Online Supplement to "An Economic Analysis of Rebates Conditional on Positive Reviews"

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A Appendix

A.1 Proof of Lemma 1

Proof. A satisfied consumer posts a positive review if $v + s \geq c$; otherwise, the benefit cannot compensate for the review-posting cost and she does not post.

An unsatisfied consumer posts a negative review only if the review-sharing value, v, is greater than the value of posting a fake positive review, $s - m$, and greater than the review-posting cost c; that is, $v \geq \max\{s - m, c\}$. An unsatisfied consumer posts a fake positive review only if the benefit of doing so, $s - m$, is greater than the value of sharing a true opinion, v, and greater than the review-posting cost c; that is, $s - m \geq \max\{v, c\}$. In other cases, unsatisfied consumers post no reviews. □

A.2 Proof of Proposition 1

Proof. First, the consumers in the first period have the same expected product valuation $x + y$, and thus $p_1^* = x + y$. Similarly, $p_2^* = x + 2\hat{\lambda}y$, where $\hat{\lambda}$ is a function of s. Second, we notice that the intermediate-rebate case with $c \leq s < c + m$ cannot arise in equilibrium. When $s \in [c, c + m)$, by

Equation (3), n_g, n_b, and n_o remain the same for any s, and so do consumers' perceived expected utilities. Therefore, any $s \in (c, c+m)$ is dominated by $s = c$ to maximize the seller's profit. We next consider the low-rebate and high-rebate cases.

Low-Rebate Case ($0 \leq s < c$). By n_g, n_b, and n_o in Equation (4), we have $\hat{\lambda} = \frac{1}{2} + \frac{s}{4}$. By substituting n_g, p_1^*, and p_2^* into Equation (1), we have the seller's profit function:

$$\Pi_l = (x+y) + [x + (1+\tfrac{s}{2})y] - s \cdot \tfrac{1}{2}[1-(c-s)]. \tag{13}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_l}{\partial s} = 0$), we have

$$s_l^* = \tfrac{c+y-1}{2} \text{ and } p_{2l}^* = x + \tfrac{(3+c+y)y}{4}. \tag{14}$$

Next, we check the constraints $0 \leq s < c$. Under the assumption $y \leq 1$, we notice that $s_l^* < c$, and $s_l^* \geq 0$ requires that $c \geq \hat{c} \equiv 1-y$. When $c < \hat{c}$, because of the concavity of Π_l, the optimal rebate is 0. The optimal second-period price is $p_{2n}^* = x+y$, and the optimal profit is $\Pi_n^* = 2(x+y)$. When $c \geq \hat{c}$, s_l^* in Equation (14) is the optimal solution to the constrained optimization problem for the low-rebate case. By substituting s_l^* into Equation (13), we derive the resulting profit as

$$\Pi_l^* = 2(x+y) + \tfrac{(c+y-1)^2}{8} = \Pi_n^* + \tfrac{(c+y-1)^2}{8}. \tag{15}$$

High-Rebate Case ($s \geq c+m$). By n_g, n_b, and n_o in Equation (2), we have $\hat{\lambda} = \frac{1}{2} + \frac{s-m}{2}$. By substituting n_g, p_1^*, and p_2^* into Equation (1), we have the seller's profit function:

$$\Pi_h = (x+y) + [x + (1+s-m)y] - s \cdot \tfrac{1}{2}[1+(s-m)], \tag{16}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_h}{\partial s} = 0$), we have

$$s_h^* = \tfrac{m+2y-1}{2} \text{ and } p_h^* = x + \tfrac{(1-m+2y)y}{2}. \tag{17}$$

Next, we check the constraint $s \geq c+m$. Under the assumption $y \leq 1$, first, we notice that $s_h^* - m = \frac{2y-m-1}{2} \leq 1$, so that n_g, n_b, and n_o in Equation (2) are well behaved. Further, $s_h^* \geq c+m$

if $m \leq \bar{m}(c) \equiv 2y-1-2c$. Accordingly, when $m \leq \bar{m}(c)$, s_h^* in Equation (17) is the optimal solution to the constrained optimization problem for the high-rebate case. By substituting s_h^* into Equation (16), we derive the resulting profit as

$$\Pi_h^* = 2(x+y) + \tfrac{(m+2y-1)^2-8my}{8} = \Pi_n^* + \tfrac{(m+2y-1)^2-8my}{8}, \tag{18}$$

When $m > \bar{m}(c)$, because of the concavity of Π_h, the optimal rebate is $c+m$, leading to the optimal second-period price $p_{2hc}^* = x + (1+c)y$ and the optimal profit as follows:

$$\Pi_{hc}^* = 2(x+y) + \tfrac{2cy-(1+c)(c+m)}{2} = \Pi_n^* + \tfrac{2cy-(1+c)(c+m)}{2}. \tag{19}$$

Globally Optimal Price and Rebate Decisions. We next compare the optimal solutions in the low-rebate and high-rebate cases to derive the globally optimal prices and rebate. Based on Equations (15), (18), and (19), we can derive that $\Pi_n^* \geq \Pi_h^*$ if and only if $m \geq m_1$, $\Pi_n^* \geq \Pi_{hc}^*$ if and only if $m \geq m_2$, $\Pi_l^* \geq \Pi_h^*$ if and only if $m \geq m_3$, and $\Pi_l^* \geq \Pi_{hc}^*$ if and only if $m \geq m_4$, where

$$\begin{cases} m_1 = (\sqrt{2y}-1)^2 \\ m_2 = \frac{(2y-c-1)c}{1+c} \\ m_3 = 1 + 2y - \sqrt{(1-c)^2 + 2(3+c)y + y^2} \\ m_4 = \frac{c(6y-2)-5c^2-(1-y)^2}{4(c+1)} \end{cases} \tag{20}$$

Notice that m_1 and $\bar{m}$ intersect at $\sqrt{2y}-1$, which is less than $\hat{c}$ if and only if $y \leq 3-\sqrt{5}$. Accordingly, we next distinguish two cases:

(a) Case with $y \leq 3-\sqrt{5}$: When $c < \sqrt{2y}-1$ and $m \geq m_1$ or when $c \in [\sqrt{2y}-1, \hat{c}]$ and $m \geq m_2$, $s = 0$ is optimal. When $c > \hat{c}$ and $m \geq m_4$, $s = s_l^*$ is optimal. In other parameter regions, if $m \leq \bar{m}(c)$, $s = s_h^*$ is optimal by the definition of $\bar{m}(c)$, and $s = s_{hc}^*$ is optimal otherwise.

(b) Case with $y > 3-\sqrt{5}$: When $c < \hat{c}$ and $m \geq m_1$, $s = 0$ is optimal. When $c \in [\hat{c}, \check{c}]$ and $m \geq m_3$ or when $c \geq \check{c}$ and $m \geq m_4$, $s = s_l^*$ is optimal, where

$$\check{c} = \tfrac{1}{3}(y - 5 + 2\sqrt{y(y+2)+4}) \tag{21}$$

is the intersection of m_3 and $\bar{m}$. In other parameter regions, if $m \leq \bar{m}(c)$, $s = s_h^*$ is optimal by the definition of $\bar{m}(c)$, and $s = s_{hc}^*$ is optimal otherwise.

Altogether, we can define $\hat{m}(c)$ as follows:

$$\begin{cases} \text{when } y \leq 3-\sqrt{5}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \sqrt{2y}-1 \\ m_2 & \text{if } \sqrt{2y}-1 < c \leq \hat{c} \\ m_4 & \text{if } \hat{c} < c \end{cases} \\ \text{when } y > 3-\sqrt{5}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \hat{c} \\ m_3 & \text{if } \hat{c} < c \leq \check{c} \\ m_4 & \text{if } \check{c} < c \end{cases} \end{cases} \tag{22}$$

where m_i, $i \in \{1,2,3,4\}$, is defined in Equation (20), and $\check{c}$ is defined in Equation (21). Noting whether $\sqrt{2y}-1 \leq \hat{c}$ and whether $\hat{c} \geq \check{c}$ depend on whether $y \leq 3-\sqrt{5}$, we can also organize Equation (22) as Equation (6). □

A.3 Proof of Corollary 1

Proof. (a) By Equations (20) and (22), $\hat{m}(c)$ weakly decreases in c because $\frac{dm_1}{dc} = 0$, $\frac{dm_2}{dc} = \frac{2y}{(1+c)^2} - 1 < 0$ for $c > \sqrt{2y}-1$, $\frac{dm_3}{dc} = \frac{1-c-y}{\sqrt{(1-c)^2+2y(3+c)+y^2}} < 0$ for $c > \hat{c}$, and $\frac{dm_4}{dc} = \frac{-1-5c(2+c)+y(4+y)}{4(c+1)^2} < 0$ for $c > \hat{c}$.

(b) By Equations (20) and (22), if $y \leq \frac{1}{2}$, $\hat{m}(c) \leq 0$ for all $c \in [0,1]$, and thus the seller does not offer a high rebate. If $y > \frac{1}{2}$, because $\hat{m}(0) > 0$ and $\hat{m}(1) < 0$, $\hat{m}(c)$ crosses zero from the positive to the negative side once over $c \in (0,1)$. We distinguish two cases:

(b.1) Case with $y \leq 3-\sqrt{5}$: Notice that m_2 might cross zero at $2y-1$ over its support. Therefore, if $2y-1 \leq \hat{c} = 1-y$, or, equivalently, if $y \leq \frac{2}{3}$, $\hat{m}(c)$ intersects zero at $2y-1$, beyond which the seller does not offer a high rebate. Otherwise, $\hat{m}(c)$ intersects zero at the point satisfying $m_4 = 0$, leading to $\bar{c}$ in the corollary, beyond which the seller does not offer a high rebate.

(b.2) Case with $y > 3-\sqrt{5}$: Although m_3 might cross zero at y, $y > \check{c}$ and it is out of its support. Therefore, $\hat{m}(c)$ intersects zero at the point satisfying $m_4 = 0$, leading to $\bar{c}$ in the corollary, beyond which the seller does not offer a high rebate. □

A.4 Proof of Corollary 2

Proof. (a) For $\hat{c}$ in Proposition 1, we have $\frac{d\hat{c}}{dy} = -1 < 0$.

By Equations (20) and (22), $\hat{m}(c)$ increases in y because $\frac{\partial m_2}{\partial y} = \frac{2c}{c+1} > 0$ and $\frac{\partial m_4}{\partial y} = \frac{3c+1-y}{2(c+1)} > 0$. In addition, m_1 increases in y for $y \geq \frac{1}{2}$. When $y < \frac{1}{2}$, $\sqrt{2y} - 1 < 0$ and m_1 is irrelevant. Further, we can show m_3 increases in y because $\frac{\partial^2 m_3}{\partial y^2} > 0$ and $\frac{\partial m_3}{\partial y}|_{y=3-\sqrt{5}} > 0$.

(b) Notice that the seller does not offer a rebate when $m > \hat{m}(c)$ and $c < \hat{c}$. When y decreases, $\hat{m}(c)$ decreases and $\hat{c}$ increases, and thus the firm is more likely not to offer a rebate, or, equivalently, less likely to offer a rebate.

(c) As in the proof of Corollary 1, if $y \leq \frac{1}{2}$, $\hat{m}(c) \leq 0$ for all $c \in [0, 1]$, so the seller does not offer a high rebate. □

A.5 Proof of Proposition 2

Proof. The equilibrium profits are derived in the proof of Proposition 1 as in Equations (15), (18), and (19). □

A.6 Proof of Proposition 3

Proof. (a) The proof of Proposition 1 guarantees that the equilibrium profit under the conditional-rebate strategy is higher than that in the benchmark case.

(b) In the benchmark, $p^*_{2b} = p^*_{2n} = (x + y)$. By Equation (5), $p^*_2 \geq p^*_{2n}$ because $\hat{\lambda} > \frac{1}{2}$ when rebates are offered.

In addition, by Equation (5), when $m \geq \hat{m}(c)$ and $c > \hat{c}$,

$$(p^*_2 - s^*) - p^*_{2n} = -\frac{(c+y-1)(2-y)}{4} < 0.$$

When $m < \hat{m}(c)$ (which is possible only if $y > \frac{1}{2}$) and $m \leq \bar{m}(c)$,

$$(p^*_2 - s^*) - p^*_{2n} = \frac{(2y-1)(y-1)-m(1+y)}{2} < 0.$$

When $m < \hat{m}(c)$ and $m > \bar{m}(c)$, $(p^*_2 - s^*) - p^*_{2n} = -m + c(-1 + y) < 0$. □

A.7 Proof of Proposition 4

Proof. (a) By Equation (5), $\frac{\partial s^*}{\partial c} = \frac{1}{2} > 0$, $\frac{\partial p_2^*}{\partial c} = \frac{y}{4} > 0$, $\frac{\partial s^*}{\partial m} = 0$, and $\frac{\partial p_2^*}{\partial m} = 0$. By Equation (7), $\frac{\partial \Pi^*}{\partial m} = 0$, and $\frac{\partial \Pi^*}{\partial c} = \frac{c+y-1}{4} > 0$ because $c > \hat{c} = 1 - y$.

(b) By Equation (5), for $m \leq \bar{m}(c)$, $\frac{\partial s^*}{\partial c} = 0$, $\frac{\partial p_2^*}{\partial c} = 0$, $\frac{\partial s^*}{\partial m} = \frac{1}{2} > 0$, and $\frac{\partial p_2^*}{\partial m} = -\frac{y}{2} < 0$; For $m > \bar{m}(c)$, $\frac{\partial s^*}{\partial c} = 1 > 0$, $\frac{\partial p_2^*}{\partial c} = 1 > 0$, $\frac{\partial s^*}{\partial m} = 0$, and $\frac{\partial p_2^*}{\partial m} = 0$. By Equation (7), for $m \leq \bar{m}(c)$, $\frac{\partial \Pi^*}{\partial c} = 0$, and $\frac{\partial \Pi^*}{\partial m} = \frac{m-1-2y}{4} < 0$. For $m > \bar{m}(c)$, $\frac{\partial \Pi^*}{\partial m} = \frac{-1-c}{2} < 0$, and $\frac{\partial \Pi^*}{\partial c} = \frac{2y-(c+m)-(1+c)}{2} < 0$ because $m > \bar{m}(c) = 2y - 1 - 2c$. □

A.8 Proof of Proposition 5

Proof. First, in the first period, the seller considers either charging a low price to sell to all consumers (i.e., the low-price strategy) or charging a high price to sell to those with high digital-attribute value only (i.e., the high-price strategy). The optimal profit under the low-price strategy is y and under the high-price strategy is $\frac{1}{2}(2x + y)$. When $x \geq \frac{y}{2}$, $p_1^* = 2x + y$. Similarly, in the second period, the seller chooses between the low-price strategy with an optimal price $2\hat{\lambda}y$ and the high-price strategy with an optimal price $2x + 2\hat{\lambda}y$. Second, as in the baseline model, the intermediate-rebate case with $c \leq s < c + m$ cannot arise in equilibrium. We next consider the low-rebate and high-rebate cases.

Low-Rebate Case ($0 \leq s < c$). As in the baseline model, we have $\hat{\lambda} = \frac{1}{2} + \frac{s}{4}$. Under the high-price strategy, the seller's profit function is

$$\Pi_l^h = \frac{(2x+y)}{2} + \frac{1}{2}[2x + (1 + \frac{s}{2})y] - s \cdot \frac{1}{4}[1 - (c - s)] \tag{23}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_l^h}{\partial s} = 0$), we have

$$s_l^h = \frac{c+y-1}{2} \text{ and } p_{2l}^h = 2x + \frac{(3+c+y)y}{4} \tag{24}$$

We now check the constraints $0 \leq s < c$. Under the assumption $y \leq 1$, we notice that $s_l^h < c$, and $s_l^h \geq 0$ requires that $c \geq \hat{c} \equiv 1 - y$. When $c < \hat{c}$, because of the concavity of Π_l^h, the optimal rebate is 0. The optimal second-period price is $p_{2n}^* = 2x + y$, and the optimal profit is $\Pi_n^* = 2x + y$. When $c \geq \hat{c}$, s_l^h in Equation (24) is the optimal solution to the constrained optimization problem for the low-rebate high-price case. By substituting s_l^h into Equation (23), we derive the resulting profit as

$$\Pi_l^h = (2x+y) + \tfrac{(c+y-1)^2}{16} = \Pi_n^* + \tfrac{(c+y-1)^2}{16}. \tag{25}$$

Under the low-price strategy, the seller's profit function is

$$\Pi_l^l = \tfrac{(2x+y)}{2} + (1+\tfrac{s}{2})y - \tfrac{s}{4}[1-(c-s)] \tag{26}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_l^l}{\partial s} = 0$), we have

$$s_l^l = \tfrac{c+2y-1}{2} \text{ and } p_{2l}^l = \tfrac{(3+c+2y)y}{4}. \tag{27}$$

We now check the constraints $0 \le s < c$. If $y \ge \frac{1}{2}$, we notice that $s_l^l \ge 0$, and $s_l^l < c$ requires that $c \ge c_l = 2y-1$. When $c < c_l$, because of the concavity of Π_l, the optimal rebate is $s_{lc}^l = c$. The optimal second-period price is $p_{2lc}^l = (1+\frac{c}{2})y$, and the optimal profit is $\Pi_{lc}^l = \frac{2x+3y}{2} + \frac{c(2y-1)}{4}$. If $y < \frac{1}{2}$, we notice that $s_l^l < c$, and $s_l^l > 0$ requires that $c \ge c_l' = 1-2y$. When $c < c_l'$, because of the concavity of Π_l, the optimal rebate is 0. In this case, the low-price strategy is dominated by the high-price strategy. Therefor, the optimal second-period price is $p_{2n}^* = 2x+y$, and the optimal profit is $\Pi_n^* = 2x+y$. When $c \ge c_l$ (if $y \ge \frac{1}{2}$) or when $c \ge c_l'$ (if $y < \frac{1}{2}$), s_l^l in Equation (27) is the optimal solution to the constrained optimization problem for the low-rebate low-price case. By substituting s_l^l into Equation (26), we derive the resulting profit as

$$\Pi_l^l = \tfrac{2x+3y}{2} + \tfrac{(c+2y-1)^2}{16} \tag{28}$$

We notice that $\hat{c} > c_l$ if and only if $y \le \frac{2}{3}$, and $\hat{c} > c_l'$. Comparing the different cases under low rebates, under the assumptions that $x \ge \frac{8y+y^2}{16}$ (when $y \le \frac{2}{3}$) and $x \ge \frac{4y+7y^2}{16}$ (when $y > \frac{2}{3}$), we can summarize the optimal second-period price and rebate as follows:

$$(p_2^*, s^*) = \begin{cases} (2x+y, 0) & \text{if } c \le \hat{c} \\ (2x+y+\frac{(c+y-1)y}{4}, \frac{c+y-1}{2}) & \text{if } \hat{c} < c \le \tilde{c} \\ (y+\frac{(c+2y-1)y}{4}, \frac{c+2y-1}{2}) & \text{if } c > \tilde{c} \end{cases} \tag{29}$$

where $\tilde{c}$ is defined as in the proposition.

High-Rebate Case ($s \geq c+m$). As in the baseline model, we have $\hat{\lambda} = \frac{1}{2} + \frac{s-m}{2}$. We consider the cases that the seller optimally chooses a low-price strategy under high rebates. Under a low-price strategy, the seller's profit function is

$$\Pi_h = \tfrac{2x+y}{2} + (1+s-m)y - s \cdot \tfrac{1}{4}[1+(s-m)], \tag{30}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_h}{\partial s} = 0$), we have

$$s_h^* = \tfrac{m+4y-1}{2} \text{ and } p_h^* = y + \tfrac{(-m+4y-1)y}{2}. \tag{31}$$

We notice that $s_h^* \geq c+m$ if $m \leq \bar{m}(c) \equiv 4y-1-2c$. Accordingly, when $m \leq \bar{m}(c)$, s_h^* in Equation (31) is the optimal solution to the constrained optimization problem for the high-rebate case. By substituting s_h^* into Equation (30), we derive the resulting profit as

$$\Pi_h^* = \tfrac{2x+3y}{2} + \tfrac{(m+4y-1)^2-16my}{16} \tag{32}$$

When $m > \bar{m}(c)$, because of the concavity of Π_h, the optimal rebate is $c+m$, leading to the optimal second-period price $p_{2hc}^* = y + cy$ and the optimal profit as follows:

$$\Pi_{hc}^* = \tfrac{2x+3y}{2} + \tfrac{4cy-(1+c)(c+m)}{4} \tag{33}$$

Globally Optimal Price and Rebate Decisions. We next compare the optimal solutions in the low-rebate and high-rebate cases to derive the globally optimal prices and rebate. Based on Equations (25), (28), (32), and (33), we can derive $\Pi_n^* \geq \Pi_h^*$ if and only if $m \geq m_1$, $\Pi_l^h \geq \Pi_h^*$ if and only if $m \geq m_2$, $\Pi_l^h \geq \Pi_{hc}^*$ if and only if $m \geq m_3$, and $\Pi_l^l \geq \Pi_{hc}^*$ if and only if $m \geq m_4$, where

$$\begin{cases} m_1 = 1+4y-2\sqrt{4x+2y)} \\ m_2 = 1+4y-\sqrt{(c+y-1)^2+16x+8y} \\ m_3 = \frac{c(14y-2)-1-5c^2-16x+(10-y)y}{4(1+c)} \\ m_4 = \frac{c(12y-2)-5c^2-(1-2y)^2}{4(1+c)} \end{cases}$$

Notice that m_4 and $\bar{m}$ intersect at $c^* = \frac{-5+2y+4\sqrt{1+y+y^2}}{3}$, which is less than $\tilde{c}$ if and only if $x > \frac{8y+13y^2+8y\sqrt{1+y+y^2}}{48}$. We can verify that $\frac{4y+7y^2}{16} < \frac{8y+13y^2+8y\sqrt{1+y+y^2}}{48}$. Therefore, under the condition specified in the proposition, when $c < \hat{c}$ and $m \geq m_1$, $s^* = 0$; when $\hat{c} \leq c < \tilde{c}$ and $m \geq \tilde{m}(c)$, $s^* = s_l^h$; when $c > \tilde{c}$ and $m \geq m_4$, $s^* = s_l^l$. In other parameter regions (i.e., if $m < \tilde{m}(c)$), if $m \leq \bar{m}(c)$, $s^* = s_h^*$, and $s^* = s_{hc}^*$ otherwise. We can then similarly define $\hat{m}(c)$ as follows:

$$\hat{m}(c) = \begin{cases} 1+4y-2\sqrt{4x+2y} & \text{if } c \leq \hat{c} \\ 1+4y-\sqrt{(c+y-1)^2+16x+8y} & \text{if } \hat{c} < c \leq c^* \\ \frac{c(14y-2)-1-5c^2+16x+(10-y)y}{4(1+c)} & \text{if } c^* < c \leq \tilde{c} \\ \frac{c(12y-2)-5c^2-(1-2y)^2}{4(1+c)} & \text{if } c > \tilde{c} \end{cases} \tag{34}$$

□

A.9 Proof of Proposition 6

Proof. Under a low rebate or no rebate, social welfare is formulated in Equation (9). Under a high rebate, when consumers with both high and low digital-attribute value purchase the product, social welfare can be formulated as

$$\left[\tfrac{1}{2}(2x+y)+(x+y)\right]+\tfrac{1}{4}\int_0^1(v-c)dv+\tfrac{1}{4}\int_{s-m}^1(v-c)dv+\tfrac{1}{4}\int_0^{s-m}(-c-m)dv \tag{35}$$

Consumer surplus is computed as social welfare minus the seller's profit (as in the proof of Proposition 5). By substituting (p^*, s^*) from Equation (8) into Equations (9) and (35) and by simple algebra, we can derive equilibrium social welfare and consumer surplus, (sw^*, cs^*), as follows.

$$\begin{cases} (2x+y+\frac{(1-c)^2}{4}, \frac{(1-c)^2}{4}) & \text{if } m \geq \hat{m}(c) \text{ and } c \leq \hat{c} \\ (2x+y+\frac{(1-c)^2}{4}-\frac{(c+y-1)^2}{32}, \frac{(1-c)^2}{4}-\frac{3(c+y-1)^2}{32}) & \text{if } m \geq \hat{m}(c) \text{ and } \hat{c} < c \leq \tilde{c} \\ (\frac{7(1-c)^2+64x+4y(13-c-y)}{32}, \frac{5(1-c)^2+32x+12y(1-y-c)}{32}) & \text{if } m \geq \hat{m}(c) \text{ and } c > \tilde{c} \\ (\frac{7-16c+3m^2+2m+64x+8y(7-2y-m)}{32}, \frac{5+6m+m^2-16c+8(m+3)y+32x-48y^2}{32}) & \text{if } m < \hat{m}(c) \text{ and } m \leq \bar{m}(c) \\ (\frac{2(1+8x+6y)-c(4+c+2m)}{8}, \frac{2(1+m+4x)-c(2-c+8y)}{8}) & \text{otherwise} \end{cases}$$

Note that the first case is equivalent to the benchmark case. We use subscript b to denote the results under the benchmark case. The second and third cases are those in which the seller offers a low rebate in equilibrium, and the fourth and fifth are those in which the seller offers a high rebate.

(a) When $m \geq \hat{m}(c)$ and $\hat{c} < c \leq \tilde{c}$, it is easy to see that $sw^* < sw_b^*$ and $cs^* < cs_b^*$.

When $m \geq \hat{m}(c)$ and $c > \tilde{c}$, we have

$$\frac{\partial(sw^* - sw_b^*)}{\partial c} = \frac{1-c-2y}{16} < 0 \text{ and } \frac{\partial(cs^* - cs_b^*)}{\partial c} = -\frac{3(-1+c+2y)}{16} < 0$$

In addition, $(sw^* - sw_b^*)|_{c=1} = \frac{(4-y)y}{8} > 0$, and $(cs^* - cs_b^*)|_{c=1} = \frac{8x-3y^2}{8} > 0$. Therefore, $sw^* > sw_b^*$, and $cs^* > cs_b^*$.

(b) When $m < \hat{m}(c)$ and $m \leq \bar{m}(c)$, we have $\frac{\partial(sw^* - sw_b^*)}{\partial c} = -\frac{c}{2} < 0$, and

$$(sw^* - sw_b^*)|_{c=\bar{m}^{-1}(m)} = \frac{m^2+(8y-2)m-3+8(5-6y)y}{32}$$

We can verify that $(sw^* - sw_b^*)|_{c=\bar{m}^{-1}(m)}$ increases in m in the high-rebate low-price equilibrium and is positive at $m = 0$. Therefore, $sw^* > sw_b^*$. Moreover,

$$cs^* - cs_b^* = \frac{1}{32}\left(-8c^2 + m^2 + m(8y+6) + 32x - 3(1-4y)^2\right)$$

which is positive if and only if $c < \frac{\sqrt{m^2+m(8y+6)+32x-3(1-4y)^2}}{2\sqrt{2}} \equiv \bar{c}_{cs}^*$.

When $m < \hat{m}(c)$ and $m > \bar{m}(c)$, we have $sw^* - sw_b^* = \frac{1}{8}\left(-3c^2 - 2cm + 4y\right)$, which is positive if and only if $c < \frac{-m+\sqrt{m^2+12y}}{3}$. Moreover,

$$cs^* - cs_b^* = \frac{1}{8}[2(m+4x) - c(c+8y-2)]$$

which is positive if and only if $c < \sqrt{2m+8x+(4y-1)^2} - 4y + 1 \equiv \bar{c}_{cs}^{**}$.

We notice that $\bar{c}_{cs}^*$ and $\bar{c}_{cs}^{**}$ intersect with $\bar{m}$ at $1 + 12y - 2\sqrt{8x+16y(y+1)-1}$. Combining the conditions in these two cases, we conclude that $cs^* > cs_b^*$ if and only if $c < \bar{c}_{cs}$, where

$$\bar{c}_{cs} = \begin{cases} \frac{\sqrt{m^2+m(8y+6)+32x-3(1-4y)^2}}{2\sqrt{2}} & \text{if } m < 1 + 12y - 2\sqrt{8x+16y(y+1)-1} \\ \sqrt{2m+8x+(4y-1)^2} - 4y + 1 & \text{otherwise} \end{cases} \quad (36)$$

□

Proof of Proposition 7

Proof. Review-posting behavior of the consumers in the first period remains the same as in the baseline model. We next consider the review-posting behavior of early arrivals in the second period. A satisfied early arrival derives value $v - 2ky(\hat{\lambda} - \frac{1}{2})$ by sharing true opinion with review-posting cost c. Therefore, satisfied early arrivals with $v \geq c + 2ky(\hat{\lambda} - \frac{1}{2})$ post positive reviews, and the others post no reviews. An unsatisfied early arrival derives value $v + 2ky(\hat{\lambda} - \frac{1}{2})$ by sharing true opinions with cost c. Therefore, unsatisfied early arrivals with $v \geq c - 2ky(\hat{\lambda} - \frac{1}{2})$ post negative reviews, and the others post no reviews. As a result, the numbers of early arrivals who post positive, negative, and no reviews are

$$n'_g = \frac{\theta}{2}[1 - (c + 2ky(\hat{\lambda} - \frac{1}{2}))],\ n'_b = \frac{\theta}{2}[1 - (c - 2ky(\hat{\lambda} - \frac{1}{2}))],\ \text{and}\ n'_o = \theta - n'_g - n'_b \qquad (37)$$

The rest of the proof is similar to that of Proposition 1. We next provide a sketch of the proof, highlighting the differences. The seller's optimal first-period price remains the same as $p_1^* = x + y$. We denote $\hat{\lambda}'$ as late arrivals' perceived likelihood of being satisfied. In the second-period, $p_2^* = x + 2\hat{\lambda}' y$, because late arrivals have lower perceived likelihood than early arrivals due to the negative effect of review manipulation.

Low-Rebate Case ($\mathbf{0} \leq s < \mathbf{c}$). As in the baseline model, $\hat{\lambda} = \frac{1}{2} + \frac{s}{4}$. By Equations (4) and (37), we have

$$\hat{\lambda}' = \frac{n_g + n'_g + 0.5(n_o + n'_o)}{1+\theta} = \frac{1}{2} + \frac{s(1-ky\theta)}{4(1+\theta)} = \frac{1}{2} + \frac{\phi s}{4}$$

where $\phi \equiv \frac{1-ky\theta}{1+\theta}$. We thus have the seller's profit function:

$$\Pi_l = (x+y) + [x + (1 + \frac{\phi s}{2})y] - s \cdot \frac{1}{2}[1 - (c - s)] \qquad (38)$$

By the first-order condition, we have

$$s_l^* = \frac{c + \phi y - 1}{2} \text{ and } p_{2l}^* = x + y + \frac{(-1 + c + \phi y)\phi y}{4} \qquad (39)$$

We can similarly derive $\hat{c} \equiv 1 - \phi y$ which segments the regions with low rebates and no rebates. If $c < \hat{c}$, the optimal rebate is 0, $p_{2n}^* = x + y$, and $\Pi_n^* = 2(x + y)$. If $c \geq \hat{c}$, s_l^* in Equation (39) is the optimal rebate, resulting in the optimal profit

$$\Pi_l^* = 2(x + y) + \tfrac{(c+\phi y-1)^2}{8} = \Pi_n^* + \tfrac{(c+\phi y-1)^2}{8} \tag{40}$$

High-Rebate Case ($s \geq c + m$). As in the baseline model, $\hat{\lambda} = \frac{1}{2} + \frac{s-m}{2}$. By Equations (4) and (37), we have

$$\hat{\lambda}' = \tfrac{n_g+n_g'+0.5(n_o+n_o')}{1+\theta} = \tfrac{1}{2} + \tfrac{(1-ky\theta)(s-m)}{2(1+\theta)} = \tfrac{1}{2} + \tfrac{\phi(s-m)}{2}$$

We thus have the seller's profit function

$$\Pi_h = (x + y) + [x + y + \phi(s - m)y] - s \cdot \tfrac{1}{2}[1 + (s - m)] \tag{41}$$

By the first-order condition, we have

$$s_h^* = \tfrac{m+2\phi y-1}{2} \text{ and } p_h^* = x + y + \tfrac{(-m+2\phi y-1)\phi y}{2} \tag{42}$$

We can similarly derive $\bar{m}(c) \equiv 2\phi y - 1 - 2c$ which segments the regions with a corner solution and a interior solution. When $m \leq \bar{m}(c)$, s_h^* in Equation (42) is the optimal rebate, resulting in the optimal profit with the interior solution

$$\Pi_h^* = 2(x + y) + \tfrac{(m+2\phi y-1)^2-8m\phi y}{8} = \Pi_n^* + \tfrac{(m+2\phi y-1)^2-8m\phi y}{8} \tag{43}$$

When $m > \bar{m}(c)$, the optimal rebate is $c + m$, leading to the optimal second-period price $p_{2hc}^* = x + y + c\phi y$ and the optimal profit with the corner solution

$$\Pi_{hc}^* = 2(x + y) + \tfrac{2c\phi y-(1+c)(c+m)}{2} = \Pi_n^* + \tfrac{2c\phi y-(1+c)(c+m)}{2} \tag{44}$$

Globally Optimal Price and Rebate Decisions. We can derive that $\Pi_n^* \geq \Pi_h^*$ if and only if $m \geq m_1$, $\Pi_n^* \geq \Pi_{hc}^*$ if and only if $m \geq m_2$, $\Pi_l^* \geq \Pi_h^*$ if and only if $m \geq m_3$, and $\Pi_l^* \geq \Pi_{hc}^*$ if and only if $m \geq m_4$, where

$$\begin{cases} m_1 = (\sqrt{2\phi y} - 1)^2 \\ m_2 = \frac{(2\phi y - c - 1)c}{1+c} \\ m_3 = 1 + 2\phi y - \sqrt{(1-c)^2 + 2(3+c)\phi y + \phi^2 y^2} \\ m_4 = \frac{c(6\phi y - 2) - 5c^2 - (1-\phi y)^2}{4(c+1)} \end{cases} \tag{45}$$

We can then similarly define $\hat{m}(c)$ as follows:

$$\begin{cases} \text{when } y \leq \frac{3-\sqrt{5}}{\phi}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \sqrt{2\phi y} - 1 \\ m_2 & \text{if } \sqrt{2\phi y} - 1 < c \leq \hat{c} \\ m_4 & \text{if } \hat{c} < c, \end{cases} \\ \text{when } y > \frac{3-\sqrt{5}}{\phi}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \hat{c} \\ m_3 & \text{if } \hat{c} < c \leq \frac{1}{3}(\phi y - 5 + 2\sqrt{\phi y(\phi y + 2) + 4}) \\ m_4 & \text{if } \frac{1}{3}(\phi y - 5 + 2\sqrt{\phi y(\phi y + 2) + 4}) < c. \end{cases} \end{cases} \tag{46}$$

□

Proof of Proposition 8

Proof. The proof is similar to that of Proposition 7. Note that $\hat{\lambda}$ in Equation (11) can be simplified to $\hat{\lambda} = \frac{\alpha(n_g + n_o/2)}{n_g + n_b + n_o} + \frac{1-\alpha}{2}$. In the low-rebate and high-rebate cases, we have $\hat{\lambda} = \frac{1}{2} + \frac{\alpha s}{4}$ and $\hat{\lambda} = \frac{1}{2} + \frac{\alpha(s-m)}{2}$, respectively. Therefore, $\hat{\lambda}$ in this case differs from $\hat{\lambda}'$ in the proof of Proposition 7 only in the coefficient α. By replacing ϕ in the proof of Proposition 7 with α, we have the equilibrium results. We can similarly define $\hat{m}(c)$ as follows:

$$\begin{cases} \text{when } y \le \frac{3-\sqrt{5}}{\alpha}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \le \sqrt{2\alpha y} - 1 \\ m_2 & \text{if } \sqrt{2\alpha y} - 1 < c \le \hat{c} \\ m_4 & \text{if } \hat{c} < c, \end{cases} \\ \text{when } y > \frac{3-\sqrt{5}}{\alpha}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \le \hat{c} \\ m_3 & \text{if } \hat{c} < c \le \frac{1}{3}(\alpha y - 5 + 2\sqrt{\alpha y(\alpha y + 2) + 4}) \\ m_4 & \text{if } \frac{1}{3}(\alpha y - 5 + 2\sqrt{\alpha y(\alpha y + 2) + 4}) < c. \end{cases} \end{cases} \tag{47}$$

□

Proof of Proposition 9

Proof. As in the baseline model, $p_1^* = x + 2\lambda y$ and $p_2^* = x + 2\hat{\lambda} y$, where $\hat{\lambda}$ is a function of s. We next consider the low-rebate and high-rebate cases.

Low-Rebate Case ($0 \le s < c$). As in the baseline model, we have $\hat{\lambda} = \lambda(1 + s - \lambda s)$, and the seller's profit function is

$$\Pi_l = (x + 2\lambda y) + [x + 2\lambda(1 + s - \lambda s)y] - s \cdot \lambda[1 - (c - s)] \tag{48}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_l}{\partial s} = 0$), we have

$$s_l^* = \frac{c + 2(1-\lambda)y - 1}{2} \text{ and } p_{2l}^* = x + \lambda y[1 - c\lambda + c + \lambda + 2(1-\lambda)^2 y] \tag{49}$$

Similar to that in the baseline model, $s_l^* < c$ under the assumption $\lambda \ge \frac{1}{3}$, and $s_l^* \ge 0$ requires that $c \ge \hat{c} \equiv 1 - 2(1-\lambda)y$. When $c < \hat{c}$, the optimal rebate is 0, the optimal second-period price $p_{2n}^* = x + 2\lambda y$, and the optimal profit is $\Pi_n^* = 2x + 4\lambda y$. When $c \ge \hat{c}$, s_l^* in Equation (49) is the optimal solution to the constrained optimization problem for the low-rebate case. By substituting s_l^* into Equation (48), we derive the resulting profit as

$$\Pi_l^* = 2x + 4\lambda y + \frac{\lambda}{4}[c + 2(1-\lambda)y - 1]^2 \tag{50}$$

High-Rebate Case ($s \geq c + m$). As in the baseline model, we have $\hat{\lambda} = \lambda + (1-\lambda)(s-m)$, and the seller's profit function is

$$\Pi_h = (x + 2\lambda y) + [x + 2\lambda y + 2(1-\lambda)(s-m)y] - s \cdot [\lambda + (1-\lambda)(s-m)] \tag{51}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_h}{\partial s} = 0$), we have

$$s_h^* = \tfrac{m+2y+1}{2} - \tfrac{1}{2(1-\lambda)} \text{ and } p_h^* = x + y[\lambda + (1-\lambda)(2y-m)] \tag{52}$$

Similar to that in the baseline model, $s_h^* \geq c + m$ if $m \leq \bar{m}(c) \equiv 2y + 1 - 2c - \frac{1}{1-\lambda}$. When $m \leq \bar{m}(c)$, s_h^* in Equation (52) is the optimal solution to the constrained optimization problem for the high-rebate case. By substituting s_h^* into Equation (51), we derive the resulting profit as

$$\Pi_h^* = 2x + 2\lambda y + \tfrac{1}{4}[\tfrac{\lambda}{1-\lambda} + (m-2y)^2 - \lambda(m-2y+1)^2] \tag{53}$$

When $m > \bar{m}(c)$, the optimal rebate is $c + m$, leading to the optimal second-period price $p_{2hc}^* = x + 2(c + \lambda - \lambda c)y$ and the optimal profit as follows:

$$\Pi_{hc}^* = 2x + 2\lambda y - [\lambda + (1-\lambda)c](c + m - 2y) \tag{54}$$

Globally Optimal Price and Rebate Decisions. As in the baseline model, based on Equations (50), (53), and (54), we can derive that $\Pi_n^* \geq \Pi_h^*$ if and only if $m \geq m_1$, $\Pi_n^* \geq \Pi_{hc}^*$ if and only if $m \geq m_2$, $\Pi_l^* \geq \Pi_h^*$ if and only if $m \geq m_3$, and $\Pi_l^* \geq \Pi_{hc}^*$ if and only if $m \geq m_4$, where

$$\begin{cases} m_1 = 2y + \frac{\lambda - 2\sqrt{2\lambda(1-\lambda)y}}{1-\lambda} \\ m_2 = \frac{2(1-\lambda)cy}{c+\lambda-c\lambda} - c \\ m_3 = 2y + \frac{\lambda - \sqrt{\lambda(1-\lambda)[(2y-2\lambda y+c-1)^2+8y]}}{1-\lambda} \\ m_4 = \frac{[2(1-\lambda)y-c][c(4-3\lambda)+2\lambda(1-(1-\lambda)y)]-\lambda}{4c(1-\lambda)+4\lambda} \end{cases} \tag{55}$$

Notice that m_1 and $\bar{m}$ intersect at $\frac{\sqrt{2y\lambda(1-\lambda)}-\lambda}{1-\lambda}$, which is less than $\hat{c}$ if and only if $y \leq \frac{2-\lambda-\sqrt{(4-3\lambda)\lambda}}{4(1-\lambda)^3}$.

Therefore, we can define $\hat{m}(c)$ as follows:

$$\begin{cases} \text{when } y \leq \frac{2-\lambda-\sqrt{(4-3\lambda)\lambda}}{4(1-\lambda)^3}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq c_1 \\ m_2 & \text{if } c_1 < c \leq \hat{c} \\ m_4 & \text{if } \hat{c} < c \end{cases} \\ \text{when } y > \frac{2-\lambda-\sqrt{(4-3\lambda)\lambda}}{4(1-\lambda)^3}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \hat{c} \\ m_3 & \text{if } \hat{c} < c \leq c_2 \\ m_4 & \text{if } c_2 < c \end{cases} \end{cases} \tag{56}$$

where m_1, m_2, m_3, and m_4 are defined in Equation (55), and

$$\begin{cases} c_1 = \frac{-\lambda+\sqrt{2\lambda(1-\lambda)y}}{1-\lambda} \\ c_2 = \frac{-[5-2(1-\lambda)y](1-\lambda)\lambda+2\sqrt{\lambda(1-\lambda)[1+2y(1-\lambda)(2+2y(1-\lambda)^3-3\lambda)]}}{(1-\lambda)(4-5\lambda)} \end{cases}$$

are the intersection between m_1 and m_2 and the intersection between m_3 and m_4, respectively. When $m < \hat{m}(c)$, the seller offers a high rebate. Altogether, the equilibrium rebate and price decisions can be summarized as in the proposition. □

Proof of Proposition 10

Proof. As in the baseline model, $p_1^* = x + y$ and $p_2^* = x + 2\hat{\lambda}y$, where $\hat{\lambda}$ is a function of s. For the high-rebate case, because $n_o = 0$, we have the same results as in the baseline model. We next consider the low-rebate case ($0 \leq s < c$). As in the baseline model, we have $\hat{\lambda} = \frac{1-c+s}{2-2c+s}$, and the seller's profit function is

$$\Pi_l = (x+y) + [x + \tfrac{2(1-c+s)}{2-2c+s}y] - s \cdot \tfrac{1}{2}[1-(c-s)] \tag{57}$$

By the first-order condition (i.e., letting $\frac{\partial \Pi_l}{\partial s} = 0$), we have the optimal rebate to be either 0 or as

$$s_l^* = \{s \in (0,1) | (1-c+2s)(2-2c+s)^2 - 4y(1-c) = 0\} \tag{58}$$

We notice that $s_l^* > 0$ requires $c > \hat{c} \equiv 1 - \sqrt{y}$, and we can verify that $s_l^* < c$. When $c \leq \hat{c}$, the optimal rebate is 0, the optimal second-period price is $p_{2n}^* = x + y$, and the optimal profit is $\Pi_n^* = 2(x + y)$. When $c > \hat{c}$, s_l^* in Equation (58) is the optimal solution to the constrained optimization problem for the low-rebate case. By substituting s_l^* into Equation (57), we derive the resulting profit as $\Pi_l^*(s_l^*)$.

As in the baseline model, $\Pi_n^* \geq \Pi_h^*$ if and only if $m \geq m_1$, and $\Pi_n^* \geq \Pi_{hc}^*$ if and only if $m \geq m_2$, where m_1 and m_2 are defined in Equation (20). Similarly, $\Pi_l^* \geq \Pi_h^*$ if and only if $m \geq m_3$, and $\Pi_l^* \geq \Pi_{hc}^*$ if and only if $m \geq m_4$, where $m_3 = \{m | \Pi_l^* = \Pi_h^*\}$ and $m_4 = \{m | \Pi_l^* = \Pi_{hc}^*\}$. Notice that m_1 and $\bar{m}$ intersect at $-1 + \sqrt{2y}$, which is less than $\hat{c}$ if and only if $y < 12 - 8\sqrt{2}$. Therefore, we can define $\hat{m}(c)$ as follows:

$$\begin{cases} \text{when } y < 12 - 8\sqrt{2}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \sqrt{2y} - 1 \\ m_2 & \text{if } \sqrt{2y} - 1 < c \leq \hat{c} \\ m_4 & \text{if } \hat{c} < c \end{cases} \\ \text{when } y \geq 12 - 8\sqrt{2}, \quad \hat{m}(c) = \begin{cases} m_1 & \text{if } c \leq \hat{c} \\ m_3 & \text{if } \hat{c} < c \leq \tilde{c} \\ m_4 & \text{if } \tilde{c} < c \end{cases} \end{cases} \tag{59}$$

where $\tilde{c} = \{c \in (0, 1) | \bar{m}(c) = m_4\}$. Consequently, the equilibrium rebate and price decisions can be summarized as in the proposition. □