s) - \beta_{sa}Y_{sa} - \beta_{sa}y_{sa}^{\#2} - \alpha_s X_s - \alpha_s y_{sa}^{\#2}
\end{aligned}$$

Because $\Pi_2(y)$ is concave, we obtain the following necessary condition for $\Pi_2(y)$ to be optimal:

$$\text{if } y_{sa}^{\#2} > 0 \text{ then } \frac{\partial \Pi_2(y)}{\partial y_{sa}^{\#2}} = 0 \text{ and if } \frac{\partial \Pi_2(y)}{\partial y_{sa}^{\#2}} < 0, \text{ then } y_{sa}^{\#2} = 0. \quad (1)$$

Similarly, for intermediary #1, with $x_s^{\#1} = y_{sa}^{\#1} + y_{sb}^{\#1}$, his payoff is:

$$\begin{aligned}
\Pi_1(y) &= (\mu_{sa} - \beta_{sa}Y_{sa})y_{sa}^{\#1} + (\mu_{sb} - \beta_{sb}Y_{sb})y_{sb}^{\#1} - (\theta_s + \alpha_s X_s)x_s^{\#1} \\
&= (\mu_{sa} - \theta_s)y_{sa}^{\#1} + (\mu_{sb} - \theta_s)y_{sb}^{\#1} - \beta_{sa}Y_{sa}y_{sa}^{\#1} - \beta_{sb}Y_{sb}y_{sb}^{\#1} - \alpha_s X_s (y_{sa}^{\#1} + y_{sb}^{\#1}).
\end{aligned}$$

Taking derivative of Π_1 according to $y_{sa}^{\#1}$ and $y_{sb}^{\#1}$ we have

$$\begin{aligned}
\frac{\partial \Pi_1(y)}{\partial y_{sa}^{\#1}} &= (\mu_{sa} - \theta_s) - \beta_{sa} Y_{sa} - \beta_{sa} y_{sa}^{\#1} \frac{\partial Y_{sa}}{\partial y_{sa}^{\#1}} - \alpha_s X_s - \alpha_s (y_{sa}^{\#1} + y_{sb}^{\#1}) \frac{\partial X_s}{\partial y_{sa}^{\#1}} \\
&= (\mu_{sa} - \theta_s) - \beta_{sa} Y_{sa} - \beta_{sa} y_{sa}^{\#1} - \alpha_s X_s - \alpha_s (y_{sa}^{\#1} + y_{sb}^{\#1}) \\
&= (\mu_{sa} - \theta_s) - \beta_{sa} Y_{sa} - \beta_{sa} y_{sa}^{\#1} - \alpha_s X_s - \alpha_s x_s^{\#1}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Pi_1(y)}{\partial y_{sb}^{\#1}} &= (\mu_{sb} - \theta_s) - \beta_{sb} Y_{sb} - \beta_{sb} y_{sb}^{\#1} \frac{\partial Y_{sb}}{\partial y_{sb}^{\#1}} - \alpha_s X_s - \alpha_s (y_{sa}^{\#1} + y_{sb}^{\#1}) \frac{\partial X_s}{\partial y_{sb}^{\#1}} \\
&= (\mu_{sb} - \theta_s) - \beta_{sb} Y_{sb} - \beta_{sb} y_{sb}^{\#1} - \alpha_s X_s - \alpha_s (y_{sa}^{\#1} + y_{sb}^{\#1}) \\
&= (\mu_{sb} - \theta_s) - \beta_{sb} Y_{sb} - \beta_{sb} y_{sb}^{\#1} - \alpha_s X_s - \alpha_s x_s^{\#1}.
\end{aligned}$$

The first order conditions for $\Pi_1(y)$ to be optimal are

$$\text{if } y_{sa}^{\#1} > 0 \text{ then } \frac{\partial \Pi_1(y)}{\partial y_{sa}^{\#1}} = 0 \text{ and if } \frac{\partial \Pi_1(y)}{\partial y_{sa}^{\#1}} < 0, \text{ then } y_{sa}^{\#1} = 0 \quad (2)$$

$$\text{if } y_{sb}^{\#1} > 0 \text{ then } \frac{\partial \Pi_1(y)}{\partial y_{sb}^{\#1}} = 0 \text{ and if } \frac{\partial \Pi_1(y)}{\partial y_{sb}^{\#1}} < 0, \text{ then } y_{sb}^{\#1} = 0. \quad (3)$$

$$\text{Let } \Omega = \frac{\alpha_s}{2} \left((X_s)^2 + (x_s^{\#2})^2 + (x_s^{\#1})^2 \right) + \frac{\beta_{sa}}{2} \left((Y_{sa})^2 + (y_{sa}^{\#2})^2 + (y_{sa}^{\#1})^2 \right) + \frac{\beta_{sb}}{2} \left((Y_{sb})^2 + (y_{sb}^{\#1})^2 \right).$$

Next, consider the optimal solution of the following quadratic program:

$$\min : \quad \Omega \quad (4)$$

$$s.t : \quad \alpha_s X_s + \alpha_s x_s^{\#2} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#2} \geq \mu_{sa} - \theta_s. \quad (5)$$

$$\alpha_s X_s + \alpha_s x_s^{\#1} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#1} \geq \mu_{sa} - \theta_s. \quad (6)$$

$$\alpha_s X_s + \alpha_s x_s^{\#1} + \beta_{sb} Y_{sb} + \beta_{sb} y_{sb}^{\#1} \geq \mu_{sb} - \theta_s. \quad (7)$$

Let $z_{sa}^{\#2}, z_{sa}^{\#1}, z_{sb}^{\#1}$ be the dual variables of the constraints (5), (6), and (7), respectively. The

Lagrangian relaxation of this convex program is

$$\begin{aligned}
& \Omega - z_{sa}^{\#2}(\alpha_s X_s + \alpha_s x_s^{\#2} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#2} - (\mu_{sa} - \theta_s)) - \\
& z_{sa}^{\#1}(\alpha_s X_s + \alpha_s x_s^{\#1} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#1} - (\mu_{sa} - \theta_s)) - \\
& z_{sb}^{\#1}(\alpha_s X_s + \alpha_s x_s^{\#1} + \beta_{sb} Y_{sb} + \beta_{sb} y_{sb}^{\#1} - (\mu_{sb} - \theta_s)).
\end{aligned} \tag{8}$$

The first order conditions of the Lagrangian relaxation according to X_s give $X_s = z_{sa}^{\#2} + z_{sa}^{\#1} + z_{sb}^{\#1}$; according to $x_s^{\#2}$ give $x_s^{\#2} = z_{sa}^{\#2}$; according to $x_s^{\#1}$ give $x_s^{\#1} = z_{sa}^{\#1} + z_{sb}^{\#1}$; according to Y_{sa} give $Y_{sa} = z_{sa}^{\#2} + z_{sa}^{\#1}$; according to Y_{sb} give $Y_{sb} = z_{sb}^{\#1}$; and according to $y_{sa}^{\#2}$ give $y_{sa}^{\#2} = z_{sa}^{\#2}$; $y_{sa}^{\#1}$ gives $y_{sa}^{\#1} = z_{sa}^{\#1}$; and $y_{sb}^{\#1}$ give $y_{sb}^{\#1} = z_{sb}^{\#1}$.

Replacing z with y , we obtain a set of conditions that are exactly the equations of the game captured in Figure 1. We need to show that the first order conditions of the payoff for intermediaries, (1), (2), and (3) are also satisfied. Consider the first constraint (5). Notice that the KKT conditions of the optimal solution of the convex program (4)-(6) imply that if the dual variable $z_{sa}^{\#2} > 0$, then the constraint binds, that is $\alpha_s X_s + \alpha_s x_s^{\#2} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#2} = \mu_{sa} - \theta_s$. Furthermore, if the constraint does not bind, i.e, $\alpha_s X_s + \alpha_s x_s^{\#2} + \beta_{sa} Y_{sa} + \beta_{sa} y_{sa}^{\#2} > \mu_{sa} - \theta_s$, then $z_{sa}^{\#2} = 0$. Notice that because $z_{sa}^{\#2} = y_{sa}^{\#2}$, this condition is exactly the first order condition of for the payoff of intermediary #1 in (1). Similarly, we can also have the equivalence between the equilibrium condition of intermediary #2, (2) and (3), and the constraints (6) and (7). The example above illustrates the equivalence between the equilibrium condition and the optimality of the quadratic program (4) to (7). The same idea can be generalized to more general networks. In what follows, we provide the formal proof.

The payoff for intermediary i is

$$\begin{aligned}\Pi_i(y) &= \sum_{i,j,k} (\mu_{jk} - \theta_j) y_{jk}^i - \sum_{jk} \beta_{jk} Y_{jk} y_{jk}^i - \sum_{j,k} \alpha_j X_j y_{jk}^i \\ &= \sum_{i,j,k} (\mu_{jk} - \theta_j) y_{jk}^i - \sum_{jk} \beta_{jk} Y_{jk} y_{jk}^i - \sum_j \alpha_j X_j \sum_k y_{jk}^i.\end{aligned}$$

The derivative of $\Pi_i(y)$ with respect to y_{jk}^i is

$$\begin{aligned}\frac{\partial \Pi_i(y)}{\partial y_{jk}^i} &= (\mu_{jk} - \theta_j) - \beta_{j,k} Y_{jk} - \beta_{jk} y_{jk}^i - \alpha_j X_j - \alpha_j \sum_{l \in K} y_{jl}^i \\ &= (\mu_{jk} - \theta_j) - \beta_{j,k} Y_{jk} - \beta_{jk} y_{jk}^i - \alpha_j X_j - \alpha_j x_j^i\end{aligned}$$

The above equation holds because $x_j^i = \sum_{l \in K} y_{jl}^i$. Hence, the first order condition for y to be a Nash equilibrium is the following:

$$(\mu_{jk} - \theta_j) - (\beta_{jk} Y_{jk} + \alpha_j X_j + \alpha_j x_j^i + \beta_{jk} y_{jk}^i) \leq 0; \text{ and equality occurs if } y_{jk}^i > 0. \quad (9)$$

We will show that (9) has a special property that allows us to characterize its unique solution by a quadratic convex program. First, consider the unique solution of (5). By the complementarity slackness condition, x, y, X, Y is the solution if for every i, j, k such that ij and ik are connected in G , there exists a dual variable z_{jk}^i satisfying

$$\frac{\alpha_j}{2} 2X_j = \sum_{ik} \alpha_j z_{jk}^i \quad (10)$$

$$\frac{\alpha_j}{2} 2x_j^i = \sum_k \alpha_j z_{jk}^i \quad (11)$$

$$\frac{\beta_{jk}}{2} 2Y_{jk} = \sum_i \beta_{jk} z_{jk}^i \quad (12)$$

$$\frac{\beta_{jk}}{2} 2y_{jk}^i = (\beta_{jk}) z_{jk}^i. \quad (13)$$

Furthermore,

$$\text{if } \alpha_j X_j + \alpha_j x_j^i + \beta_{jk} Y_{jk} + \beta_{jk} y_{jk}^i > \mu_{jk} - \theta_j, \text{ then } z_{jk}^i = 0. \quad (14)$$

Observe that (13) implies that $z_{jk}^i = y_{jk}^i$. Therefore, from (10), (11), and (12)

$$X_j = \sum_{ik} y_{jk}^i; x_j^i = \sum_k y_{jk}^i \text{ and } Y_{jk} = \sum_i y_{jk}^i.$$

Given this, (14) is equivalent to the first order condition in (9).

To see the reverse, given a \vec{z} satisfying (9), we introduce $z_{jk}^i := y_{jk}^i$; $X_j = \sum_{ik} z_{jk}^i$; $x_j^i = \sum_k z_{jk}^i$ and $Y_{jk} = \sum_i z_{jk}^i$. It is straightforward to see that z, x, y, X, Y satisfy (10-14). Therefore, z, x, y, X, Y are actually the unique solution of the program (5). \square

B Proof of Corollary 5.1, 5.2 and Numerical examples

B.1 Proof of Corollary 5.1

First, if $\mu_b \geq 2\mu_a$, then according to the equilibrium characterization, $y_{sa}^{\#2} = 0$ and intermediary 1 will be the only active players. Thus the welfare of the system is independent of the number of intermediaries $I + 1$.

It remains to consider the case $\mu_a < \mu_b < 2\mu_a$. According to the equilibrium characterization, the amount of goods delivered to a and b are

$$\begin{aligned} Y_{sa} &= y_{sa}^{\#1} + I y_{sa}^{\#2} = \frac{I}{I+2} (2\mu_a - \mu_b) = 1 - 2 \frac{(2\mu_a - \mu_b)}{I+2} \\ Y_{sb} &= \frac{(I+1)\mu_b - I\mu_a}{I+2} = (\mu_b - \mu_a) + \frac{2\mu_a - \mu_b}{I+2}. \end{aligned}$$

The total number of goods delivered is

$$X_s = Y_{sa} + Y_{sb} = \frac{I\mu_a + \mu_b}{I+2} = \mu_a - \frac{2\mu_a - \mu_b}{I+2}$$

Denote $t := \frac{2\mu_a - \mu_b}{I+2}$, the welfare is

$$\begin{aligned} \mu_a Y_{sa} + \mu_b Y_{sb} - \frac{1}{2}(X_s)^2 &= \mu_a(1 - 2t) + \mu_b(\mu_b - \mu_a + t) - \frac{1}{2}(\mu_a - t)^2 \\ &= \mu_a + \mu_b(\mu_b - \mu_a) - \frac{1}{2}\mu_a^2 + (\mu_b - \mu_a)t - \frac{1}{2}t^2 =: f(t). \end{aligned}$$

When I increases from 1 to ∞ , $t = \frac{2\mu_a - \mu_b}{I+2}$ decreases from $t_1 := \frac{2\mu_a - \mu_b}{3}$ to $t_\infty := 0$.

Notice that $f(t)$ is a quadratic function and $f'(\mu_b - \mu_a) = 0$. Thus, if $t_1 \leq \mu_b - \mu_a$, then $f(t)$ decreases if I increases; if $t_1 > \mu_b - \mu_a$, then as I increases $f(t)$ will first increase, then decrease.

Now, $t_1 = \mu_b - \mu_a$ implies $\mu_b = 5/4\mu_a$, which implies our result. \square

We consider two numerical examples where both α, β are positive.

Example 1: $\alpha = 1; \beta = 1; \mu_a = 3; \mu_b = 9$ See **Figure 2**

Without intermediaries #2, 3, ..., $I+1$, intermediary #1 will sell exclusively to purchaser b . $y_{sa}^{\#1} = 0; y_{sb}^{\#1} = 2.25$ and the welfare is 15.1875. With intermediary #2, the equilibrium is $y_{sa}^{\#2} = 0.2; y_{sb}^{\#1} = 0; y_{sb}^{\#1} = 2.2$ and the welfare is 15.08. As I increases, the social welfare decreases.

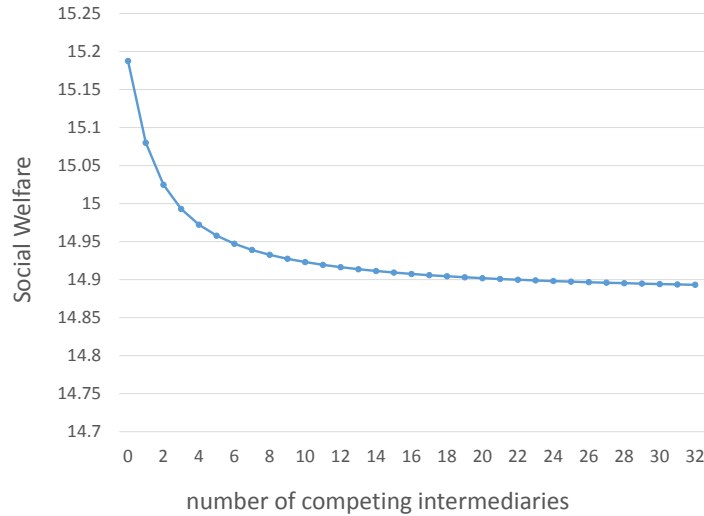


Figure 2: Welfare decreases

Example 2: $\mu_a = 3; \mu_b = 4; \alpha = 1; \beta = 0.1$ See Figure 3

Here, as I increases, the welfare initially increases then decreases below the monopolistic scenario.

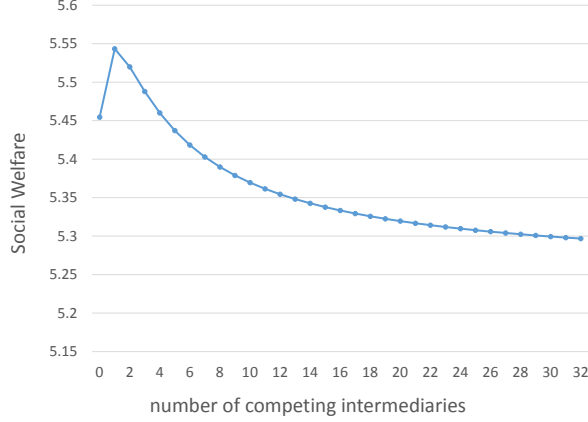


Figure 3: welfare increases then decreases

B.2 Proof of Corollary 5.2

Proof. According to the equilibrium computation (6) and (7), as I increases, the total amount of goods s provides and the amount of goods a receives both increase. This shows that s and b 's payoff increases. The amount of goods delivered to b , on the other hand decreases. As argued above, this is a natural consequence of increasing competition for trade between s and a .

The formula for the payoff of intermediary 1 is $\mu_a y_{sa}^{\#1} + \mu_b y_{sb}^{\#1} - (y_{sa}^{\#1} + y_{sb}^{\#1})(Y_{sa} + Y_{sb})$. Replacing the solution of (6), we obtain that the payoff of intermediary 1 is

$$\left(\mu_b - \mu_a + \frac{2\mu_a - \mu_b}{I + 2}\right)^2.$$

Thus, the payoff of intermediary 1 decreases as I increases. The payoff of each intermediary 2, 3, ..., $I + 1$ is $\mu_a y_{sa}^{\#2} - y_{sa}^{\#2}(Y_{sa} + Y_{sb}) = \left(\frac{2\mu_a - \mu_b}{I + 2}\right)^2$, which also decreases as I increases. \square

C Proof of Theorem 6.2

Proof. Let y_{jk}^i be the Nash equilibrium. The social welfare with y_{jk}^i as the decision is

$$SW_{Nash} = \sum_{jk} (\mu_{jk} \sum_i y_{jk}^i - \frac{\beta_{jk}}{2} Y_{jk}^2) - \sum_j (\theta_j X_j + \frac{\alpha_j}{2} X_j^2).$$

Replacing $X_j = \sum_{i,k} y_{jk}^i$ and $Y_{jk} = \sum_i y_{jk}^i$, we have

$$SW_{Nash} = \sum_{i,j,k} (\mu_{ik} - \theta_j) z_{ik}^i - \sum_{jk} \frac{\beta_{jk}}{2} Y_{jk}^2 - \sum_j \frac{\alpha_j}{2} X_j^2. \quad (15)$$

Let c_{jk}^i be the solution generating the optimal welfare, $A_j = \sum_{i,k} c_{jk}^i$, $B_{jk} = \sum_i c_{jk}^i$ and $c_j^i = \sum_k c_{jk}^i$. The optimal social welfare is

$$SW_{Opt} = \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_{jk} \frac{\beta_{jk}}{2} B_{jk}^2 - \sum_j \frac{\alpha_j}{2} A_j^2. \quad (16)$$

Comparing SW_{Nash} and SW_{Opt} , we observe that because of (9), we have

$$c_{jk}^i ((\mu_{jk} - \theta_j) - (\alpha_j X_j + \alpha_j x_j^i + \beta_{jk} Y_{jk} + \beta_{jk} y_{jk}^i)) \leq 0.$$

Summing over all i, j, k we have

$$\sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \alpha_j A_j X_j - \sum_{ij} \alpha_j x_j^i c_j^i - \sum_{jk} \beta_{jk} B_{jk} Y_{jk} - \sum_{i,j,k} \beta_{jk} c_{jk}^i y_{jk}^i \leq 0. \quad (17)$$

We use the following inequalities¹

$$B_{jk} Y_{jk} \leq \frac{1}{3} B_{jk}^2 + \frac{3}{4} Y_{jk}^2; A_j X_j \leq \frac{1}{3} A_j^2 + \frac{3}{4} X_j^2 \text{ and}$$

¹These inequalities come from the fact that $(\gamma x - \frac{1}{2\gamma} y)^2 \geq 0$ implies $xy \leq \gamma^2 x^2 + \frac{1}{4\gamma^2} y^2$.

$$c_j^i x_j^i \leq \frac{1}{6}(c_j^i)^2 + \frac{3}{2}(x_j^i)^2; \quad c_{jk}^i y_{jk}^i \leq \frac{1}{6}(c_{jk}^i)^2 + \frac{3}{2}(y_{jk}^i)^2$$

to apply to (17) to obtain the following inequality:

$$\begin{aligned} & \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \frac{\alpha_j}{3} A_j^2 - \sum_{ij} \frac{\alpha_j}{6} (c_j^i)^2 - \sum_{jk} \frac{\beta_{jk}}{3} B_{jk}^2 - \sum_{i,j,k} \frac{\beta_{jk}}{6} (c_{jk}^i)^2 \\ & \leq \frac{3}{2} \left(\sum_j \frac{\alpha_j}{2} X_j^2 + \sum_{ij} \alpha_j (x_j^i)^2 + \sum_{jk} \frac{\beta_{jk}}{2} Y_{jk}^2 + \sum_{i,j,k} \beta_{jk} (y_{jk}^i)^2 \right). \end{aligned} \quad (18)$$

To prove the theorem, we will show that the Left hand side of (18) is at least SW_{Opt} and the Right hand side of (18) is $\frac{3}{2}SW_{Nash}$. Focusing on the Left hand side of (18), first observe that

$$\begin{aligned} \sum_{ij} \frac{\alpha_j}{6} (c_j^i)^2 & \leq \sum_j \frac{\alpha_j}{6} \left(\sum_i (c_j^i) \right)^2 = \sum_j \frac{\alpha_j}{6} A_j^2, \text{ and} \\ \sum_{i,j,k} \frac{\beta_{jk}}{6} (c_{jk}^i)^2 & \leq \sum_{jk} \frac{\beta_{jk}}{6} \left(\sum_i c_{jk}^i \right)^2 = \sum_{jk} \frac{\beta_{jk}}{6} B_{jk}^2. \end{aligned}$$

Thus,

$$\text{the left hand side of (18)} \underset{\substack{\geq \\ \text{from (16)}}}{\geq} \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \frac{\alpha_j}{2} A_j^2 - \sum_{jk} \frac{\beta_{jk}}{2} B_{jk}^2 = SW_{Opt}. \quad (19)$$

Next, we will show that the right hand side of (18) is equal to SW_{Nash} . To see this, observe that from the Nash equilibrium condition (9), we have

$$\sum_{i,j,k} y_{jk}^i \cdot \left((\mu_{jk} - \theta_j) - (\alpha_j X_j + \alpha_j x_j^i + \beta_{jk} Y_{jk} + \beta_{jk} y_{jk}^i) \right) = 0.$$

This is equivalent to

$$\sum_{i,j,k} (\mu_{jk} - \theta_j) y_{jk}^i = \sum_j \alpha_j X_j^2 + \sum_{ij} \alpha_j (x_j^i)^2 + \sum_{jk} \beta_{jk} Y_{jk}^2 + \sum_{i,j,k} \beta_{jk} (y_{jk}^i)^2 \quad (20)$$

Replacing $\sum_{i,j,k}(\mu_{jk} - \theta_j)y_{jk}^i$ as in (20) to the formulation of the social welfare at Nash (15),

$$SW_{Nash} = \sum_j \frac{\alpha_j}{2} X_j^2 + \sum_{ij} \alpha_j (x_j^i)^2 + \sum_{jk} \frac{\beta_{jk}}{2} Y_{jk}^2 + \sum_{i,j,k} \beta_{jk} (y_{jk}^i)^2. \quad \square$$

D Proof of Theorem 6.4

Let Y_{jk}^* be the amount of goods k shown on provider j that gives the optimal social welfare $OPT(E_\omega)$ of the ω -subnetwork. That is Y^* maximizes

$$\sum_{j,k} (\mu_{jk} - \theta_j) Y_{jk} - \sum_{jk} \frac{\beta_{jk}}{2} Y_{jk}^2 - \sum_j \frac{\alpha_j}{2} \left(\sum_k Y_{jk} \right)^2,$$

where Y are non-negative and $Y_{jk} = 0$ if j and k are connected by fewer than ω intermediaries.

Denote $X_j^* = \sum_k Y_{jk}^*$. We introduce the following notations. Let ω_{jk} be the number of intermediaries that connects j and k . For any pair j, k that are connected by at least ω intermediaries, let $c_{jk} := \frac{Y_{jk}^*}{\omega_{jk}}$, and let $c_{jk}^i := c_{jk}$ if i is connected to both j and k ; and 0 otherwise. We also denote $c_j^i := \sum_k c_{jk}^i$.

To compare SW_{Nash} and $OPT(E_\omega)$, we observe that because of (9), we have

$$c_{jk}^i ((\mu_{jk} - \theta_j) - (\alpha_j X_j + \alpha_j x_j^i + \beta_{jk} Y_{jk} + \beta_{jk} y_{jk}^i)) \leq 0.$$

Summing over all i, j, k we have

$$\sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \alpha_j X_j^* X_j - \sum_{ij} \alpha_j c_j^i x_j^i - \sum_{jk} \beta_{jk} Y_{jk}^* Y_{jk} - \sum_{i,j,k} \beta_{jk} c_{jk}^i y_{jk}^i \leq 0. \quad (21)$$

We use the following inequalities

$$Y_{jk}^* Y_{jk} \leq \frac{\omega}{2\omega + 1} Y_{jk}^{*2} + \frac{2\omega + 1}{4\omega} Y_{jk}^2; \quad X_j^* X_j \leq \frac{\omega}{2\omega + 1} X_j^{*2} + \frac{2\omega + 1}{4\omega} X_j^2 \text{ and}$$

$$c_j^i y_j^i \leq \frac{\omega}{2(2\omega + 1)} (c_j^i)^2 + \frac{2\omega + 1}{2\omega} (y_j^i)^2; \quad c_{jk}^i y_{jk}^i \leq \frac{\omega}{2(2\omega + 1)} (c_{jk}^i)^2 + \frac{2\omega + 1}{2\omega} (y_{jk}^i)^2$$

to apply to (21) and obtain the following inequality.

$$\begin{aligned} & \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \frac{\omega}{2\omega+1} \alpha_j X_j^{*2} - \sum_{ij} \frac{\omega}{2(2\omega+1)} \alpha_j c_j^{i2} - \sum_{jk} \frac{\omega}{2\omega+1} \beta_{jk} Y_{jk}^{*2} \\ & - \sum_{i,j,k} \frac{\omega}{2(2\omega+1)} \beta_{jk} (c_{jk}^i)^2 \leq \frac{2\omega+1}{2\omega} \left(\sum_j \frac{\alpha_j}{2} X_j^2 + \sum_{ij} \alpha_j (x_j^i)^2 + \sum_{jk} \frac{\beta_{jk}}{2} Y_{jk}^2 + \sum_{i,j,k} \beta_{jk} (y_{jk}^i)^2 \right) \end{aligned} \quad (22)$$

Similar to the proof of Theorem 6.2, the right hand side of (22) is $\frac{2\omega+1}{2\omega} SW_{Nash}$. In order to bound the left hand side of (22), we show the following.

$$\omega \sum_{i,j} (c_j^i)^2 \leq \sum_j (X_j^*)^2. \quad (23)$$

$$\omega \sum_{i,j,k} (c_{jk}^i)^2 \leq \sum_{jk} (Y_{jk}^*)^2. \quad (24)$$

Assuming that (23) and (24) are true then the right hand side of (22) is at least

$$\begin{aligned} & \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_j \frac{\omega}{2\omega+1} \alpha_j X_j^{*2} - \sum_j \frac{1}{2(2\omega+1)} \alpha_j X_j^{*2} - \sum_{jk} \frac{\omega}{2\omega+1} \beta_{jk} Y_{jk}^{*2} \\ & - \sum_{jk} \frac{1}{2(2\omega+1)} \beta_{jk} Y_{jk}^{*2} = \sum_{i,j,k} (\mu_{jk} - \theta_j) c_{jk}^i - \sum_{jk} \frac{1}{2} \beta_{jk} Y_{jk}^{*2} - \sum_j \frac{1}{2} \alpha_j X_j^{*2} = OPT(E_\omega). \end{aligned} \quad (25)$$

Hence, we conclude that $OPT(E_\omega) \leq \frac{2\omega+1}{2\omega} SW_{Nash}$, which is what we need to prove.

Thus, it remains to show that (23) and (24) are true. To see (24), observe that j, k are connected by $\omega_{jk} \geq \omega$ intermediaries, and by definition $c_{jk}^i = \frac{Y_{jk}^*}{\omega_{jk}}$. Thus,

$$Y_{jk}^{*2} = \omega_{jk} \cdot \sum_{i,k} (c_{jk}^i)^2 \geq \omega \cdot \sum_{i,k} (c_{jk}^i)^2.$$

To see (23), observe that

$$(X_j^*)^2 = \left(\sum_k Y_{jk}^* \right)^2 = \left(\sum_k \omega_{jk} \frac{Y_{jk}^*}{\omega_{jk}} \right)^2 = \left(\sum_k \omega_{jk} c_{jk} \right)^2 = \sum_k \omega_{jk}^2 c_{jk}^2 + 2 \sum_{k<l} \omega_{jk} \omega_{jl} c_{jk} c_{jl}. \quad (26)$$

On the other hand,

$$\sum_i (c_j^i)^2 = \sum_i \left(\sum_k c_{jk}^i \right)^2. \quad (27)$$

Notice that $c_{jk}^i = c_{jk}$ if i is connected to both j and k and 0 otherwise. Thus, to bound $\sum_i (c_j^i)^2$, we can bound the number of terms c_{jk}^2 and $c_{jk}c_{jl}$ that appear in (27). There are ω_{jk} intermediaries that connect to both j and k , thus c_{jk}^2 appears exactly ω_{jk} in (27). For $k < l$; the number of terms $c_{jk}c_{jl}$ that appear in (27) is twice the number of intermediaries who is connected to all three nodes j , k and l . Hence the number of terms $c_{jk}c_{jl}$ appearing in (27) is at most $2 \min\{\omega_{jk}, \omega_{jl}\}$. So,

$$\sum_i (c_j^i)^2 = \sum_i \left(\sum_k c_{jk}^i \right)^2 \leq \sum_k \omega_{jk} c_{jk}^2 + 2 \sum_{k < l} \min\{\omega_{jk}, \omega_{jl}\} c_{jk} c_{jl}.$$

Comparing this and (26), we have

$$\sum_i (c_j^i)^2 \leq \omega(X_j^*)^2,$$

which implies (23). With this we finish the proof. \square

E Proof of 7.1

Consider the convex program

$$\min \sum_j \alpha_j (X_j)^2 + \sum_{j,i} \alpha_j (x_j^i)^2 + \sum_k \beta_k (Y_k)^2 + \sum_{i,k} \beta_k (Y_k^i)^2 \quad (28)$$

$$s.t. : \alpha_j X_j + \alpha_j x_j^i + \beta_k Y_k + \beta_k Y_k^i \geq \mu_k - \theta_j - c_{jk} \quad \forall j, k, i \text{ where } i \text{ connects } j \text{ and } k. \quad (29)$$

Let \vec{X}, \vec{Y} be its unique optimal solution. Let z_{jk}^i be the dual variables and set $y_{jk}^i := \frac{z_{jk}^i}{2}$. We will show that \vec{y} together with \vec{X} and \vec{Y} satisfies the equilibrium condition. First, observe that the first

order condition of the convex program implies that

$$X_j = \sum_{i,k} \frac{z_{jk}^i}{2} = \sum_{i,k} y_{jk}^i; \quad Y_k = \sum_{i,j} \frac{z_{jk}^i}{2} = \sum_{i,j} y_{jk}^i;$$

$$X_j^i = \sum_k \frac{z_{jk}^i}{2} = \sum_k y_{jk}^i; \quad Y_k^i = \sum_j \frac{z_{jk}^i}{2} = \sum_j y_{jk}^i.$$

These equations capture exactly the relationship between the quantities of amount of goods traded.

Second, the complementarity condition for an optimal solution of the convex program implies that if (29) holds with strict inequality, then $\frac{z_{jk}^i}{2} = y_{jk}^i = 0$. This condition is exactly the equilibrium condition to guarantee each intermediary maximizes his own payoff.

To see the reverse direction, let \vec{X}, \vec{Y} , and \vec{y} be the quantities traded at equilibrium. Define $z_{jk}^i := 2y_{jk}^i$. We need to show that \vec{X}, \vec{Y} is the optimal solution of the convex program and \vec{z} is the dual. First, the relationships among \vec{X}, \vec{Y} , and \vec{y} implies the exactly the first order condition of the convex program among \vec{X}, \vec{Y} , and \vec{z} . Second, the following equilibrium condition $\alpha_j X_j + \alpha_j x_j^i + \beta_k Y_k + \beta_k Y_k^i \geq \mu_k - \theta_j - c_{jk}$ if strict inequality holds then $y_{jk}^i = 0$ implies the complementarity and slackness condition between the primal \vec{X}, \vec{Y} and the dual \vec{z} . This shows that \vec{X}, \vec{Y} is the unique optimal solution for the convex program (28-29).