

## Appendix for Dynamics of Shared Security in the Cloud

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### Appendix A: Proofs for Section 2

#### A.1. Proof of Proposition 1

*Proposition 1* Recall

$$u_{i,t} = (1 - p(V_{j,t})c)b(y_{i,t}) - ks_{i,t} - \lambda S_{j,t-1}$$

Taking the first derivative with respect to other users' investment decisions,

$$\frac{\partial u_{i,t}}{\partial s_{i',t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{i',t}} - \lambda \frac{\partial S_{j,t}}{\partial s_{i',t}}$$

In a symmetric MPE equations (1) and (2) become

$$V_{j,t} = \sum_t \delta_V^t [(n-1)(y_{i',t} - s_{i',t}) + (y_{i,t} - s_{i,t}) - \alpha s_{cspj,t}]$$

$$S_{j,t} = \sum_t \delta_S^t [\alpha s_{cspj,t} + (n-1)s_{i',t} + s_{i,t}]$$

First, without switching costs ( $\lambda = 0$ ) the first-order derivative is

$$\frac{\partial u_{i,t}}{\partial s_{i',t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{i',t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} [-(n-1)\delta_V^t] > 0$$

Without switching costs, users' security investments are plain complements.

With switching costs, plain complements requires

$$\frac{\partial u_{i,t}}{\partial s_{i',t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{i',t}} - \lambda \frac{\partial S_{j,t}}{\partial s_{i',t}} > 0$$

Appealing again to the expressions for  $V_{j,t}$  and  $S_{j,t}$  in a symmetric MPE, it becomes

$$b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \delta_V^t > \lambda \delta_S^t$$

Without switching costs, a CSP's security investment is a plain complement for users,

$$\frac{\partial u_{i,t}}{\partial s_{cspj,t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{cspj,t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} [-\alpha \delta_V^t] > 0$$

With switching costs, plain complements requires

$$\frac{\partial u_{i,t}}{\partial s_{cspj,t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{cspj,t}} - \lambda \frac{\partial S_{j,t}}{\partial s_{cspj,t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} [-\alpha \delta_V^t] - \lambda \frac{\partial S_{j,t}}{\partial s_{cspj,t}} > 0$$

Given  $\frac{\partial S_{j,t}}{\partial s_{cspj,t}} = \delta_S^t \alpha$ , this reduces to

$$b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \delta_V^t > \lambda \delta_S^t$$

The condition is the same as for other user's security investments.

## A.2. Proof of Proposition 2

*Proposition 2* From the proof of Proposition 1

$$\frac{\partial u_{i,t}}{\partial s_{i',t}} = b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} (\delta_V(n-1)) - \lambda \delta_S^t (n-1)$$

The cross derivative is

$$\frac{\partial u_{i,t}^2}{\partial s_{i,t} \partial s_{i',t}} = \frac{\partial u_{i,t}^2}{\partial s_{i',t} \partial s_{i,t}} = -b(y_{i,t}) \delta_V \frac{\partial p^2(V_{j,t})}{\partial V_{j,t}^2} (\delta_V(n-1)) < 0$$

## A.3. Proof of Proposition 3

*Proposition 3* The first derivative is

$$\frac{\partial u_{i,t}}{\partial s_{i,t}} = -b(y_{i,t}) \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{i,t}} - k - \lambda \frac{\partial S_{j,t}}{\partial s_{i,t}}$$

The cross derivative is

$$\frac{\partial u_{i,t}^2}{\partial s_{i,t} \partial s_{cspj,t}} = b(y_{i,t}) \delta_V \frac{\partial p^2(V_{j,t})}{\partial V_{j,t}^2} (-\alpha \delta^t) < 0$$

## Appendix B: Proofs for Section 4

### B.1. Proof of Lemma 1

*Lemma 1* Following the argument in B-S, differentiating Bellman equation (8) with respect to state  $V_{j,t-1}$

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial V_{j,t-1}} = -cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial V_{j,t-1}} + \delta \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial V_{j,t-1}}$$

Given  $\partial V_{j,t} / \partial V_{j,t-1} = \delta_V$ ,

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial V_{j,t-1}} = \delta_V \left\{ -cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} + \delta \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right\}$$

Solving the best reply for  $y_{i,t}$  in (10) for  $\frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}}$  and substituting into the above equation

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial V_{j,t-1}} = -\delta_V (1 - p(V_{j,t})c) b' < 0$$

### B.2. Proof of Proposition 4

*Proposition 4* From lemma 1,  $(1 - p(V_{j,t})c) b' = -\frac{1}{\delta_V} \frac{\partial U_i(V_{t-1}, S_{t-1})}{\partial V_{j,t-1}}$ . Substituting this into the first term on the right-hand side of (10) and solving for  $\frac{\partial U_i(V_{t-1}, S_{t-1})}{\partial V_{j,t-1}}$  results in the Euler equation.

### B.3. Proof of Proposition 5

*Proposition 5* Totally differentiating equation (12) with respect to  $V_{j,t}^*$ :

$$\left[ -cb' \frac{\partial p(V_{j,t-1}^*)}{\partial V_{j,t-1}^*} - cb \frac{\partial^2 p(V_{j,t-1}^*)}{\partial^2 V_{j,t-1}^*} \right] \frac{\partial V_{j,t-1}^*}{\partial V_{j,t}^*} = -\delta \delta_V cb' \left[ \frac{\partial p(V_{j,t}^*)}{\partial V_{j,t}^*} \right] \frac{\partial V_{j,t}^*}{\partial V_{j,t}^*}$$

Given  $\frac{\partial p(V_{j,t})}{\partial V_{j,t}} > 0$ ,  $\frac{\partial p^2(V_{j,t})}{\partial V_{j,t}^2} > 0$ , and rearranging terms

$$\frac{\partial V_{j,t}^*}{\partial V_{j,t-1}^*} = \frac{cb' \frac{\partial p(V_{j,t-1}^*)}{\partial V_{j,t-1}^*} + cb \frac{\partial^2 p(V_{j,t-1}^*)}{\partial^2 V_{j,t-1}^*}}{\delta \delta_V cb' \frac{\partial p(V_{j,t}^*)}{\partial V_{j,t}^*}} > 0 \quad (21)$$

All terms in the numerator and the denominator of equation (21) are positive. Hence,  $\partial V_{j,t}^* / \partial V_{j,t-1}^* > 0$ . This and a positive second derivative of  $p(\cdot)$  implies the  $\partial p(V_{j,t}^*) / \partial V_{j,t}^*$  in the denominator becomes larger. It is, therefore, possible the denominator is greater than the first term in the numerator. Hence, whether  $\partial V_{j,t}^* / \partial V_{j,t-1}^* > 1$  depends on  $\delta$ ,  $\delta_V$ , and the second term in the numerator.

#### B.4. Proof of Proposition 6

*Proposition 6* From equations (12), the Euler equation of optimal  $V_{j,t-1}^*$  and  $V_{j,t}^*$  is:

$$[1 - p(V_{j,t-1}^*)c]b'(y_{i,t-1}) - b(y_{i,t-1}) \frac{\partial p(V_{j,t-1}^*)}{\partial V_{j,t-1}^*} = \delta \delta_V [1 - p(V_{j,t}^*)c]b'(y_{i,t})$$

Substituting in the components of  $V_{j,t}^*$  as a function of  $V_{j,t-1}^*$ ,  $S_{j,t}$  and  $y_{i,t}$ , (with  $y_{i',t} = y_{i,t}$  in a symmetric MPE):

$$[1 - p(V_{j,t-1}^*)c]b'(y_{i,t-1}) - b(y_{i,t-1}) \frac{\partial p(V_{j,t-1}^*)}{\partial V_{j,t-1}^*} = \delta \delta_V [1 - p(\delta_V V_{j,t-1}^* + ny_{i,t} - S_{j,t})c]b'(y_{i,t}) \quad (22)$$

For any given value of the left-hand side of equation (22), if  $S_{j,t}$  increases,  $y_{i,t}$  must commensurately increase, holding all other variables constant:

$$\frac{\partial y_{i,t}}{\partial S_{j,t}} > 0$$

#### B.5. Proof of Proposition 7

*Proposition 7* From Proposition 6,

$$\frac{\partial y_{i,t}}{\partial S_{j,t}} > 0$$

where  $S_{j,t} \equiv nS_{i,t} + \alpha S_{cspj,t} = S_{i,t} + (n-1)S_{i',t} + \alpha S_{cspj,t}$ .

Suppose the increase in  $S_{j,t}$  is solely from other users. If  $S_{i',t}$ ,  $i' \neq i$  increases but  $S_{i,t}$  is constant ( $\Rightarrow s_{i,t} = 0$ ) and  $S_{cspj,t}$  is constant ( $\Rightarrow s_{cspj,t} = 0$ ), then  $v_{i,t} = y_{i,t} - s_{i,t}$  must increase because  $y_{i,t}$  increases in  $S_{j,t}$ .

#### B.6. Proof of Lemma 2

*Lemma 2* The rationale for this lemma and its proof is as follows. By backward induction, given the optimal  $y_{i,t}$ 's, from Bellman equation (8),

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial s_{i,t}} = -cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{i,t}} \frac{\partial S_{i,t}}{\partial s_{i,t}} - k + \delta \left[ \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} \frac{\partial S_{i,t}}{\partial s_{i,t}} + \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{i,t}} \frac{\partial S_{i,t}}{\partial s_{i,t}} \right] = 0$$

Substituting in  $\frac{\partial V_{j,t}}{\partial S_{i,t}} = -1$  and  $\frac{\partial S_{j,t}}{\partial s_{i,t}} = 1$ :

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial s_{i,t}} = cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} - k + \delta \left[ \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right] = 0 \quad (23)$$

where  $\partial U_i(V_{j,t}, S_{j,t})/\partial S_{j,t}$  is unknown. Hence, the need for the lemma.

Once again, characterizing the implicit best reply function for  $s_{i,t}$  in (23) requires analyzing the Bellman equation (8) using the B-S procedure with respect to state  $S_{j,t}$ . Given the optimal  $s_{i,t}$  from (23), differentiating Bellman equation (8) with respect to state  $S_{j,t-1}$ :

$$\begin{aligned} \frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} &= -bc \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} - \frac{k}{n} \left( \frac{\partial S_{j,t}}{\partial S_{j,t-1}} - \delta_S \right) \\ &+ \delta \left( \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} + \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} \right) \end{aligned}$$

As  $\frac{\partial S_{j,t}}{\partial S_{j,t-1}} = \delta_S$ , and  $\frac{\partial V_{j,t}}{\partial S_{j,t}} = -1$ :

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = \delta_S bc \frac{\partial p(V_{j,t})}{\partial V_{j,t}} + \delta_S \delta \left( \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right)$$

Substituting the value for  $\frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}}$  derived from the implicit best reply function for  $s_{i,t}$  in (23) yields:

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = \delta_S k$$

Thus proving the lemma. Upon iterating one period, it is the unknown term in first-order condition (23).

## Appendix C: Proofs for Section 5

### C.1. Proof of Lemma 4

*Lemma 4* Given an infinite horizon, the characterizations in lemmas 1 and 2 continue to hold for all  $i \in \{stay, switch\}$ , where  $j$  is the relevant host CSP  $j \in \{1, 2\}$ :

$$\begin{aligned} \frac{\partial U_i}{\partial V_{j,t-1}} &= -\delta_V [1 - p(V_{j,t})c]b', \quad \frac{\partial^2 U_i}{\partial V_{j,t-1}^2} = \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \delta_V^2 cb' \\ \frac{\partial U_i}{\partial S_{j,t-1}} &= \delta_S k \Rightarrow \frac{\partial^2 U_i}{\partial S_{j,t-1}^2} = 0, \quad \frac{\partial U_i}{\partial S_{j,t-1} \partial V_{j,t-1}} = 0 \end{aligned}$$

In addition, from lemma 3:  $\frac{\partial U_{switch}}{\partial S_{1,t-1}} = -\lambda \Rightarrow \frac{\partial^2 U_{switch}}{\partial S_{1,t-1}^2} = 0$ .

As  $U_{stay}$  is a function of  $V_{1,t-1}, S_{1,t-1}$ , and  $U_{switch}$  is a function of  $V_{2,t-1}, S_{2,t-1}$  and  $S_{1,t-1}$ , we employ the partial derivatives expressed above in order to generate a second-order Taylor series as approximations of  $U_{stay}$  and  $U_{switch}$ . Such a quadratic approximation is appropriate because two of the second-order partials above equal zero, as does the cross-partial. Higher-order partial derivatives are equal to zero as well. Furthermore, a Taylor expansion is around zero, and  $U_{stay}(0, 0) = 0$ ,  $U_{switch}(0, 0, 0) = 0$ . It follows that the Taylor expansions are

$$\begin{aligned} U_{stay}(V_{1,t-1}, S_{1,t-1}) &= \delta_S k S_{1,t-1} - \delta_V [1 - p(V_{1,t})c]b' V_{1,t-1} + \frac{1}{2} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 + \zeta_{stay} \\ U_{switch}(V_{2,t-1}, S_{2,t-1}, S_{1,t-1}) &= -\lambda S_{1,t-1} + \delta_S k S_{2,t-1} - \delta_V [1 - p(V_{2,t})c]b' V_{2,t-1} \\ &\quad + \frac{1}{2} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 + \zeta_{switch} \end{aligned}$$

where  $\zeta_{stay}$  and  $\zeta_{switch}$  represent the residuals. From here on we ignore the residuals and focus on the quadratic approximations.

CSP 1 only needs to make a sufficient security investment such that the no-switching constraint is binding. Call this  $S_{1,t}^{min}$  such that  $U_{stay} = U_{switch}$ . That is,

$$\begin{aligned} &\delta_S k S_{1,t-1}^{min} - \delta_V [1 - p(V_{1,t})c]b' V_{1,t-1} + \frac{1}{2} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 \\ &= -\lambda S_{1,t-1}^{min} + \delta_S k S_{2,t-1} - \delta_V [1 - p(V_{2,t})c]b' V_{2,t-1} + \frac{1}{2} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 \\ S_{1,t-1}^{min} &= \frac{1}{\lambda} \{ \delta_S k (S_{2,t-1} - S_{1,t-1}^{min}) - \delta_V b' \{ [1 - p(V_{2,t})c]V_{2,t-1} - [1 - p(V_{1,t})c]V_{1,t-1} \} \\ &\quad + \frac{1}{2} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 - \frac{1}{2} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 \} \end{aligned}$$

this is the expression for  $S_{1,t-1}^{min}$  given in Lemma 4.

We now turn to the claim that the no-switching condition can hold even when  $S_{1,t-1} < S_{2,t-1}$  and  $V_{1,t-1} > V_{2,t-1}$ . Pulling all  $S_{1,t-1}^{min}$  terms to the left-hand side of the above equality:

$$\begin{aligned} \left(1 + \frac{1}{\lambda} \delta_S k\right) S_{1,t-1}^{min} &= \frac{1}{\lambda} \delta_S k S_{2,t-1} - \frac{1}{\lambda} \{ \delta_V b' \{ [1 - p(V_{2,t})c]V_{2,t-1} - [1 - p(V_{1,t})c]V_{1,t-1} \} \\ &\quad + \frac{1}{2\lambda} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 - \frac{1}{2\lambda} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 \} \end{aligned} \quad (24)$$

Since  $p'(V_{j,t}) > 0$  and  $p''(V_{j,t}) > 0$  and we assume  $V_{2,t-1} < V_{1,t-1}$ :

$$\Omega = \frac{1}{2\lambda} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 - \frac{1}{2\lambda} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 < 0$$

If we assume  $1 - p'(V_{j,t})\delta_V c < 0$ ,<sup>8</sup>

$$\Psi = -\frac{\delta_V b'}{\lambda} \{[1 - p(V_{2,t})c]V_{2,t-1} - [1 - p(V_{1,t})c]V_{1,t-1}\} < 0$$

Then the above expression for  $S_{1,t-1}^{\min}$  in (27) can be written as:

$$\left(1 + \frac{1}{\lambda} \delta_S k\right) S_{1,t-1}^{\min} = \frac{1}{\lambda} \delta_S k S_{2,t-1} + \underbrace{[\Psi + \Omega]}_{(-)} \quad (25)$$

For this to hold it must be the case that  $S_{1,t-1}^{\min} < S_{2,t-1}$

### C.2. Proof of Proposition 8

$$\begin{aligned} \left(1 + \frac{1}{\lambda} \delta_S k\right) S_{1,t-1}^{\min} &= \frac{1}{\lambda} \delta_S k S_{2,t-1} - \frac{1}{\lambda} \delta_V b' \{[1 - p(V_{2,t})c]V_{2,t-1} - [1 - p(V_{1,t})c]V_{1,t-1}\} \\ &\quad + \frac{1}{2\lambda} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 - \frac{1}{2\lambda} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 \end{aligned} \quad (26)$$

$$\begin{aligned} S_{1,t-1}^{\min} &= \frac{1}{(\lambda + \delta_S k)} \{ \delta_S k S_{2,t-1} - \delta_V b' \{[1 - p(V_{2,t})c]V_{2,t-1} - [1 - p(V_{1,t})c]V_{1,t-1}\} \\ &\quad + \frac{1}{2} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 cb' V_{2,t-1}^2 - \frac{1}{2} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 cb' V_{1,t-1}^2 \} \end{aligned} \quad (27)$$

### C.3. Proof of Proposition 9

The no-switching condition in equation (27) can be written as a function of participation constraints (15) and (16),

$$\left(1 + \frac{1}{\lambda} \delta_S k\right) S_{1,t-1}^{\min} = \frac{1}{\lambda} \delta_S k S_{2,t-1}^{pc} + \frac{1}{\lambda} \delta_S k S_{1,t-1}^{pc} \quad (28)$$

By symmetry, if users of CSP 1's participation constraint is met, it is met for users of CSP 2 as well. It follows that  $S_{2,t-1}^{pc} = S_{1,t-1}^{pc}$ . Substituting into equation (28):

$$\left(1 + \frac{1}{\lambda} \delta_S k\right) S_{1,t-1}^{\min} = \frac{2}{\lambda} \delta_S k S_{1,t-1}^{pc}$$

We can see  $S_{1,t-1}^{\min} < S_{1,t-1}^{pc}$  if:

$$1 + \frac{1}{\lambda} \delta_S k > \frac{2}{\lambda} \delta_S k$$

which is:

$$\lambda > \delta_S k$$

The left hand is the per-unit marginal cost of accumulated security for users because of the lock-in effect. The right hand is the depreciation speed of security investment times the per-unit cost of security investment for users. And from lemma 1, it is the marginal benefit of accumulated security for users.

<sup>8</sup> Recall  $\delta_V, c \in (0, 1)$ .

#### C.4. Proof of Proposition 10

By backward induction, given the optimal  $y_{i,t}$  and optimal  $s_{i,t}$ :

$$\begin{aligned} \frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial s_{cspj,t}} &= -CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{cspj,t}} \frac{\partial S_{cspj,t}}{\partial s_{cspj,t}} - K \\ &+ \delta \left[ \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial s_{cspj,t}} + \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial s_{cspj,t}} \frac{\partial S_{cspj,t}}{\partial s_{cspj,t}} \right] = 0 \end{aligned}$$

Substituting in  $\frac{\partial V_{j,t}}{\partial s_{cspj,t}} = -\alpha$ ,  $\frac{\partial S_{j,t}}{\partial s_{cspj,t}} = \alpha$ , and  $\frac{\partial S_{cspj,t}}{\partial s_{cspj,t}} = 1$ :

$$\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial s_{cspj,t}} = CnB\alpha \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} - K + \delta\alpha \left[ \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right] = 0 \quad (29)$$

where  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}}$  and  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}}$  are unknowns.

At the optimum the term in brackets takes the value

$$\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} = \frac{K}{\delta\alpha} - \frac{CnB}{\delta} \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \quad (30)$$

While it is asserted  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}}$  is an unknown, in actuality  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} = 0$  because, as we argue in the text, by the process of backward induction, once the CSP selects its  $s_{cspj,t}$  there are no degrees of freedom left for determining  $S_{j,t}$ . Selecting  $s_{cspj,t}$  is equivalent to selecting  $S_{j,t}$  and, at the optimum,  $\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial s_{cspj,t}} = 0$ . For purposes of completeness, we prove  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} = 0$ . Once this is established, equation (30) is used to solve for  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}}$  along the optimal path.

Employing the B-S procedure by differentiating (17) with respect to state  $S_{j,t-1}$ :

$$\begin{aligned} \frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} &= -CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} - K \frac{\partial s_{cspj,t}}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} \\ &+ \delta \left[ \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} + \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{j,t}} \frac{\partial S_{j,t}}{\partial S_{j,t-1}} \right] \end{aligned}$$

Directly differentiating,  $\frac{\partial S_{j,t}}{\partial S_{j,t-1}} = \delta_S$ ,  $\frac{\partial V_{j,t}}{\partial S_{j,t}} = -1$  and  $\frac{\partial S_{j,t}}{\partial s_{cspj,t}} = \alpha$ , which is a constant. Hence, by the inverse function rule for partial differentiation,  $\frac{\partial s_{cspj,t}}{\partial S_{j,t}} = \frac{1}{\alpha}$ . The above equation becomes

$$\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = \delta_S CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} - \frac{\delta_S}{\alpha} K + \delta_S \delta \left[ \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right]$$

Substituting in the value,  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} = \frac{K}{\delta\alpha} - \frac{CnB}{\delta} \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}}$ , from equation (30),

$$\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = 0$$

That is, our intuition with respect to  $\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial s_{cspj,t}}$  and  $\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial S_{j,t}}$  at the optimum holds. A CSP's security investment only impacts its optimal value function by reducing total vulnerability. Substituting this value into equation (30),

$$\frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} = \frac{1}{\delta} \left[ CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} - \frac{K}{\alpha} \right] \quad (31)$$

Returning again to the B-S procedure,

$$\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial V_{j,t-1}} = -CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial V_{j,t-1}} + \delta \frac{\partial \Pi_j(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial V_{j,t-1}}$$

Substituting in  $\frac{\partial V_{j,t}}{\partial V_{j,t-1}} = \delta_V$  and the value from equation (31),

$$\frac{\partial \Pi_j(V_{j,t-1}, S_{j,t-1})}{\partial V_{j,t-1}} = -CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \delta_V + \delta \frac{1}{\delta} CnB \frac{\partial \tilde{p}(V_{j,t})}{\partial V_{j,t}} \delta_V - \frac{1}{\delta} \frac{K}{\alpha} \delta_V \delta = -\delta_V \frac{K}{\alpha}$$

## Appendix D: Proof for Section 6

### D.1. Proof of Lemma 6

$$U_i(V_{j,t-1}, S_{j,t-1}) = \max_{s_{i,t}, y_{i,t}} \{u_{i,t}(\cdot) + \delta U_i(V_{j,t}, S_{j,t})\}$$

Given the optimal  $y_{i,t}$ ,  $\frac{\partial U_i}{\partial s_{i,t}} = 0$ .

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial s_{i,t}} = -cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{i,t}} \frac{\partial S_{i,t}}{\partial S_{i,t}} - k S_{j,t} \frac{\partial S_{j,t}}{\partial S_{i,t}} + \delta \left[ \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} \frac{\partial S_{i,t}}{\partial S_{i,t}} + \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \frac{\partial V_{j,t}}{\partial S_{i,t}} \frac{\partial S_{i,t}}{\partial S_{i,t}} \right] = 0$$

Substituting in  $\frac{\partial V_{j,t}}{\partial S_{i,t}} = -1$  and  $\frac{\partial S_{j,t}}{\partial S_{i,t}} = 1$ :

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{i,t}} = cb \frac{\partial p(V_{j,t})}{\partial V_{j,t}} - k S_{j,t} + \delta \left[ \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right] = 0 \quad (32)$$

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = \delta_S bc \frac{\partial p(V_{j,t})}{\partial V_{j,t}} - \frac{k}{2} (2S_{j,t} \delta_S - \delta_S^2 2S_{j,t-1}) + \delta_S \delta \left( \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}} \right)$$

Substituting the value for  $\frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial S_{j,t}} - \frac{\partial U_i(V_{j,t}, S_{j,t})}{\partial V_{j,t}}$  derived from the implicit best reply function for  $s_{i,t}$  in (32) yields:

$$\frac{\partial U_i(V_{j,t-1}, S_{j,t-1})}{\partial S_{j,t-1}} = \delta_S^2 k S_{j,t-1}$$

### D.2. Proof of Lemma 7

$$\frac{\partial U_i}{\partial S_{j,t-1}} = \delta_S^2 k S_{j,t-1} \Rightarrow \frac{\partial^2 U_i}{\partial S_{j,t-1}^2} = \delta_S^2 k, \quad \frac{\partial U_i}{\partial S_{j,t-1} \partial V_{j,t-1}} = 0$$

$\frac{\partial U_{switch}}{\partial S_{1,t-1}} = -\lambda \Rightarrow \frac{\partial^2 U_{switch}}{\partial S_{1,t-1}^2} = 0$  does not change.

The second-order Taylor series as approximations of  $U_{stay}$  changes,

$$U_{stay}(V_{1,t-1}, S_{1,t-1}) = \delta_S^2 k S_{1,t-1}^2 + \delta_S^2 k S_{1,t-1} - \delta_V [1 - p(V_{1,t})] c b' V_{1,t-1} + \frac{1}{2} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 c b' V_{1,t-1}^2 + \zeta_{stay}$$

$U_{switch}$  does not change,

$$U_{switch}(V_{2,t-1}, S_{2,t-1}, S_{1,t-1}) = -\lambda S_{1,t-1} + \delta_S k S_{2,t-1} - \delta_V [1 - p(V_{2,t})] c b' V_{2,t-1} + \frac{1}{2} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 c b' V_{2,t-1}^2 + \zeta_{switch}$$

where  $\zeta_{stay}$  and  $\zeta_{switch}$  represent the residuals. Terms about  $V$  do not change. In the previous proof, since  $p'(V_{j,t}) > 0$  and  $p''(V_{j,t}) > 0$  and we assume  $V_{2,t-1} < V_{1,t-1}$ :

$$\Omega = \frac{1}{2\lambda} \frac{\partial p(V_{2,t})}{\partial V_{2,t}} \delta_V^2 c b' V_{2,t-1}^2 - \frac{1}{2\lambda} \frac{\partial p(V_{1,t})}{\partial V_{1,t}} \delta_V^2 c b' V_{1,t-1}^2 < 0$$

If we assume  $1 - p'(V_{j,t}) \delta_V c < 0$ ,<sup>9</sup>

$$\Psi = -\frac{1}{\lambda} \{ \delta_V b' \{ [1 - p(V_{2,t})] c \} V_{2,t-1} - [1 - p(V_{1,t})] c \} V_{1,t-1} < 0$$

Then the  $S_{1,t-1}^{minq}$  can be written as,

$$2\delta_S^2 k S_{1,t-1}^{minq^2} + \lambda S_{1,t-1}^{minq} = \delta_S k S_{2,t-1} + [\Psi + \Omega]$$

$$\left( 1 + 2\frac{1}{\lambda} \delta_S^2 k S_{1,t-1}^{minq} \right) S_{1,t-1}^{minq} = \frac{1}{\lambda} \delta_S k S_{2,t-1} + [\Psi + \Omega]$$

<sup>9</sup> Recall  $\delta_V, c \in (0, 1)$ .